Phys 552 Quantum Information Theory **Fall 2005**

Homework Assignment #3 **Due Tuesday, October 4**
(40 points) **(at lecture)**

3.1 (10 points) **Operators on a real vector space.**

The linear operators on a D-dimensional vector space V over the complex numbers form a D^2-dimensional complex vector space L_V. The adjoint $A^\dagger$ is defined by

$$\langle\phi|A^\dagger|\psi\rangle = \langle\psi|A|\phi\rangle^* \quad \text{for all vectors } |\psi\rangle \text{ and } |\phi\rangle.$$

Relative to an orthonormal basis $|e_j\rangle$, $j = 1, \ldots, D$, the matrix elements of the adjoint are

$$(A^\dagger)_{jk} = \langle e_j|A^\dagger|e_k\rangle = \langle e_k|A|e_j\rangle^* = A^*_{kj} \ .$$

A natural inner product on L_V is given by

$$(A, B) = \mathrm{tr}(A^\dagger B) = \sum_{j,k} A^*_{jk} B_{jk} \ .$$

The Hermitian operators $A = A^\dagger$ form a D^2-dimensional real vector space H_V, whose complexification is L_V. In a previous homework problem, you constructed a basis consisting of D^2 pure states for L_V and H_V.

The linear operators on a D-dimensional vector space V over the real numbers form a D^2-dimensional real vector space L_V. The transpose A^T is defined by

$$\langle\phi|A^T|\psi\rangle = \langle\psi|A|\phi\rangle \quad \text{for all vectors } |\psi\rangle \text{ and } |\phi\rangle.$$

Relative to an orthonormal basis $|e_j\rangle$, $j = 1, \ldots, D$, the matrix elements of the transpose are

$$(A^T)_{jk} = \langle e_j|A^T|e_k\rangle = \langle e_k|A|e_j\rangle = A_{kj} \ .$$

A natural inner product on L_V is given by

$$(A, B) = \mathrm{tr}(A^T B) = \sum_{j,k} A_{jk} B_{jk} \ .$$

Notice that you can't define the transpose of operators on a complex vector space, because transposition is inconsistent with linearity:

$$\begin{aligned} a\langle\psi|A|\phi\rangle + b\langle\psi|A|\chi\rangle &= \langle\psi|A\big(a|\phi\rangle + b|\chi\rangle\big) \\ &= \big(\langle\phi|a^* + \langle\chi|b^*\big)A^T|\psi\rangle \\ &= a^*\langle\phi|A^T|\psi\rangle + b^*\langle\chi|A^T|\psi\rangle \\ &= a^*\langle\psi|A|\phi\rangle + b^*\langle\psi|A|\chi\rangle \ . \end{aligned}$$

The complex conjugation in the definition of the adjoint is there precisely to take care of this problem. What this means is that if you want to use the transpose of a linear operator in a complex vector space, you have to define it relative to a particular orthonormal basis. The same definition works in any other orthonormal basis that is a *real* linear combination of the vectors in the original basis.

This problem deals with linear operators L_V on a real vector space V.

(a) *Show* that the symmetric operators $A = A^T$ form a subspace S_V of L_V. What is the dimension of S_V? States and observables are symmetric operators. *Show* that the antisymmetric operators $A = -A^T$ form a subspace A_V of L_V. What is the dimension of A_V?

(b) *Show* that any operator O can be written as a sum of a symmetric operator and an antisymmetric operator. This result shows that L_V is the *direct sum* of S_V and A_V.

(c) *Show* that there is no basis of pure states that spans L_V. Given an orthonormal basis $|e_j\rangle$, $j = 1, \ldots, D$, construct a basis of pure states for S_V. *Explain* what is different in a complex vector space that allows the pure states to span L_V.

(d) A system described by a real two-dimensional vector space V is called a *rebit*. *Find* a basis of orthogonal operators for S_V and a basis of orthogonal operators for A_V.

(e) Now consider two rebits. *Find* a basis of orthonormal operators for the symmetric subspace and a basis of orthonormal operators for the antisymmetric subspace. Is it possible to determine the state of two rebits from the statistics of local measurements?

3.2 (10 points) **GHZ-Mermin violation of local realism.** Consider the three-qubit state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\big(|000\rangle - |111\rangle\big)\;,$$

which is called the *Greenberger-Horne-Zeilinger* (GHZ) state.

(a) *Show* that the GHZ state is a $+1$ eigenstate of $X \otimes Y \otimes Y$, $Y \otimes X \otimes Y$, and $Y \otimes Y \otimes X$.

(b) Use the results of part (a) to argue that each qubit has well defined values of X and Y. For qubit j, denote these values by x_j and y_j. We say that these values are *elements of reality*. What would local realism, i.e., the assumption of realistic values that are undisturbed by measurements on other qubits, predict for the product of the outcomes of measurements of X on each qubit?

(c) What does quantum mechanics predict for the product of the outcomes of measurements of X on each qubit?

3.3 (10 points) **Maximal violation of the CHSH Bell inequality.** Let $A = \boldsymbol{\sigma}\cdot\mathbf{a}$, $B = \boldsymbol{\sigma}\cdot\mathbf{b}$, $C = \boldsymbol{\sigma}\cdot\mathbf{c}$, and $D = \boldsymbol{\sigma}\cdot\mathbf{d}$, where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ are unit vectors in three dimensions, and let

$$\mathcal{B} = A\otimes B + C\otimes B + C\otimes D - A\otimes D = \boldsymbol{\sigma}\cdot\mathbf{a}\otimes\boldsymbol{\sigma}\cdot(\mathbf{b}-\mathbf{d}) + \boldsymbol{\sigma}\cdot\mathbf{c}\otimes\boldsymbol{\sigma}\cdot(\mathbf{b}+\mathbf{d})$$

be the *Bell operator*. The quantity we called S in our discussion of the CHSH inequality is the expectation value of the Bell operator, i.e., $S = \langle\mathcal{B}\rangle$.

(a) *Show* that

$$\mathcal{B}^2 = 4I \otimes I + [A, C] \otimes [B, D] = 4(I \otimes I - \boldsymbol{\sigma} \cdot \mathbf{a} \times \mathbf{c} \otimes \boldsymbol{\sigma} \cdot \mathbf{b} \times \mathbf{d}) \ .$$

(b) Use the result of part (a) to *show* that

$$|\langle \mathcal{B} \rangle| \le 2\sqrt{2} \ .$$

This result, called *T'sirelson's inequality*, determines the maximum violation of the CHSH Bell inequality.

(c) *Find* the conditions for equality in T'sirelson's inequality. (Warning: This third part is hard, which is probably why it is not included in Nielsen and Chuang's Problem 2.3.)

3.4 (10 points) Consider two systems, A of dimension d_A and B of dimension d_B. An arbitrary joint pure state $|\Psi\rangle$, when expanded in an arbitrary product basis $|e_j, f_k\rangle$, looks like

$$|\Psi\rangle = \sum_{j,k} c_{jk} |e_j, f_k\rangle \ .$$

(a) *Show* how $|\Psi\rangle$ can be brought into Schmidt form by using the singular-value decomposition of the matrix whose entries are c_{jk}, and *find* the Schmidt vectors for the two systems in terms of the unitary matrices involved in the singular-value decomposition.

(b) Now suppose the two systems have the same dimension d. A maximally entangled state of A and B is one such that the marginal density operators are maximally mixed, i.e., $\rho_A = I_A/d$ and $\rho_B = I_B/d$. *Find* the conditions on c_{jk} such that $|\Psi\rangle$ is maximally entangled, and discuss what this means for the singular values of c_{jk} and thus for the Schmidt coefficients.