Physics 521

Homework #4

Solution Set

4.1. C-T L_{III}.4

Free particle: $H = P^2/2m$

a. $$i\hbar \frac{d\langle X\rangle}{dt} = \langle [X,H]\rangle = \frac{1}{2m}\langle \underbrace{[X,P^2]}_{= 2P[X,P] = 2i\hbar P}\rangle = \frac{i\hbar}{m}P$$

$$i\hbar \frac{d\langle P\rangle}{dt} = \langle [P,H]\rangle = 0$$

$$\frac{d\langle P\rangle}{dt} = 0 \Rightarrow \boxed{\langle P\rangle = \langle P\rangle_0}$$

$$\frac{d\langle X\rangle}{dt} = \frac{\langle P\rangle}{m} = \frac{\langle P\rangle_0}{m} \Rightarrow \boxed{\langle X\rangle = \langle X\rangle_0 + \frac{\langle P\rangle_0 t}{m}}$$

b. $$i\hbar \frac{d\langle X^2\rangle}{dt} = \langle [X^2,H]\rangle = \frac{1}{2m}\langle [X^2,P^2]\rangle = \frac{i\hbar}{m}\langle XP + PX\rangle$$

$$[X^2,P^2] = X^2P^2 - P^2X^2$$

$$= X(P^2X + \underbrace{[X,P^2]}_{2Pi\hbar}) - P^2X^2$$

$$= (P^2X + [X,P^2])X + 2i\hbar XP - P^2X^2$$

$$= 2i\hbar(XP + PX)$$

$$\boxed{\frac{d\langle X^2\rangle}{dt} = \frac{1}{m}\langle XP + PX\rangle}$$

$$i\hbar \frac{d\langle XP+XP\rangle}{dt} = \langle [XP+PX, H]\rangle = \frac{1}{2m}\langle [XP+PX, P^2]\rangle$$

$$\begin{aligned}[XP+PX, P^2] &= [XP, P^2] + [PX, P^2] \\ &= \underbrace{[X, P^2]}_{2i\hbar P}P + P\underbrace{[X, P^2]}_{2i\hbar P} \\ &= 4i\hbar P^2\end{aligned}$$

$$\boxed{\frac{d\langle XP+PX\rangle}{dt} = \frac{2}{m}\langle P^2\rangle}$$

$$i\hbar \frac{d\langle P^2\rangle}{dt} = \langle [P^2, H]\rangle = 0 \implies \boxed{\langle P^2\rangle = \langle P^2\rangle_0}$$

$$\boxed{\langle XP+PX\rangle = \langle XP+PX\rangle_0 + \frac{2\langle P^2\rangle_0 t}{m}}$$

$$\frac{d\langle X^2\rangle}{dt} = \frac{1}{m}\langle XP+PX\rangle_0 + \frac{2\langle P^2\rangle_0 t}{m^2}$$

$$\boxed{\langle X^2\rangle = \langle X^2\rangle_0 + \frac{\langle XP+PX\rangle_0\, t}{m} + \frac{\langle P^2\rangle_0\, t^2}{m^2}}$$

Easier way: use Heisenberg picture

$$\frac{dX}{dt} = -\frac{i}{\hbar}[X,H] = \frac{P}{m} \qquad\qquad X(t) = X(0) + \frac{P(0)t}{m}$$

$$\Longrightarrow$$

$$\frac{dP}{dt} = -\frac{i}{\hbar}[P,H] = 0 \qquad\qquad P(t) = P(0)$$

$$\langle X(t)\rangle = \langle X(0)\rangle + \frac{\langle P(0)\rangle t}{m}$$

$$\langle P(t)\rangle = \langle P(0)\rangle$$

$$\langle X^2(t)\rangle = \langle X^2(0)\rangle + \frac{\langle X(0)P(0) + P(0)X(0)\rangle t}{m} + \frac{\langle P^2(0)\rangle t^2}{m^2}$$

c. $$(\Delta X)_t^2 = \langle X^2\rangle_t - \langle X\rangle_t^2$$

$$= \langle X^2\rangle_0 + \frac{\langle XP + PX\rangle_0 t}{m} + \frac{\langle P^2\rangle_0 t^2}{m^2}$$

$$- \langle X\rangle_0^2 - \frac{2\langle X\rangle_0\langle P\rangle_0 t}{m} - \frac{\langle P\rangle_0^2 t^2}{m^2}$$

$$(\Delta X)_t^2 = (\Delta X)_0^2 + \frac{(\langle XP+PX\rangle_0 - 2\langle X\rangle_0\langle P_0\rangle)t}{m} + \frac{(\Delta P)_0^2 t^2}{m^2}$$

$$\langle XP + PX\rangle_t - 2\langle X\rangle_t\langle P\rangle_t = (\langle XP + PX\rangle_0 - 2\langle X\rangle_0\langle P\rangle_0) + \frac{2(\Delta P)_0^2 t}{m}$$

Let t_0 be the time such that

$$0 = \langle XP + PX\rangle_{t_0} - 2\langle X\rangle_{t_0}\langle P\rangle_{t_0} = (\langle XP + PX\rangle_0 - 2\langle X\rangle_0\langle P_0\rangle) + \frac{2(\Delta P)_0^2 t_0}{m}$$

$\uparrow$

X and P are uncorrelated at $t = t_0$

Then

$$\boxed{(\Delta X)^2_{t} = (\Delta X)^2_{t_0} + \frac{(\Delta P)^2_{t_0}(t-t_0)^2}{m^2}}$$

Wave packet spreads from its minimum extent at $t = t_0$ due to momentum uncertainty.

4.2. C-T L_{III}.5

$$H = P^2/2m + V(X) = P^2/2m - fX$$

a. $$i\hbar \frac{d\langle X\rangle}{dt} = \langle [X,H]\rangle = \frac{1}{2m}\langle \underbrace{[X,P^2]}_{2i\hbar P}\rangle = \frac{i\hbar}{m}\langle P\rangle$$

$$\boxed{\frac{d\langle X\rangle}{dt} = \frac{\langle P\rangle}{m}}$$

$$i\hbar \frac{d\langle P\rangle}{dt} = \langle [P,H]\rangle = -f\langle \underbrace{[P,X]}_{-i\hbar}\rangle = i\hbar f$$

$$\boxed{\frac{d\langle P\rangle}{dt} = f}$$

These are the classical equations of motion

b. $$i\hbar \frac{d\langle P^2\rangle}{dt} = \langle [P^2,H]\rangle = -f\langle \underbrace{[P^2,X]}_{-2i\hbar P}\rangle = 2i\hbar f\langle P\rangle$$

$$\frac{d\langle P^2\rangle}{dt} = 2f\langle P\rangle$$

$$\frac{d(\Delta P)^2}{dt} = \frac{d}{dt}\left(\langle P^2\rangle - \langle P\rangle^2\right) = 2f\langle P\rangle - 2\langle P\rangle f = 0$$

$$\Longrightarrow (\Delta P)^2 = \text{constant}$$

c. $$i\hbar \frac{\partial \langle p|\psi(t)\rangle}{\partial t} = \langle p|H|\psi(t)\rangle$$

$$= \underbrace{\langle p|\frac{P^2}{2m}|\psi(t)\rangle}_{\frac{p^2}{2m}\langle p|\psi(t)\rangle} - f\underbrace{\langle p|X|\psi\rangle}_{-\frac{\hbar}{i}\frac{\partial}{\partial p}\langle p|\psi(t)\rangle}$$

$$i\hbar \frac{\partial \langle p|\psi(t)\rangle}{\partial t} = -i\hbar f \frac{\partial}{\partial p}\langle p|\psi(t)\rangle + \frac{p^2}{2m}\langle p|\psi(t)\rangle$$

$$i\hbar\left(\frac{\partial}{\partial t} + f\frac{\partial}{\partial p}\right)\langle p|\psi(t)\rangle = \frac{p^2}{2m}\langle p|\psi(t)\rangle$$

We want an equation for the momentum probability density, $|\langle p|\psi(t)\rangle|^2$:

$$\left(\frac{\partial}{\partial t} + f\frac{\partial}{\partial p}\right)|\langle p|\psi(t)\rangle|^2 = \langle p|\psi(t)\rangle^* \overbrace{\left(\frac{\partial}{\partial t} + f\frac{\partial}{\partial p}\right)\langle p|\psi(t)\rangle}^{-\frac{i}{\hbar}\frac{p^2}{2m}\langle p|\psi(t)\rangle}$$

$$+ \langle p|\psi(t)\rangle \underbrace{\left(\frac{\partial}{\partial t} + f\frac{\partial}{\partial p}\right)\langle p|\psi(t)\rangle^*}_{+\frac{i}{\hbar}\frac{p^2}{2m}\langle p|\psi(t)\rangle^*}$$

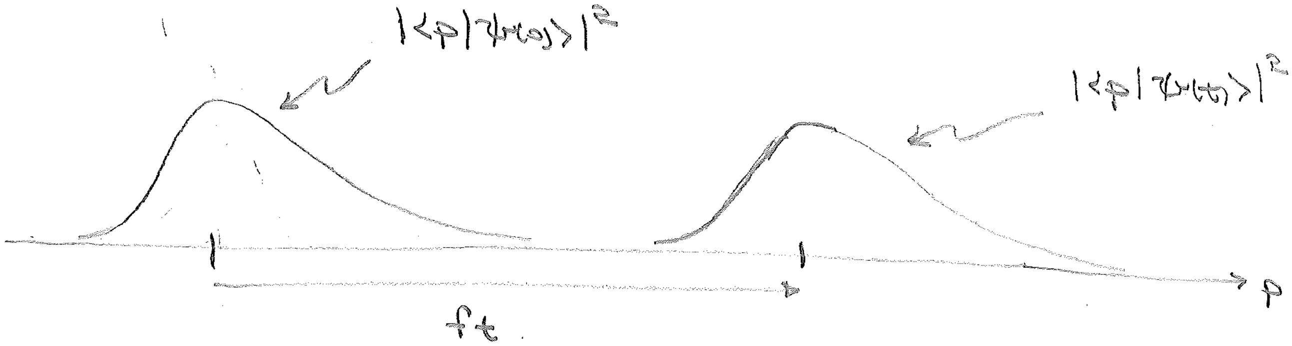
$$\therefore \boxed{\left(\frac{\partial}{\partial t} + f\frac{\partial}{\partial p}\right)|\langle p|\psi(t)\rangle|^2 = 0}$$

This equation describes motion along trajectories defined by the classical equation $dp/dt = f$, with $|\langle p|\psi(t)\rangle|^2$ being constant along those trajectories.

$$\frac{dp}{dt} = f \Rightarrow p = p_0 + ft$$

$$0 = \underbrace{\left(\frac{\partial}{\partial t} + \overset{dp/dt}{\overset{\downarrow}{f}} \frac{\partial}{\partial p}\right)}_{d/dt} |\langle p|\psi(t)\rangle|^2 = \frac{d}{dt}|\langle p|\psi(t)\rangle|^2$$

$$\Rightarrow \boxed{|\langle p|\psi(t)\rangle|^2 = |\langle p_0|\psi(0)\rangle|^2 = |\langle p-ft|\psi(0)\rangle|^2}$$



$|\langle p|\psi(t)\rangle|^2$ moves to right with "velocity" f without changing shape.

4.3. C-T L_{III}.10(a)

$$H = \underbrace{\frac{P^2}{2m}}_{= T} + V(X) = \frac{P^2}{2m} + \lambda X^n$$

$$[H, XP] = \frac{1}{2m}[P^2, XP] + \lambda [X^n, XP]$$

$$[P^2, XP] = P^2XP - XPP^2 = \underbrace{[P^2, X]}_{= -[X,P^2] = -2P[X,P] = -2i\hbar P} P = -2i\hbar P^2$$

$$[X^n, XP] = X^nXP - XPX^n = X\underbrace{[X^n, P]}_{\substack{-[P,X^n] = -nX^{n-1}[P,X] \\ = i\hbar n X^{n-1}}} = i\hbar n X^n$$

$$\boxed{[H, XP] = i\hbar\left(-\frac{P^2}{m} + n\lambda X^n\right) = i\hbar\left(-2T + nV\right)}$$

$$i\hbar \frac{d\langle XP\rangle}{dt} = \langle [XP, H]\rangle = i\hbar\left(+2\langle T\rangle - n\langle V\rangle\right)$$

$$\boxed{\frac{d\langle XP\rangle}{dt} = -2\langle T\rangle - n\langle V\rangle}$$

Why is this real even though XP is not Hermitian?

Now suppose that the particle is in a stationary state $|\varphi\rangle$. Expectation values are constant in a stationary state, because the evolution is multiplication by a phase factor, which leaves expectation values unchanged. (Formally, $i\hbar\, d\langle\varphi|A|\varphi\rangle/dt = \langle\varphi|[A, H]|\varphi\rangle$ $= (E-E)\langle\varphi|A|\varphi\rangle = 0$.)

$$\therefore \quad 0 = \frac{d\langle xp\rangle}{dt} \Rightarrow \boxed{2\langle T\rangle = n\langle V\rangle}$$; stationary state

Virial Theorem

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$$H = \hbar\omega_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$A = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$B = b \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

matrix reps. in the $|u_1\rangle, |u_2\rangle, |u_3\rangle$ basis

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{1}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle$$

a. H is diagonal in the u-representation, so its eigenstates are $|u_1\rangle, |u_2\rangle, |u_3\rangle$, and its eigenvalues are $\hbar\omega_0$, $2\hbar\omega_0$, $2\hbar\omega_0$.

Possible results: $\hbar\omega_0$, $2\hbar\omega_0$, $2\hbar\omega_0$

Probabilities: $\frac{1}{2}$ $\frac{1}{4}$ $\frac{1}{4}$

$$\langle\psi_0|H|\psi_0\rangle = \langle H\rangle = \hbar\omega_0\left(\frac{1}{2} + \frac{1}{4}2 + \frac{1}{4}2\right) = \frac{3}{2}\hbar\omega_0$$

$$\langle\psi_0|H^2|\psi_0\rangle = \langle H^2\rangle = (\hbar\omega_0)^2\left(\frac{1}{2} + \frac{1}{4}4 + \frac{1}{4}4\right) = \frac{5}{2}(\hbar\omega_0)^2$$

$$(\Delta H)^2 = \langle H^2\rangle - \langle H\rangle^2 = (\hbar\omega_0)^2\left(\frac{5}{2} - \frac{9}{4}\right) = \frac{1}{4}(\hbar\omega_0)^2$$

$$\Delta H = \frac{1}{2}\hbar\omega_0$$

b. Need to find eigenvalues and eigenvectors of A.

$$A|a_n\rangle = \lambda_n|a_n\rangle$$

$$A = a\Big(|u_1\rangle\langle u_1| + \underbrace{|u_2\rangle\langle u_3| + |u_3\rangle\langle u_2|}\Big) \qquad \text{u-rep of A}$$

$$\frac{1}{2}\big(|u_2\rangle + |u_3\rangle\big)\big(\langle u_2| + \langle u_3|\big)$$

$$- \frac{1}{2}\big(|u_2\rangle - |u_3\rangle\big)\big(\langle u_2| - \langle u_3|\big)$$

Eigenvectors of A — Eigenvalues

Define $|a_1\rangle = |u_1\rangle$ — a

$|a_2\rangle = \frac{1}{\sqrt{2}}\big(|u_2\rangle + |u_3\rangle\big)$ — a

$|a_3\rangle = \frac{1}{\sqrt{2}}\big(|u_2\rangle - |u_3\rangle\big)$ — $-a$

$$A = a\big(|a_1\rangle\langle a_1| + |a_2\rangle\langle a_2| - |a_3\rangle\langle a_3|\big)$$

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}|u_1\rangle + \frac{1}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle$$

$$= \frac{1}{\sqrt{2}}|a_1\rangle + \frac{1}{2}\frac{1}{\sqrt{2}}\big(|a_2\rangle + |a_3\rangle\big) + \frac{1}{2}\frac{1}{\sqrt{2}}\big(|a_2\rangle - |a_3\rangle\big)$$

$$= \frac{1}{\sqrt{2}}|a_1\rangle + \frac{1}{\sqrt{2}}|a_2\rangle$$

Possible results : a

Probability : 1

$-a$ cannot occur because $|a_3\rangle$ does not contribute to $|\psi(0)\rangle$

State vector after measurement: $|\psi(0)\rangle$

$|\psi(0)\rangle$ is in a degenerate subspace so post-measurement projection into that subspace does not affect the state

c. $$|\psi(t)\rangle = e^{-\frac{i}{\hbar}Ht}\,|\psi(0)\rangle$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-i\omega_0 t}|u_1\rangle + \frac{1}{2}e^{-2i\omega_0 t}|u_2\rangle + \frac{1}{2}e^{-2i\omega_0 t}|u_3\rangle$$

d.

only nonzero elements are $(n,m) = (1,1)$, $(2,3)$, and $(3,2)$

$$\langle A\rangle_t = \langle\psi(t)|A|\psi(t)\rangle$$

$$= \sum_{n,m}\langle\psi(t)|u_n\rangle\underbrace{\langle u_n|A|u_m\rangle}\langle u_m|\psi(t)\rangle$$

$$= \langle\psi(t)|u_1\rangle\overbrace{\langle u_1|A|u_1\rangle}^{a}\langle u_1|\psi(t)\rangle$$

$$+ \langle\psi(t)|u_2\rangle\overbrace{\langle u_2|A|u_3\rangle}^{a}\langle u_3|\psi(t)\rangle$$

$$+ \langle\psi(t)|u_3\rangle\underbrace{\langle u_3|A|u_2\rangle}_{a}\langle u_2|\psi(t)\rangle$$

$$= a\left(|\langle u_1|\psi(t)\rangle|^2 + \langle u_2|\psi(t)\rangle\langle u_3|\psi(t)\rangle^* + \langle u_3|\psi(t)\rangle\langle u_2|\psi(t)\rangle^*\right)$$

$$= a\left[\underbrace{|\langle u_1|\psi(t)\rangle|^2}_{\frac{1}{2}} + 2\,\mathrm{Re}\left(\underbrace{\langle u_2|\psi(t)\rangle}_{\frac{1}{2}e^{-2i\omega_0 t}}\underbrace{\langle u_3|\psi(t)\rangle^*}_{\frac{1}{2}e^{+2i\omega_0 t}}\right)\right]$$

$$\langle A\rangle_t = a$$

One can also get this from

$$|\psi(t)\rangle = \underbrace{\frac{1}{\sqrt{2}} e^{-i\omega_0 t} |u_1\rangle}_{\overset{\shortparallel}{|a_1\rangle}} + e^{-2i\omega_0 t} \underbrace{\left(\frac{1}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle \right)}_{\frac{1}{\sqrt{2}}|a_2\rangle}$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-i\omega_0 t} |a_1\rangle + \frac{1}{\sqrt{2}} e^{-2i\omega_0 t} |a_2\rangle$$

Thus the result of a measurement of A at <u>any</u> time t is a with probability 1.

$$\langle B \rangle_t = \langle \psi(t) | B | \psi(t) \rangle$$

$$= \sum_{n,m} \langle \psi(t) | u_n \rangle \langle u_n | B | u_m \rangle \langle u_m | \psi(t) \rangle$$

$$= b \Big(\langle \psi(t) | u_1 \rangle \langle u_2 | \psi(t) \rangle + \langle \psi(t) | u_2 \rangle \langle u_1 | \psi(t) \rangle + |\langle \psi(t) | u_3 \rangle|^2 \Big)$$

$$= b \Big(\underbrace{2 \,\mathrm{Re} \big(\underbrace{\langle u_1 | \psi(t) \rangle}_{\frac{1}{\sqrt{2}} e^{-i\omega_0 t}} \underbrace{\langle u_2 | \psi(t) \rangle^*}_{\frac{1}{2} e^{+2i\omega_0 t}} \big)}_{\frac{1}{\sqrt{2}} \cos\omega_0 t} + \underbrace{|\langle \psi(t) | u_3 \rangle|^2}_{\frac{1}{4}} \Big)$$

$$\langle B \rangle_t = b \left(\frac{1}{4} + \frac{1}{\sqrt{2}} \cos\omega_0 t \right) = \frac{1}{4} b \left(1 + 2\sqrt{2} \cos\omega_0 t \right)$$

(d)

A has eigenvalues $a, a, -a$

B has eigenvalues $b, -b, b$

That $\langle A \rangle_t = a$ means that a measurement of A at time yields result a with probability 1.

That $\langle B_t \rangle = \frac{1}{4} b (1 + 2\sqrt{2} \cos \omega_0 t)$ means that both b and $-b$ are possible results at all times t

4.5. $\hat{U}(t,0) = e^{-(i/\hbar)\hat{H}t}$

$$\hat{A}_S|A\rangle = A|A\rangle$$

$$\hat{A}_H(t) = \hat{U}^\dagger(t,0)\,\hat{A}_S\,\hat{U}(t,0)$$

(a) $$\hat{A}_H(t)|A,t\rangle = \hat{U}^\dagger(t,0)\,\hat{A}_S\,\underbrace{\hat{U}(t,0)\,\hat{U}^\dagger(t,0)}_{\hat{1}}|A\rangle$$

$$= \hat{U}^\dagger(t,0)\,\underbrace{\hat{A}_S|A\rangle}_{A|A\rangle}$$

$$= A\,\hat{U}^\dagger(t,0)|A\rangle$$

$$\boxed{\hat{A}_H(t)|A,t\rangle = A|A,t\rangle}$$

(b) $$p(A,t) = |\langle A|\psi_S(t)\rangle|^2$$

$$= \langle\psi_S(t)|A\rangle\langle A|\psi_S(t)\rangle$$

$$= \langle\psi_S(0)|\underbrace{\hat{U}^\dagger(t,0)|A\rangle\langle A|\hat{U}(t,0)}_{|A,t\rangle\langle A,t| = \hat{P}_A(t)}\underbrace{|\psi_S(0)\rangle}_{|\psi_H\rangle}$$

$$\boxed{p(A,t) = \langle\psi_H|\hat{P}_A(t)|\psi_H\rangle}$$

(c) $$\hat{A}_S = \sum_{A,k} A\,|A,k\rangle\langle A,k| = \sum_A A\,\hat{P}_A$$

$$\hat{P}_A = \sum_k |A,k\rangle\langle A,k|$$

$$p(A,t) = \langle\psi_S(t)|\,\hat{P}_A\,|\psi_S(t)\rangle$$

$$= \langle\psi_S(0)|\,\underbrace{\hat{U}^\dagger(t,0)\,\hat{P}_A\,\hat{U}(t,0)}_{\equiv\,\hat{P}_A(t)}\,\underbrace{|\psi_S(0)\rangle}_{|\psi_H\rangle}$$

$$\boxed{\begin{aligned} p(A,t) &= \langle\psi_H|\,\hat{P}_A(t)\,|\psi_H\rangle \\ \hat{P}_A(t) &= \hat{U}^\dagger(t,0)\,\hat{P}_A\,\hat{U}(t,0) = \sum_k |A,k,t\rangle\langle A,k,t| \end{aligned}}$$

4.6. $\psi(x,0) = \sqrt{a}\, e^{-a|x|}$, $a > 0$.

Normalized? $\displaystyle\int_{-\infty}^{\infty} dx\, |\psi(x,0)|^2 = a \int_{-\infty}^{\infty} dx\, e^{-2a|x|}$

$$= 2a \int_0^{\infty} dx\, e^{-2ax}$$

$$= \underline{1}$$

(a) $\tilde{\psi}(p,0) = \langle p | \psi(0) \rangle$

$$= \int_{-\infty}^{\infty} dx\, \underbrace{\langle p | x \rangle}_{\frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar}px}} \langle x | \psi(0) \rangle$$

$$= \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi\hbar}}\, \psi(x,0)\, e^{-\frac{i}{\hbar}px}$$

$$= \sqrt{\frac{a}{2\pi\hbar}} \underbrace{\int_{-\infty}^{\infty} dx\, e^{-a|x|}\, e^{-ipx/\hbar}}$$

$$= \underbrace{\int_0^{\infty} dx\, e^{-(a+ip/\hbar)x}}_{\frac{1}{a+ip/\hbar}} + \underbrace{\int_{-\infty}^{0} dx\, e^{(a-ip/\hbar)x}}_{= \int_0^{\infty} dx\, e^{-(a-ip/\hbar)x} = \frac{1}{a-ip/\hbar}}$$

$$= \frac{1}{a+ip/\hbar} + \frac{1}{a-ip/\hbar}$$

$$= \frac{2a}{a^2 + (p/\hbar)^2}$$

$$= \frac{2a\hbar^2}{p^2 + (a\hbar)^2}$$

$$\boxed{\bar{\psi}(p,0) = \sqrt{\frac{2}{\pi(a\hbar)}}\,\frac{(a\hbar)^2}{p^2 + (a\hbar)^2} = \sqrt{\frac{2}{\pi(a\hbar)}}\,\frac{1}{1 + (p/a\hbar)^2}}$$

(b)
$$\boxed{\langle \hat{x} \rangle_0 = \int_{-\infty}^{\infty} dx\, \underbrace{x}_{\text{odd}}\, \underbrace{|\psi(x,0)|^2}_{\text{even}} = 0}$$

$$\boxed{\langle \hat{p} \rangle_0 = \int_{-\infty}^{\infty} dp\, \underbrace{p}_{\text{odd}}\, \underbrace{|\bar{\psi}(p,0)|^2}_{\text{even}} = 0}$$

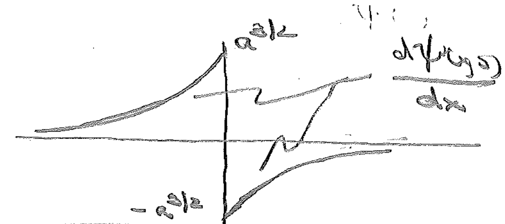
Alternative approach:

$$\langle \hat{p} \rangle_0 = \int_{-\infty}^{\infty} dx\, \underbrace{\psi^*(x,0)}_{\text{even}}\, \frac{\hbar}{i}\, \frac{d}{dx}\psi(x,0)$$

$$\frac{d}{dx}\psi(x,0) = \begin{cases} -a^{3/2} e^{-ax}, & x>0 \\ a^{3/2} e^{ax}, & x<0 \end{cases}$$

odd function with a discontinuity at $x=0$

$$= 0$$



$$\langle \hat{x}^2 \rangle_0 = \int_{-\infty}^{\infty} dx\, x^2 |\psi(x,0)|^2$$

$$= 2a \int_0^{\infty} dx\, x^2 e^{-2ax}$$

$u = 2ax$
$du = 2a\,dx$

$$= \frac{1}{4a^2} \underbrace{\int_0^{\infty} du\, u^2 e^{-u}}_{2}$$

$$\boxed{\langle \hat{x}^2 \rangle_0 = \frac{1}{2a^2}}$$

$$(\Delta x)^2 = \langle \hat{x}^2 \rangle_0 - \langle \hat{x} \rangle_0^2 = \frac{1}{2a^2}$$

$$\boxed{\Delta x = \frac{1}{\sqrt{2}\,a}}$$

$$\langle \hat{p}^2 \rangle_0 = \int_{-\infty}^{\infty} dp\, p^2 |\bar{\psi}(p,0)|^2$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{dp}{a\hbar} \frac{p^2}{\left(1 + (p/a\hbar)^2\right)^2}$$

$u = \frac{p}{a\hbar}$

$du = \frac{dp}{a\hbar}$

$$= \frac{2}{\pi} (a\hbar)^2 \underbrace{\int_{-\infty}^{\infty} du \frac{u^2}{(1+u^2)^2}}_{\frac{\pi}{2}}$$

$$\boxed{\langle \hat{p}^2 \rangle_0 = (a\hbar)^2}$$

Alternative approach: (3)

$$\langle \hat{p}^2 \rangle_0 = \int_{-\infty}^{\infty} dx\, \psi^*(x,0) \left(-\hbar^2 \frac{d^2}{dx^2}\right) \psi(x,0)$$

$$\frac{d^2}{dx^2} \psi(x,0) = -2a^{3/2} \delta(x) + a^{5/2} e^{-a|x|}$$

$$\langle \hat{p}^2 \rangle_0 = \int_{-\infty}^{\infty} dx \left(2\hbar^2 a^2 \delta(x) - \hbar^2 a^3 e^{-2a|x|}\right)$$

$$= 2\hbar^2 a^2 - \hbar^2 a^2$$

$$= \hbar^2 a^2$$

$$(\Delta p)^2 = \langle \hat{p}^2 \rangle_0 - \langle \hat{p} \rangle_0^2 = (a\hbar)^2$$

$$\boxed{\Delta p = a\hbar}$$

$$\langle \hat{x}\hat{p} + \underset{\hat{x}\hat{p} - i\hbar}{\hat{p}\hat{x}} \rangle_0 = 2\langle \hat{x}\hat{p} \rangle_0 - i\hbar$$

$$= 2\int_{-\infty}^{\infty} dx\, \psi^*(x,0)\, x \frac{\hbar}{i} \frac{d}{dx} \psi(x,0) - i\hbar$$

$$= -i\hbar \left(1 + 2 \underbrace{\int_{-\infty}^{\infty} dx\, \underbrace{\psi^*(x,0)\, x \frac{d}{dx} \psi(x,0)}_{\text{even fcn}}}\right)$$

$$= 4 \int_0^{\infty} dx\, \underbrace{\psi^*(x,0)\, x \frac{d}{dx} \psi(x,0)}_{-a^2 x e^{-2ax}}$$

$$= -4a^2 \int_0^{\infty} dx\, x e^{-2ax}$$

$$= -\int_0^{\infty} du\, u e^{-u}$$

$$= -1$$

$$\boxed{\langle \hat{x}\hat{p} + \hat{p}\hat{x} \rangle_0 = 0}$$

Another approach:

$$\langle \hat{x}\hat{p} \rangle_0 = \int_{-\infty}^{\infty} dx\, \psi^*(x,0)\, x \frac{\hbar}{i}\frac{d}{dx}\psi(x,0)$$

$$= \frac{\hbar}{i}\Big(\underbrace{\psi^*(x,0)\, x\, \psi(x,0)\Big|_{-\infty}^{\infty}}_{0} - \int_{-\infty}^{\infty} dx\, \frac{d}{dx}\left(x\psi^*(x,0)\right)\psi(x,0) \Big)$$

$$= - \int_{-\infty}^{\infty} dx\, \psi(x,0)\frac{\hbar}{i}\frac{d}{dx}\left(x\psi^*(x,0)\right)$$

Since $\psi(x,0)$ is real $\rightarrow$

$$= - \int_{-\infty}^{\infty} dx\, \psi^*(x,0)\frac{\hbar}{i}\frac{d}{dx}\left(x\psi(x,0)\right)$$

$$= - \langle \hat{p}\hat{x} \rangle_0$$

$\Longrightarrow \quad \langle \hat{x}\hat{p} + \hat{p}\hat{x} \rangle_0 = 0$ for any <u>real</u> wave function

(c) For a free particle, the HP operator evolution is the same as the classical evolution:

$$\hat{x}(t) = \hat{x}(0) + \frac{\hat{p}(0)t}{m}$$

$$\hat{p}(t) = \hat{p}(0)$$

$$\Longrightarrow \boxed{\begin{array}{l} \langle \hat{x} \rangle_t = \langle \hat{x} \rangle_0 + \dfrac{\langle \hat{p} \rangle_0 t}{m} = 0 \\ \langle \hat{p} \rangle_t = \langle \hat{p} \rangle_0 = 0 \end{array}}$$

$$\hat{p}^2(t) = \hat{p}^2(0) \Longrightarrow \boxed{\langle \hat{p}^2 \rangle_t = \langle \hat{p}^2 \rangle_0 = (a\hbar)^2}$$

$$\hat{x}(t)\hat{p}(t)+\hat{p}(t)\hat{x}(t) = \hat{x}(0)\hat{p}(0)+\hat{p}(0)\hat{x}(0)+\frac{2\hat{p}^2(0)t}{m}$$

$$\Rightarrow \boxed{\langle \hat{x}\hat{p}+\hat{p}\hat{x}\rangle_t = \langle \hat{x}\hat{p}+\hat{p}\hat{x}\rangle_0 + \frac{2\langle \hat{p}^2\rangle_0 t}{m} = \frac{2(a\hbar)^2 t}{m}}$$

$$\hat{x}^2(t) = \hat{x}^2(0)+\frac{(\hat{x}(0)\hat{p}(0)+\hat{p}(0)\hat{x}(0))t}{m}+\frac{\hat{p}^2(0)t^2}{m^2}$$

$$\Rightarrow \boxed{\begin{aligned}\langle \hat{x}^2\rangle_t &= \langle \hat{x}^2\rangle_0 + \frac{\langle \hat{x}\hat{p}+\hat{p}\hat{x}\rangle_0 t}{m} + \frac{\langle \hat{p}^2\rangle_0 t^2}{m^2}\\ &= \frac{1}{2a^2}+\left(\frac{a\hbar t}{m}\right)^2\end{aligned}}$$

Another calculation of $\langle \hat{x}\hat{p}+\hat{p}\hat{x}\rangle_0$:

$$\langle \hat{x}\hat{p}+\hat{p}\hat{x}\rangle_0 = 2\langle \hat{p}\hat{x}\rangle_0 + i\hbar$$

$$= i\hbar + 2\int_{-\infty}^{\infty} dp\, \bar{\Psi}^*(p,0)\, p\left(-\frac{\hbar}{i}\frac{d}{dp}\right)\bar{\Psi}(p,0)$$

$$= i\hbar\left(1+2\int_{-\infty}^{\infty} dp\, \bar{\Psi}^*(p,0)\, p\,\underbrace{\frac{d}{dp}\bar{\Psi}(p,0)}\right)$$

$$= -\sqrt{\frac{2}{\pi(a\hbar)}}\;\frac{2p/(a\hbar)^2}{(1+(p/a\hbar)^2)^2}$$

$$= -\frac{2p/(a\hbar)^2}{1+(p/a\hbar)^2}\,\bar{\Psi}(p,0)$$

$$2\int_{-\infty}^{\infty} dp\, \bar{\Psi}^*(p,0)\, p\frac{d}{dp}\bar{\Psi}(p,0) = -4\int_{-\infty}^{\infty} dp\, \frac{(p/a\hbar)^2}{1+(p/a\hbar)^2}\,|\bar{\Psi}(p,0)|^2$$

$$= -\frac{8}{\pi}\int_{-\infty}^{\infty}\frac{dp}{a\hbar}\frac{(p/a\hbar)^2}{(1+(p/a\hbar)^2)^3}$$

$p/a\hbar = \tan u$

$\frac{dp}{a\hbar} = \frac{du}{\cos^2 u}$

$$= -\frac{8}{\pi}\int_{-\pi/2}^{\pi/2} du \frac{1}{\cos^2 u}\frac{\tan^2 u}{\left(\frac{1}{\cos^2 u}\right)^3}$$

$$= -\frac{8}{\pi}\int_{-\pi/2}^{\pi/2} du \cos^4 u \tan^2 u$$

$$= -\frac{8}{\pi}\int_{-\pi/2}^{\pi/2} du \underbrace{\cos^2 u \sin^2 u}$$

$$\frac{1}{4}\sin^2 2u = \frac{1}{8}(1-\cos 4u)$$

$$= -\frac{1}{\pi}\int_{-\pi/2}^{\pi/2} du \cos 4u$$

$$= -1$$

$$\therefore \langle \hat{x}\hat{p} + \hat{p}\hat{x}\rangle_0 = 0$$