Phys 503

Lectures 20-23

Canonical transformations

Canonical transformations

(1) Time-independent canonical transformations (generalize below)

Coordinate transformations that leave the form of Hamilton's equations fixed for all possible Hamiltonians

$$Q_j = Q_j(q,p) \qquad \text{with} \qquad \dot{Q}_j = \frac{\partial H}{\partial P_j} \text{ and } \dot{P}_j = -\frac{\partial H}{\partial Q_j}$$

AND

$$P_j = P_j(q,p) \qquad \dot{q}_j = \frac{\partial H}{\partial p_j} \text{ and } \dot{p}_j = -\frac{\partial H}{\partial q_j}$$

Much broader than linear transformations

Symplectic notation:

$$\vec{\zeta} = \begin{pmatrix} Q_1 \\ \vdots \\ Q_n \\ P_1 \\ \vdots \\ P_n \end{pmatrix}, \qquad \vec{\eta} = \begin{pmatrix} q_1 \\ \vdots \\ q_n \\ p_1 \\ \vdots \\ p_n \end{pmatrix} \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\dot{\vec{\zeta}} = J \frac{\partial H}{\partial \vec{\zeta}} \qquad \dot{\vec{\eta}} = J \frac{\partial H}{\partial \vec{\eta}}$$

OR

$$\dot{\zeta}_J = J_{JK} \frac{\partial H}{\partial \zeta_K} \qquad \dot{\eta}_J = J_{JK} \frac{\partial H}{\partial \eta_K}$$

caps

not a linear transformation

Do the transformation: $\dot{\zeta}_J = \underbrace{\frac{\partial \zeta_J}{\partial \eta_K}}_{\equiv M_{JK}} \dot{\eta}_K \iff \dot{\vec{\zeta}} = M \dot{\vec{\eta}}$

$\equiv M_{JK}$ ← fn. of coordinates

$$\frac{\partial H}{\partial \eta_J} = \frac{\partial H}{\partial \zeta_K} \underbrace{\frac{\partial \zeta_K}{\partial \eta_J}}_{M_{KJ}} \iff \frac{\partial H}{\partial \vec{\eta}} = \tilde{M} \frac{\partial H}{\partial \vec{\zeta}}$$

$$\dot{\vec{\mathcal{J}}} = M\dot{\vec{\eta}} = MJ\frac{\partial H}{\partial \vec{\eta}} = \underbrace{MJ\tilde{M}}_{=J}\frac{\partial H}{\partial \vec{\mathcal{J}}}$$

Canonical transformation $\Longleftrightarrow$ symplectic transformation :

Independent of the form of the Hamiltonian

$$\boxed{MJ\tilde{M} = J \quad\Longleftrightarrow\quad MJ = J\tilde{M}^{-1}}$$

preservation of the canonical 2-form

quadratic relation

linear relation

Take transpose: $\tilde{J}\tilde{M} = M^{-1}\tilde{J}$ $\Rightarrow$ $JM = \tilde{M}^{-1}J$ $\Rightarrow$

What's M^{-1}? $\quad \dot{\eta}_J = \underbrace{\frac{\partial \eta_J}{\partial \mathcal{J}_K}}_{(M^{-1})_{JK}}\dot{\mathcal{J}}_K \Longleftrightarrow \dot{\vec{\eta}} = M^{-1}\dot{\vec{\mathcal{J}}}$

$$\Rightarrow JM = \tilde{M}^{-1}J$$

$$\Rightarrow \boxed{\tilde{M}JM = J}$$

$$\boxed{\frac{\partial \mathcal{J}_J}{\partial \eta_L}J_{LK} = J_{JL}(M^{-1})_{KL} = J_{JL}\frac{\partial \eta_K}{\partial \mathcal{J}_L}}$$

4 equations:

$$\frac{\partial Q_j}{\partial p_k} = \frac{\partial q_k}{\partial P_j} \qquad \frac{\partial Q_j}{\partial q_k} = \frac{\partial p_k}{\partial P_j}$$

$$\frac{\partial P_j}{\partial p_k} = -\frac{\partial q_k}{\partial Q_j} \qquad \frac{\partial P_j}{\partial q_k} = -\frac{\partial p_k}{\partial Q_j}$$

Direct conditions

We never use these. Why?

1. If we're going to use the symplectic approach, we use the efficient matrix notation
2. To find canonical transformation, we use a generating function, which can be specified freely.

Constructive procedure

Show $\boxed{p_j dq_j = P_j dQ_j + dF}$ is equivalent to direct conditions (obvious if we used forms)

(a) Direct demonstration

(b) Can also see from Hamilton's principle

$$\hookrightarrow a_J dy_J \equiv p_j dq_j - P_j dQ_j$$

Defining $J' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\tilde{J}' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$J' - \tilde{J}' = J$

$$= J'_{JK} y_K dy_J - \underbrace{J'_{JK} \mathcal{J}_K d\mathcal{J}_J}$$

$$J'_{LK} \mathcal{J}_K \underbrace{\frac{\partial \mathcal{J}_L}{\partial y_J}}_{M_{LJ}} dy_J$$

$$= \underbrace{\left(J'_{JK} y_K - J'_{LM} \mathcal{J}_M M_{LJ} \right)}_{a_J} dy_J$$

Perfect differential: $\frac{\partial a_J}{\partial y_K} - \frac{\partial a_K}{\partial y_J} = 0$

$$\frac{\partial a_J}{\partial y_K} = J'_{JK} - J'_{LM} \underbrace{\frac{\partial \mathcal{J}_M}{\partial y_K}}_{M_{MK}} M_{LJ} - J'_{LM} \mathcal{J}_M \frac{\partial M_{LJ}}{\partial y_K}$$

$$\frac{\partial M_{LJ}}{\partial y_K} \rightarrow \frac{\partial^2 \mathcal{J}_L}{\partial y_J \partial y_K}$$

$$= J'_{JK} - J'_{LM} M_{LJ} M_{MK} - J'_{LM} \mathcal{J}_M \frac{\partial^2 \mathcal{J}_L}{\partial y_J \partial y_K}$$

$$\frac{\partial a_J}{\partial y_K} - \frac{\partial a_K}{\partial y_K} = \underbrace{J'_{JK} - J'_{KJ}}_{J_{JK}} - \underbrace{(\underbrace{J'_{LM} - J'_{ML}}_{J_{LM}}) M_{LJ} M_{MK}}_{\tilde{M}_{JL} J_{LM} M_{MK} = J_{JK}}$$

$$= J_{JK} - \tilde{M}_{JL} J_{LM} M_{MK}$$

$\therefore a_J dy_J = dF$ iff $J = \tilde{M} J M$

$$\therefore \quad \boxed{p_j\, dq_j - P_j\, dQ_j = dF}$$

Equivalent to direct conditions for a canonical transformation; this is what you want to remember

① $$p_j\, dq_j - P_j\, dQ_j = dF = \frac{\partial F_1}{\partial q_j} dq_j + \frac{\partial F_1}{\partial Q_j} dQ_j + \frac{\partial F_1}{\partial t} dt$$

$F_1(q, Q)$

$$\boxed{p_j = \frac{\partial F_1}{\partial q_j} \quad \text{and} \quad P_j = -\frac{\partial F_1}{\partial Q_j} \qquad K = H + \frac{\partial F_1}{\partial t}}$$

Ex: $F_1(q, Q) = q_j Q_j$

$p_j = Q_j$ and $P_j = -q_j$

(exchange)

② $$p_j\, dq_j + Q_j\, dP_j = d(\underbrace{F + P_j Q_j}_{F_2(q,P)}) = \frac{\partial F_2}{\partial q_j} dq_j + \frac{\partial F_2}{\partial P_j} dP_j$$

$$\boxed{p_j = \frac{\partial F_2}{\partial q_j} \quad \text{and} \quad Q_j = \frac{\partial F_2}{\partial P_j} \qquad K = H + \frac{\partial F_2}{\partial t}}$$

Ex: $F_2(q, P) = q_j P_j$

$p_j = P_j$ and $Q_j = q_j$ (identity)

$F_2 = f_j(q) P_j \Rightarrow Q_j = f_j(q)$ point transformation

$p_j = \frac{\partial f_\ell}{\partial q_j} P_\ell$

③ $$-q_j\, dp_j - P_j\, dQ_j = d(\underbrace{F - p_j q_j}_{F_3(p,Q)}) = \frac{\partial F_3}{\partial p_j} dp_j + \frac{\partial F_3}{\partial Q_j} dQ_j$$

$$\boxed{q_j = -\frac{\partial F_3}{\partial p_j} \quad \text{and} \quad P_j = -\frac{\partial F_3}{\partial Q_j} \qquad K = H + \frac{\partial F_3}{\partial t}}$$

In every case, the variables in F must cover phase space; e.g. there is no identity transformation using F_1.

(4) $-q_j\,dp_j + Q_j\,dP_j = d(\underbrace{F + P_jQ_j - p_jq_j}_{F_4(p,P)}) = \frac{\partial F_4}{\partial p_j}dp_j + \frac{\partial F_4}{\partial P_j}dP_j$

$$\boxed{\begin{gathered} q_j = -\frac{\partial F_4}{\partial p_j} \quad \text{and} \quad Q_j = \frac{\partial F_4}{\partial P_j} \\ K = H + \frac{\partial F_4}{\partial t} \end{gathered}}$$

(c) Time-dependent transformations:

$Q_j = Q_j(q,p,t)$

$P_j = P_j(q,p,t)$

→ Everything we have done applies with t held fixed, but the Hamiltonian must change

Use Hamilton's principle to find out how Hamiltonian changes:

(1) $0 = \delta\int dt\left(p_j\dot{q}_j - H\right) = \delta\int dt\left(P_j\dot{Q}_j - (\underbrace{H + \frac{\partial F_1}{\partial t}}_{K})\right)$ ← In using F_1, one is regarding q and Q as the independent variables. There is no such thing as $\partial q_j/\partial t$.

$\quad\quad\quad\quad\;\; \| $

$P_j\dot{Q}_j + \frac{dF_1}{dt}\boxed{-\frac{\partial F_1}{\partial t}}$

BC's? Why? At each time we make a canonical transformation

(2) $0 = \delta\int dt\left(p_j\dot{q}_j - H\right) = \delta\int dt\left(P_j\dot{Q}_j - (\overbrace{H + \frac{\partial F_2}{\partial t}}^{K})\right)$

$\quad\quad\quad\quad\;\; \| $

$-Q_j\dot{P}_j + \frac{dF_2}{dt} - \frac{\partial F_2}{\partial t} = P_j\dot{Q}_j - \frac{d(F_2 + P_jQ_j)}{dt} - \frac{\partial F_2}{\partial t}$ BC's?

⋮

Poisson brackets: Allows us to investigate non canonical variables

Let $u = u(q, p, t)$

must be canonically invariant

$$\frac{du}{dt} = \underbrace{\frac{\partial u}{\partial q_j}\dot{q}_j + \frac{\partial u}{\partial p_j}\dot{p}_j}_{} + \frac{\partial u}{\partial t} = [u, H] + \frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial u}{\partial p_j}\frac{\partial H}{\partial q_j} \equiv [u, H]$$

$[u, H] \equiv$ $\frac{\partial u}{\partial \eta_J}\dot{\eta}_J = \frac{\partial u}{\partial \eta_J} J_{JK} \frac{\partial H}{\partial \eta_K}$

Special case: $\dot{q}_j = [q_j, H] = \frac{\partial H}{\partial p_j}$, $\dot{p}_j = [p_j, H] = -\frac{\partial H}{\partial q_j}$

OR $\dot{\eta}_J = [\eta_J, H] = J_{JK}\frac{\partial H}{\partial \eta_K}$

Poisson bracket:

$$[u, v] = \frac{\partial u}{\partial q_j}\frac{\partial v}{\partial p_j} - \frac{\partial u}{\partial p_j}\frac{\partial v}{\partial q_j} = \frac{\partial u}{\partial \eta_J} J_{JK} \frac{\partial v}{\partial \eta_K} = \frac{\widetilde{\partial u}}{\partial \eta} J \frac{\partial v}{\partial \eta}$$

Special case: $[\eta_J, \eta_K] = J_{JK}$ ⟵ fundamental Poisson brackets ⟹ $[q_j, q_k] = [p_j, p_k] = 0$

$[q_j, p_k] = -[p_k, q_j] = \delta_{jk}$

Canonical invariance:

$$\frac{\partial u}{\partial \eta_J} J_{JK} \frac{\partial v}{\partial \eta_K} = \frac{\partial u}{\partial \zeta_L} \underbrace{\frac{\partial \zeta_L}{\partial \eta_J} J_{JK} \frac{\partial \zeta_M}{\partial \eta_K}}_{M_{LJ} J_{JK} \tilde{M}_{KM} = J_{LM}} \frac{\partial v}{\partial \zeta_M} = \frac{\partial u}{\partial \zeta_J} J_{JK} \frac{\partial v}{\partial \zeta_K}$$

Properties:

Like a matrix commutator

1. Antisymmetry: $[u, v] = -[v, u]$ ⟹ $[u, u] = 0$
2. Bilinearity: $[au + bv, w] = a[u, w] + b[v, w]$
3. $[uv, w] = [u, w]v + u[v, w]$
4. Jacobi's identity: $[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$

Simple but tedious to show

Constants of motion:

$$0 = \frac{du}{dt} = [u,H] + \frac{\partial u}{\partial t}$$

Example: free particle

$$H = \frac{p^2}{2m}$$

① $[p,H] = \frac{\partial p}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial p}{\partial p}\frac{\partial H}{\partial q} = 0$

② $u = q - \frac{pt}{m} = q(0)$

$$[u,H] + \frac{\partial u}{\partial t} = \underbrace{\frac{\partial u}{\partial q}\frac{\partial H}{\partial p}}_{\frac{p}{m}} - \underbrace{\frac{\partial u}{\partial p}\frac{\partial H}{\partial q}}_{0} + \underbrace{\frac{\partial u}{\partial t}}_{-\frac{p}{m}} = 0$$

$$\eta_J \to x_J \to \zeta_J$$

$$M_{JK} = \frac{\partial \zeta_J}{\partial \eta_K} = \frac{\partial \zeta_J}{\partial x_L}\frac{\partial x_L}{\partial \eta_K} = N_{JL}P_{LK}$$

Infinitesimal canonical transformations:

Why? Group structure

$$Q_j = q_j + \delta q_j$$
$$P_j = p_j + \delta p_j \Longrightarrow \zeta_J = \eta_J + \delta\eta_J = \eta_J + \epsilon J_{JK} u_K(\eta)$$

$$M_{JK} = \frac{\partial \zeta_J}{\partial \eta_K} = \delta_{JK} + \epsilon J_{JL}\frac{\partial u_L}{\partial \eta_K}$$

Condition for a canonical transformation:

$$J_{JK} = M_{JL} J_{LM} M_{KM}$$

$$= \left(\delta_{JL} + \epsilon J_{JN}\frac{\partial u_N}{\partial \eta_L}\right) J_{LM} \left(\delta_{KM} + \epsilon J_{KP}\frac{\partial u_P}{\partial \eta_M}\right)$$

$$= J_{JK} + \epsilon J_{J\cancel{M}_L} J_{K\cancel{P}_M} \frac{\partial u_{\cancel{P}}^{M}}{\partial \eta_{\cancel{M}_L}} + \epsilon J_{\cancel{L}_M K} J_{J\cancel{N}_L} \frac{\partial u_{\cancel{N}}^{L}}{\partial \eta_{\cancel{L}_M}}$$

$$\Longrightarrow 0 = J_{JL} J_{KM}\left(\frac{\partial u_M}{\partial \eta_L} - \frac{\partial u_L}{\partial \eta_M}\right)$$

$$\Longrightarrow 0 = \frac{\partial u_M}{\partial \eta_L} - \frac{\partial u_L}{\partial \eta_M} \Longrightarrow u_J = \frac{\partial G}{\partial \eta_J} \longleftarrow$$

infinitesimal version of earlier perfect differential result

$$\delta\eta_J = \epsilon J_{JK}\frac{\partial G}{\partial \eta_K} \Longleftrightarrow \vec{\delta\eta} = \epsilon J \frac{\partial G}{\partial \vec{\eta}}$$

$$= \epsilon[\eta_J, G]$$

G is the generator of the infinitesimal canonical transformation. (relation to finite generators?)

$$\delta\eta^J = \epsilon\gamma^{JK}\frac{\partial G}{\partial \eta^K}$$

$$\vec{\delta\eta} = \epsilon\,\Gamma\, dG$$

cf.

$$\vec{v}\,dt = dt\,\Gamma\, dH$$

$$F_2(q,P) = q_j P_j + \epsilon G(q,P)$$

$$Q_j = \frac{\partial F_2}{\partial P_j} = q_j + \epsilon\frac{\partial G}{\partial P_j}$$

$$p_j = \frac{\partial F_2}{\partial q_j} = P_j + \epsilon\frac{\partial G}{\partial q_j}$$

Do example of $G = q_j$ and $G = p_j$

cf. $\dot{\eta}_J\, dt = dt\, J_{JK}\frac{\partial H}{\partial \eta_K} = dt[\eta_J, H] = \delta\eta^J$

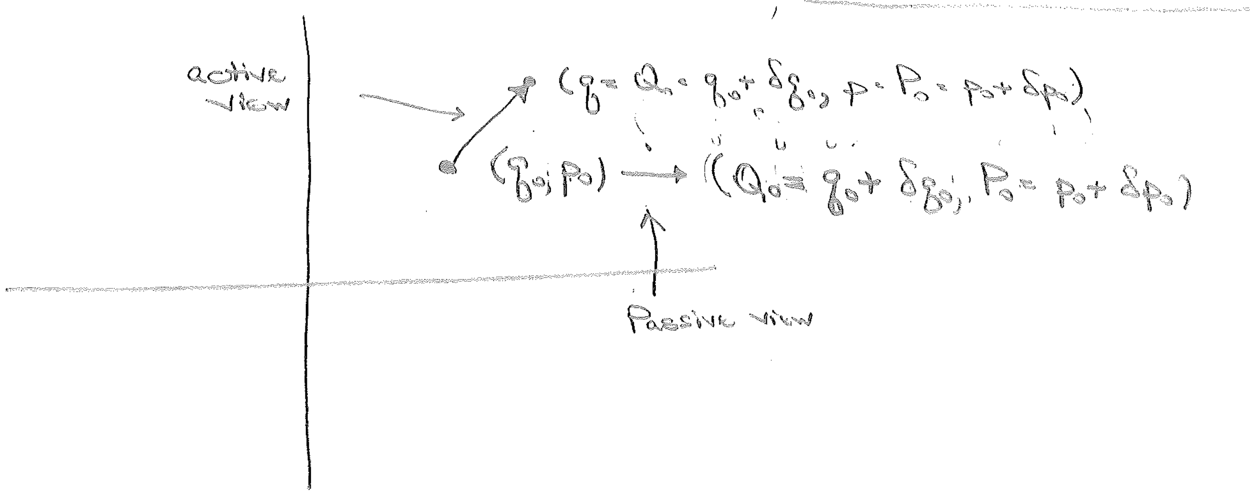
$\dot{\vec{\eta}}\, dt = \epsilon J\frac{\partial H}{\partial \vec{\eta}} = \vec{\delta\eta}$ ⟵ This is what happens if $G = H$

What does it mean? Dynamical evolution generates a canonical transformation. Suggests an active view of all canonical transformations.

Passive view vs. active view.

Do fns. of q and p change under this canonical transformation?

Passive vs. active view



Active view:

$$\delta u = u(q, p) - u(q_0, p_0)$$ ⟵ Must do something more general for Hamiltonian if t-d canonical transformation

$$= u(\vec{\eta} + \delta\vec{\eta}) - u(\vec{\eta})$$

$$\partial u = \frac{\partial u}{\partial \eta_J}\,\delta\eta_J = \epsilon \frac{\partial u}{\partial \eta_J} J_{JK} \frac{\partial G}{\partial \eta_K} = \boxed{\epsilon [u, G] = \partial u}$$

Note $\partial\eta_J = \epsilon[\eta_J, G] = \delta\eta_J \rightarrow$ Refer to examples $G = q_j$ and $G = p_j$

(relates active and passive)

$$\boxed{\begin{aligned} \partial u &= \epsilon\langle du, \vec{\delta\eta}\rangle \\ &= \epsilon\langle du, \Gamma\, dG\rangle \\ &= \epsilon\, \gamma(du, dG) \\ &= \epsilon[u, G] \end{aligned}}$$

$$\partial H = \epsilon[H, G] = -\epsilon \frac{dG}{dt}$$

Intermediate step requires that $\partial G/\partial t = 0$, but outer relation is always true

Constant of motion:

$$0 = \epsilon \frac{dG}{dt} = \epsilon[G, H] = -\partial H$$

$\rightarrow$ H is invariant under canonical transformation generated by G

$\leftarrow$ 3 different ways to view; relation to symmetries

relation to QM

Examples:

① Momentum conservation: $G(q, p) = p_j$ $\rightarrow$ doesn't have to be ordinary linear momentum

$$\delta\eta_J = \epsilon J_{JK} \frac{\partial G}{\partial \eta_K} \Longrightarrow \begin{aligned} \delta q_k &= \epsilon \frac{\partial G}{\partial p_k} = \epsilon\,\delta_{jk} \\ \delta p_k &= -\epsilon \frac{\partial G}{\partial q_k} = 0 \end{aligned}$$

② Angular momentum conservation: $G = \sum_i n_k \epsilon_{klm} x_l^{(i)} p_m^{(i)} = \vec{n}\cdot\sum_i \vec{x}^{(i)}\times\vec{p}^{(i)}$

$$\delta\eta_J = \epsilon J_{JK} \frac{\partial G}{\partial \eta_K} \Longrightarrow \delta x_j^{(i)} = \epsilon \frac{\partial G}{\partial p_j^{(i)}} = \epsilon n_k \epsilon_{klj} x_l^{(i)} = \epsilon(\vec{n}\times\vec{x}^{(i)})_j$$

$$\delta p_j^{(i)} = -\epsilon \frac{\partial G}{\partial x_j^{(i)}} = -\epsilon n_k \epsilon_{kjm} p_m^{(i)} = \epsilon(\vec{n}\times\vec{p}^{(i)})_j$$

Liouville's theorem:

$\rho(q, p, t) \longleftarrow$ phase-space density

$$\frac{d\rho}{dt} = [\rho, H] + \frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial \rho}{\partial p_j}\frac{\partial H}{\partial q_j} + \frac{\partial \rho}{\partial t}$$

Conservation of probability: $\dfrac{d(\rho \mathcal{V})}{dt} = 0$

Canonical invariance:

$$\frac{d\mathcal{V}}{dt} = \frac{1}{\Delta t}\left(\int_{\mathcal{V}(t+\Delta t)} d^{2N}\eta - \int_{\mathcal{V}(t)} d^{2N}\eta \right) = 0$$

$\Rightarrow \dfrac{d\rho}{dt} = 0$ (incompressible flow)

Another derivation: $\dfrac{1}{\mathcal{V}}\dfrac{d\mathcal{V}}{dt} = \dfrac{\partial \dot{\eta}_J}{\partial \eta_J} = \dfrac{\partial}{\partial \eta_J}\left(J_{JK}\dfrac{\partial H}{\partial \eta_K}\right) = 0$