Phys 521

Homework #7

Solution Set

7.1. C-T F_{VI} 1

$j=1$

$|\psi\rangle = \alpha|+1\rangle + \beta|0\rangle + \gamma|-1\rangle$

$\vec{J}^2 = 2\hbar^2 \mathbb{1}$

$J_z = \hbar\left(|+1\rangle\langle+1| - |-1\rangle\langle-1|\right)$

$J_+ = \sqrt{2}\hbar\left(|+1\rangle\langle 0| + |0\rangle\langle-1|\right) = J_x + iJ_y$

$J_- = J_+^\dagger = \sqrt{2}\hbar\left(|0\rangle\langle 1| + |-1\rangle\langle 0|\right) = J_x - iJ_y$

$J_x = \frac{1}{2}(J_+ + J_-)$, $J_y = -\frac{i}{2}(J_+ - J_-)$

(a) $\langle J_z\rangle = \langle\psi|J_z|\psi\rangle$

$$= \hbar\left(|\underbrace{\langle+1|\psi\rangle}_{\alpha}|^2 - |\underbrace{\langle-1|\psi\rangle}_{\gamma}|^2\right)$$

$$\boxed{\langle J_z\rangle = \hbar\left(|\alpha|^2 - |\gamma|^2\right)}$$

$\langle J_+\rangle = \langle\psi|J_+|\psi\rangle$

$$= \sqrt{2}\hbar\left(\underbrace{\langle\psi|+1\rangle}_{\alpha^*}\underbrace{\langle 0|\psi\rangle}_{\beta} + \underbrace{\langle\psi|0\rangle}_{\beta^*}\underbrace{\langle-1|\psi\rangle}_{\gamma}\right)$$

$\langle J_+\rangle = \sqrt{2}\hbar\left(\alpha^*\beta + \beta^*\gamma\right)$

$$\boxed{\langle J_x\rangle = \mathrm{Re}(\langle J_+\rangle) = \sqrt{2}\hbar\,\mathrm{Re}\left(\alpha^*\beta + \beta^*\gamma\right)}$$

$$\boxed{\langle J_y\rangle = \mathrm{Im}(\langle J_+\rangle) = \sqrt{2}\hbar\,\mathrm{Im}\left(\alpha^*\beta + \beta^*\gamma\right)}$$

(b) $\langle J_z^2\rangle = \langle\psi|J_z^2|\psi\rangle$ $\qquad J_z^2 = \hbar^2(|+1\rangle\langle+1| + |-1\rangle\langle-1|)$

$$= \hbar^2\left(|\langle+1|\psi\rangle|^2 + |\langle-1|\psi\rangle|^2\right)$$

$$\boxed{\langle J_z^2\rangle = \hbar^2(|\alpha|^2 + |\gamma|^2) = \hbar^2(1-|\beta|^2)}$$

$$\langle J_x^2\rangle + \langle J_y^2\rangle = \underbrace{\langle \vec{J}^2\rangle}_{=2\hbar^2} - \langle J_z^2\rangle = \hbar^2(1+|\beta|^2)$$

$$J_+^2 = J_x^2 - J_y^2 + i(J_xJ_y + J_yJ_x)$$

$$J_-^2 = J_x^2 - J_y^2 - i(J_xJ_y + J_yJ_x)$$

$$J_x^2 - J_y^2 = \frac{1}{2}(J_+^2 + J_-^2)$$

$$\langle J_x^2\rangle - \langle J_y^2\rangle = \mathrm{Re}(\langle J_+^2\rangle)$$

$$= \mathrm{Re}(\langle\psi|J_+^2|\psi\rangle)$$

$$J_+^2 = 2\hbar^2|+1\rangle\langle-1|$$

$$\langle J_x^2\rangle - \langle J_y^2\rangle = \mathrm{Re}\Big(2\hbar^2\underbrace{\langle\psi|+1\rangle}_{\alpha^*}\underbrace{\langle-1|\psi\rangle}_{\gamma}\Big) = 2\hbar^2\,\mathrm{Re}(\alpha^*\gamma)$$

$$\langle J_{x,y}^2\rangle = \frac{1}{2}\left[\hbar^2(1-|\beta|^2) \pm 2\hbar^2\,\mathrm{Re}(\alpha^*\gamma)\right]$$

$$\boxed{\langle J_{x,y}^2\rangle = \hbar^2\left(\frac{1}{2}(1-|\beta|^2) \pm \mathrm{Re}(\alpha^*\gamma)\right)}$$

7.2. C-T F_{VI}-2

Simultaneous eigenstates of $\vec{J}^2$ and J_z

$j=1$

$|1, m_z=+1\rangle \equiv |1,1\rangle_z$

$|1, m_z=0\rangle \equiv |1,0\rangle_z$

$|1, m_z=-1\rangle \equiv |1,-1\rangle_z$

$j=0$

$|0, m_z=0\rangle \equiv |0,0\rangle_z$

$J_+|1,1\rangle_z = 0$

$J_+|1,0\rangle_z = \sqrt{2}\hbar|1,1\rangle_z$

$J_+|1,-1\rangle_z = \sqrt{2}\hbar|1,-1\rangle_z$

$J_-|1,1\rangle_z = \sqrt{2}\hbar|1,0\rangle_z$

$J_-|1,0\rangle_z = \sqrt{2}\hbar|1,-1\rangle_z$

$J_-|1,-1\rangle_z = 0$

(a) Simultaneous eigenstates of $\vec{J}^2$ and $J_x = \frac{1}{2}(J_+ + J_-)$

$$J_x|0,0\rangle_z = \frac{1}{2}(J_+ + J_-)|0,0\rangle_z = 0$$

$$\Rightarrow \boxed{|0,0\rangle_z = |0, m_x=0\rangle \equiv |0,0\rangle_x}$$

$$J_x|1,0\rangle_z = \frac{1}{2}(J_+ + J_-)|1,0\rangle_z = \frac{1}{\sqrt{2}}\hbar(|1,1\rangle_z + |1,-1\rangle_z) = \hbar|\phi\rangle$$

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|1,1\rangle_z + |1,-1\rangle_z)$$

$$J_x|\phi\rangle = J_x\frac{1}{\sqrt{2}}(|1,1\rangle_z + |1,-1\rangle_z)$$

$$= \frac{1}{2}(J_+ + J_-)\frac{1}{\sqrt{2}}(|1,1\rangle_z + |1,-1\rangle_z)$$

$$= \hbar|1,0\rangle$$

Now consider $|\chi\rangle = \frac{1}{\sqrt{2}}(|1,1\rangle_z - |1,-1\rangle_z)$, which is $\perp$ to $|1,0\rangle_z$ and to $|\phi\rangle$.

$$J_x|\chi\rangle = \frac{1}{2}(J_+ + J_-)\frac{1}{\sqrt{2}}(|1,1\rangle_z - |1,-1\rangle_z) = 0$$

$$\Rightarrow \boxed{|\chi\rangle = \frac{1}{\sqrt{2}}(|1,1\rangle_z - |1,-1\rangle_z) = |1, m_x=0\rangle \equiv |1,0\rangle_x}$$

$$\therefore \quad J_x = \hbar\left(|\phi\rangle {}_z\langle 1,0| + |1,0\rangle_z \langle \phi| \right)$$

We only have to diagonalize J_x in a 2-d subspace, where it looks like $\hbar\sigma_x$. The eigenvectors are

$$\boxed{\begin{aligned} |1, m_x = +1\rangle &= \frac{1}{\sqrt{2}}\left(|\phi\rangle + |1,0\rangle_z\right) = \frac{1}{2}|1,1\rangle_z + \frac{1}{\sqrt{2}}|1,0\rangle_z + \frac{1}{2}|1,-1\rangle_z \equiv |1,1\rangle_x \\ |1, m_x = -1\rangle &= \frac{1}{\sqrt{2}}\left(|\phi\rangle - |1,0\rangle_z\right) = \frac{1}{2}|1,1\rangle_z - \frac{1}{\sqrt{2}}|1,0\rangle_z + \frac{1}{2}|1,-1\rangle_z \equiv |1,-1\rangle_x \end{aligned}}$$

(b) $|\psi\rangle = \alpha|1,1\rangle_z + \beta|1,0\rangle_z + \gamma|1,-1\rangle_z + \delta|0,0\rangle_z$

(i) $\boxed{P(\vec{J}^2 = 2\hbar^2, J_x = \hbar) = \left|{}_x\langle 1,1|\psi\rangle\right|^2 = \frac{1}{4}\left|\alpha + \sqrt{2}\beta + \gamma\right|^2}$

$$\begin{aligned} {}_x\langle 1,1|\psi\rangle &= \frac{1}{2}\,{}_z\langle 1,1|\psi\rangle + \frac{1}{\sqrt{2}}\,{}_z\langle 1,0|\psi\rangle + \frac{1}{2}\,{}_z\langle 1,-1|\psi\rangle \\ &= \frac{1}{2}\alpha + \frac{1}{\sqrt{2}}\beta + \frac{1}{2}\gamma \\ &= \frac{1}{2}\left(\alpha + \sqrt{2}\beta + \gamma\right) \end{aligned}$$

(ii) $\langle J_z\rangle = \langle\psi|J_z|\psi\rangle$

$$= \hbar\left(|\underbrace{{}_z\langle 1,1|\psi\rangle}_{\alpha}|^2 - |\underbrace{{}_z\langle 1,-1|\psi\rangle}_{\gamma}|^2 \right)$$

$$\boxed{\langle J_z\rangle = \hbar\left(|\alpha|^2 - |\gamma|^2\right)}$$

Measurement of J_z:

Possible results	$\hbar$	$-\hbar$	0
Probabilities	$\lvert{}_z\langle 1,1\vert\psi\rangle\rvert^2 = \lvert\alpha\rvert^2$	$\lvert{}_z\langle 1,-1\vert\psi\rangle\rvert^2 = \lvert\gamma\rvert^2$	$\lvert{}_z\langle 1,0\vert\psi\rangle\rvert^2 + \lvert{}_z\langle 0,0\vert\psi\rangle\rvert^2 = \lvert\beta\rvert^2 + \lvert\delta\rvert^2 = 1 - \lvert\alpha\rvert^2 - \lvert\gamma\rvert^2$

(iii) $\boxed{\langle \vec{J}^2 \rangle = \langle \psi | \vec{J}^2 | \psi \rangle = 2\hbar^2 \left(|\alpha|^2 + |\beta|^2 + |\gamma|^2 \right)}$

Measurement of $\vec{J}^2$:

Possible results	$2\hbar^2$	0
Probabilities	$\|\alpha\|^2 + \|\beta\|^2 + \|\gamma\|^2$	$\|\delta\|^2$

$$\langle J_x \rangle = \langle \psi | J_x | \psi \rangle$$
$$= \hbar \left(\langle \psi | \phi \rangle \, {}_z\langle 1,0 | \psi \rangle + \langle \psi | 1,0 \rangle_z \langle \phi | \psi \rangle \right)$$
$$= 2\hbar \, \mathrm{Re} \left(\langle \psi | \phi \rangle \underbrace{{}_z\langle 1,0 | \psi \rangle}_{\beta} \right)$$

$$\frac{1}{\sqrt{2}} \left(\langle \psi | 1,1 \rangle_z + \langle \psi | 1,-1 \rangle_z \right) = \frac{1}{\sqrt{2}} \left(\alpha^* + \gamma^* \right)$$

$$\langle J_x \rangle = \sqrt{2}\hbar \, \mathrm{Re} \left((\alpha^* + \gamma^*) \beta \right)$$

Measurement of J_x:

Possible results	$\hbar$	$-\hbar$	0
Probabilities	$\|{}_x\langle 1,1 \| \psi \rangle\|^2$	$\|{}_x\langle 1,-1 \| \psi \rangle\|^2$	$\|{}_x\langle 1,0 \| \psi \rangle\|^2 + \|{}_x\langle 0,0 \| \psi \rangle\|^2$
	$= \frac{1}{4} \|\alpha + \sqrt{2}\beta + \gamma\|^2$	$= \frac{1}{4} \|\alpha - \sqrt{2}\beta + \gamma\|^2$	$= \frac{1}{2} \|\alpha - \gamma\|^2 + \|\delta\|^2$

$${}_x\langle 1,1 | \psi \rangle = \frac{1}{2}\alpha + \frac{1}{\sqrt{2}}\beta + \frac{1}{2}\gamma = \frac{1}{2} \left(\alpha + \sqrt{2}\beta + \gamma \right)$$

$${}_x\langle 1,0 | \psi \rangle = \langle \chi | \psi \rangle = \frac{1}{\sqrt{2}} \left(\alpha - \gamma \right)$$

$${}_x\langle 1,-1 | \psi \rangle = \frac{1}{2}\alpha - \frac{1}{\sqrt{2}}\beta + \frac{1}{2}\gamma = \frac{1}{2} \left(\alpha - \sqrt{2}\beta + \gamma \right)$$

$${}_x\langle 0,0 | \psi \rangle = \delta$$

(4A)

(vi) $\langle J_z^2 \rangle = \langle \psi | J_z^2 | \psi \rangle = \hbar^2 (|\alpha|^2 + |\gamma|^2)$

Measurement of J_z^2		
Possible results	$\hbar^2$	0
Probabilities	$|\alpha|^2 + |\gamma|^2$	$|\beta|^2 + |\delta|^2$

7.3. C-T F_{VI}.3

$\vec{L} = \vec{R} \times \vec{P}$, $\quad L_j = \epsilon_{jkl} R_k P_l$

$$[L_i, R_j] = \epsilon_{ikl} R_k \underbrace{[P_l, R_j]}_{-i\hbar\delta_{lj}} = -i\hbar \underbrace{\epsilon_{ikj}}_{-\epsilon_{ijk}} R_k$$

$$\boxed{[L_i, R_j] = i\hbar\, \epsilon_{ijk} R_k}$$

$$[L_i, P_j] = \epsilon_{ikl} \underbrace{[R_k, P_j]}_{i\hbar\delta_{jk}} P_l = i\hbar\, \epsilon_{ijl} P_l$$

$$\boxed{[L_i, P_j] = i\hbar\, \epsilon_{ijk} P_k}$$

$$[L_i, \vec{P}^2] = [L_i, P_j P_j] = L_i P_j P_j - P_j P_j L_i$$

$$= (P_j L_i + [L_i, P_j]) P_j$$

$$- P_j (L_i P_j + [P_j, L_i])$$

$$= [L_i, P_j] P_j + P_j [L_i, P_j]$$

$$= i\hbar (\epsilon_{ijk} P_k P_j + \epsilon_{ijk} P_j P_k)$$

relabel j and k

But $\epsilon_{ijk} P_j P_k = -\epsilon_{ikj} P_j P_k = -\epsilon_{ijk} P_k P_j = -\epsilon_{ijk} P_j P_k$

switch j and k in ϵ_{ijk}

$\therefore \quad \epsilon_{ijk} P_j P_k = 0$

$$\boxed{[L_i, \vec{P}^2] = 0}$$

$$[L_i, \vec{R}^2] = [L_i, R_j]R_j + R_j[L_i, R_j]$$

$$= i\hbar(\underbrace{\epsilon_{ijk} R_k R_j}_{=0} + \underbrace{\epsilon_{ijk} R_j R_k}_{=0})$$

$$= 0$$

$$\boxed{[L_i, \vec{R}^2] = 0}$$

$$[L_i, \vec{R}\cdot\vec{P}] = [L_i, R_j P_j]$$

$$= L_i R_j P_j - R_j P_j L_i$$

$$= (R_j L_i + [L_i, R_j])P_j - R_j(L_i P_j + [P_j, L_i])$$

$$= [L_i, R_j]P_j + R_j[L_i, P_j]$$

$$= i\hbar(\epsilon_{ijk} R_k P_j + \epsilon_{ijk} R_j P_k)$$

$$\epsilon_{ijk} R_k P_j = -\epsilon_{ikj} R_k P_j = -\epsilon_{ijk} R_j P_k$$

switch j and k in ϵ_{ijk}

relabel j and k

$$= 0$$

$$\boxed{[L_i, \vec{R}\cdot\vec{P}] = 0}$$

7.4. C-T $F_{VI.5}$

$$\psi(x,y,z) = N(x+y+z)e^{-r^2/\alpha^2}, \quad \alpha = \alpha^*$$

(a) We can expand $\psi(\vec{r})$ as

$$\psi(\vec{r}) = \sum_{l,m} a_{lm}(r)\, Y_l^m(\theta,\varphi),$$

Where

$$a_{lm}(r) = \int d\Omega\, Y_l^{m*}(\theta,\varphi)\,\psi(\vec{r}).$$

The probability that $\vec{L}^2$ and L_z have particular values is

$$\mathcal{P}_{\vec{L}^2, L_z}(l,m) = \int_0^\infty r^2\, dr\, |a_{lm}(r)|^2$$

In our case we want

$$\mathcal{P}_{\vec{L}^2, L_z}(1,0) = \int_0^\infty r^2\, dr\, |a_{10}(r)|^2,$$

$$a_{10}(r) = \int d\Omega\, \overbrace{Y_1^{0*}(\theta,\varphi)}^{=\sqrt{\frac{3}{4\pi}}\cos\theta}\, \psi(\vec{r})$$

$$= Nr\,e^{-r^2/\alpha^2}\big(\underbrace{\sin\theta\cos\varphi + \sin\theta\sin\varphi}_{\text{integrate to } 0} + \cos\theta\big)$$

$$= \sqrt{\frac{3}{4\pi}}\, Nr\,e^{-r^2/\alpha^2} \underbrace{\int d\Omega\, \cos^2\theta}$$

$$= \int_0^\pi d\theta\, \sin\theta\cos^2\theta \int_0^{2\pi} d\varphi$$

$$= -2\pi \int_0^{\pi} \cos^2\theta \, d(\cos\theta)$$

$$= -\frac{2\pi}{3} \cos^3\theta \Big|_0^{\pi}$$

$$= \frac{4\pi}{3}$$

$$R_{10}(r) = \sqrt{\frac{4\pi}{3}}\, N r \, e^{-r^2/a^2}$$

$$\Rightarrow P_{\vec{L}^2, L_z}(1,0) = \frac{4\pi}{3} N^2 \int_0^{\infty} r^2 dr \; r^2 e^{-2r^2/a^2}$$

Normalization: $1 = \int d^3x \, |\psi(\vec{r})|^2$

$$= \int d^3x \, N^2 \underbrace{(x+y+z)^2}_{\underbrace{x^2+y^2+z^2}_{r^2} + \underbrace{2xy + 2xz + 2yz}_{\text{Integrates to } 0}} e^{-2r^2/a^2}$$

$$= N^2 \int r^2 dr \, \underbrace{d\Omega}_{\text{integrates to } 4\pi} \, r^2 e^{-2r^2/a^2}$$

$$= 4\pi N^2 \int r^2 dr \; r^2 e^{-2r^2/a^2}$$

$$\Rightarrow \boxed{P_{\vec{L}^2, L_z}(1,0) = \frac{1}{3}}$$

(b) $$Y_1^{\pm 1}(\theta,\varphi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\varphi} = \mp\sqrt{\frac{3}{8\pi}}\,\frac{x \pm iy}{r} = \sqrt{\frac{3}{8\pi}}\,\frac{\mp x - iy}{r}$$

$$Y_1^{0}(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\,\frac{z}{r}$$

$$\frac{z}{r} = \sqrt{\frac{4\pi}{3}}\,Y_1^0$$

$$\Rightarrow \quad \frac{x}{r} = \sqrt{\frac{4\pi}{3}}\,\frac{1}{\sqrt{2}}\left(Y_1^{-1} - Y_1^{1}\right)$$

$$\frac{y}{r} = i\sqrt{\frac{4\pi}{3}}\,\frac{1}{\sqrt{2}}\left(Y_1^{-1} + Y_1^{1}\right)$$

$$\therefore \quad \psi(x,y,z) = N r e^{-r^2/a^2}\underbrace{\left(\frac{x}{r} + \frac{y}{r} + \frac{z}{r}\right)}$$

$$= \sqrt{\frac{4\pi}{3}}\left(Y_1^0 + \frac{1}{\sqrt{2}}\left(Y_1^{-1} - Y_1^{1}\right) + \frac{i}{\sqrt{2}}\left(Y_1^{-1} + Y_1^{1}\right)\right)$$

$$= \sqrt{\frac{4\pi}{3}}\left(Y_1^0 + \frac{-1+i}{\sqrt{2}}Y_1^1 + \frac{1+i}{\sqrt{2}}Y_1^{-1}\right)$$

$$\boxed{\psi(x,y,z) = \underbrace{\sqrt{\frac{4\pi}{3}}\,N r e^{-r^2/a^2}}_{\equiv f(r)}\left(Y_1^0 + \frac{-1+i}{\sqrt{2}}Y_1^1 + \frac{1+i}{\sqrt{2}}Y_1^{-1}\right)}$$

$$R_{10}(r) = f(r)$$

$$R_{11}(r) = f(r)\,\frac{-1+i}{\sqrt{2}} \quad \longrightarrow \quad |R_{lm}(r)|^2 = |f(r)|^2$$

$$R_{1,-1}(r) = f(r)\,\frac{1+i}{\sqrt{2}}$$

$$\therefore \boxed{\begin{aligned} &P_{\vec{L}^2, L_z}(\ell, m) = 0, \quad \ell \neq 1 \\ &P_{\vec{L}^2, L_z}(1, m) = \int_0^\infty r^2 dr\, |f(r)|^2 = \frac{1}{3} \end{aligned}}$$

Because the 3 probabilities are equal

7.5. Adding spin-$\frac{1}{2}$ angular momenta

$N = 2J$ spin-$\frac{1}{2}$ particles

$$\vec{J} = \sum_{l=1}^{N} \vec{S}_l = \frac{1}{2}\hbar \sum_{l=1}^{N} \vec{\sigma}_l \qquad J_\pm = J_x \pm iJ_y = \frac{1}{2}\hbar \sum_{l=1} \sigma_{lx} \pm i\sigma_{ly} = \hbar \sum_l \sigma_{l\pm}$$

(a) First, let's notice that

$$J_z |\epsilon_1 \ldots \epsilon_N\rangle = \frac{1}{2}\hbar \sum_{l=1}^{N} \underbrace{\sigma_{lz} |\epsilon_1 \ldots \epsilon_N\rangle}_{\epsilon_l |\epsilon_1 \ldots \epsilon_N\rangle} = \frac{1}{2}\hbar \Big(\underbrace{\sum_{l=1} \epsilon_l}_{= n_+ - n_-} \Big) |\epsilon_1 \ldots \epsilon_N\rangle = \hbar \frac{1}{2}(n_+ - n_-) |\epsilon_1 \ldots \epsilon_N\rangle$$

So $|\epsilon_1 \ldots \epsilon_N\rangle$ is an eigenstate of J_z with eigenvalue $\hbar\frac{1}{2}(n_+ - n_-)$. So the state $|JM\rangle$ is a superposition of states $|\epsilon_1 \ldots \epsilon_N\rangle$ with $M = (n_+ - n_-)/2$. There is only one state with $M = J = N/2$, the state with all spins up, so we have

$$|JJ\rangle = |+ + \cdots +\rangle \longleftarrow \text{There is an arbitrary phase here, which we choose to be 1.}$$

The above argument shows that $|JM\rangle$ is a superposition of states with n_+ spins up and n_- spins down, with $M = (n_+ - n_-)/2$. We can obtain the amplitudes in the superposition by successively lowering $|JJ\rangle$ with J_-. But it is clear that the lowering treats all the spins symmetrically and only gives real, positive amplitudes, so all the amplitudes are equal and positive. There being $\binom{N}{n_+}$ states in the superposition, normalization gives

$$|JM\rangle = \sqrt{\frac{n_+!\, n_-!}{N!}} \sum_{\substack{\epsilon_1, \ldots, \epsilon_N \\ M = (n_+ - n_-)/2}} |\epsilon_1 \ldots \epsilon_N\rangle \equiv |n_+, n_-\rangle$$

(b) $N = 2$:

$$|J=1, M=1\rangle = |++\rangle \qquad (n_+ = 2, n_- = 0)$$

$$|J=1, M=0\rangle = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle) \qquad (n_+ = 1, n_- = 1)$$

$$|J=1, M=-1\rangle = |--\rangle \qquad (n_+ = 0, n_- = 2)$$

The state that is orthogonal to all three triplet states must lie in the subspace spanned by $|+-\rangle$ and $|-+\rangle$, so

$$|\psi\rangle = a|+-\rangle + b|-+\rangle.$$

We have

$$0 = \langle 10|\psi\rangle = \frac{a}{\sqrt{2}} + \frac{b}{\sqrt{2}} \implies a = -b = |a|e^{i\delta}$$

Normalization implies that $|a| = 1/\sqrt{2}$, and we can choose the phase to be anything we want. The conventional choice is $\delta = 0$, which gives

$$\boxed{|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right) = |J=0, M=0\rangle}$$

Singlet state
Antisymmetric under particle exchange

How do we know this state has $J=0$ and $M=0$. It is clear that $J_z|\psi\rangle = 0$, so this is an eigenstate of J_z with eigenvalue 0. But this state is also an eigenstate of J_x and J_y with eigenvalue 0 for both:

$$J_x = \frac{1}{2}\hbar(\sigma_{1x} + \sigma_{2x}) = \frac{1}{2}\hbar(\sigma_{1+} + \sigma_{1-} + \sigma_{2+} + \sigma_{2-})$$

$$\implies J_x|\psi\rangle = \frac{1}{2}\hbar\Big(\underbrace{\sigma_{1+}|\psi\rangle}_{-\frac{1}{\sqrt{2}}|++\rangle} + \underbrace{\sigma_{1-}|\psi\rangle}_{\frac{1}{\sqrt{2}}|--\rangle} + \underbrace{\sigma_{2+}|\psi\rangle}_{\frac{1}{\sqrt{2}}|++\rangle} + \underbrace{\sigma_{2-}|\psi\rangle}_{-\frac{1}{\sqrt{2}}|--\rangle}\Big) = 0$$

$$\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$$

$$\sigma_x = \sigma_+ + \sigma_-$$

$$\sigma_y = -i(\sigma_+ - \sigma_-)$$

$$J_y = \frac{1}{2}\hbar(\sigma_{1y} + \sigma_{2y}) = -\frac{i}{2}\hbar(\sigma_{1+} - \sigma_{1-} + \sigma_{2+} - \sigma_{2-})$$

$$\implies J_y|\psi\rangle = -\frac{i}{2}\hbar\Big(\underbrace{\sigma_{1+}|\psi\rangle}_{-\frac{1}{\sqrt{2}}|++\rangle} - \underbrace{\sigma_{1-}|\psi\rangle}_{\frac{1}{\sqrt{2}}|--\rangle} + \underbrace{\sigma_{2+}|\psi\rangle}_{\frac{1}{\sqrt{2}}|++\rangle} - \underbrace{\sigma_{2-}|\psi\rangle}_{-\frac{1}{\sqrt{2}}|--\rangle}\Big) = 0$$

This implies that $\vec{J}^2|\psi\rangle = 0$, so $|\psi\rangle$ is an eigenstate of $\vec{J}^2$ with eigenvalue 0. We have established that $|\psi\rangle = |J=0, M=0\rangle$.