Phys 552 Quantum Information Theory **Fall 2005**

Homework Assignment #4 **Due Tuesday, October 18**
(30 points) **(at lecture)**

4.1 (10 points) **Neumark extension of a rank-one POVM.** Consider a POVM for a D-dimensional quantum system, which consists of $N \geq D$ rank-one POVM elements (i.e., operators proportional to one-dimensional projectors). One way to think of this POVM is as a measurement of one-dimensional orthogonal projectors (an ODOP) on an N-dimensional space. This problem develops this way of thinking, which is called the Neumark extension.

Let the (rank-one) POVM elements be denoted by $E_\alpha = |\overline{\psi}_\alpha\rangle\langle\overline{\psi}_\alpha|$, $\alpha = 1, \ldots, N$, where the vectors $|\overline{\psi}_\alpha\rangle$ are subnormalized, i.e., $0 < \mu_\alpha = \langle\overline{\psi}_\alpha|\overline{\psi}_\alpha\rangle \leq 1$. The corresponding normalized vectors are $|\psi_\alpha\rangle = |\overline{\psi}_\alpha\rangle/\sqrt{\mu_\alpha}$. The POVM satisfies a completeness relation

$$P = \sum_{\alpha=1}^{N} E_\alpha = \sum_{\alpha=1}^{N} |\overline{\psi}_\alpha\rangle\langle\overline{\psi}_\alpha| = \sum_{\alpha=1}^{N} \mu_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| \ ,$$

where P denotes the identity operator on the D-dimensional Hilbert space.

(a) *Show* that by adding $N - D$ dimensions to the Hilbert space, you can find an orthonormal set of vectors $|\hat{\psi}_\alpha\rangle$, $\alpha = 1, \ldots, N$, that project to the POVM elements in the original D dimensions, i.e., $P|\hat{\psi}_\alpha\rangle = |\overline{\psi}_\alpha\rangle$. (Hint: Expand the POVM elements in an arbitrary arbitrary orthonormal basis, and show that the expansion coefficients form part of a unitary matrix.)

The orthonormal set $|\hat{\psi}_\alpha\rangle$ is a *Neumark extension* of the POVM. The support of a system state ρ is confined to the original D-dimensional space, i.e., $P\rho P = \rho$. For such states the POVM measurement statistics are the same as the statistics of a measurement of the Neumark extension, i.e.,

$$\begin{aligned}\langle\hat{\psi}_\alpha|\rho|\hat{\psi}_\alpha\rangle = \mathrm{tr}(\rho|\hat{\psi}_\alpha\rangle\langle\hat{\psi}_\alpha|) = \mathrm{tr}(P\rho P|\hat{\psi}_\alpha\rangle\langle\hat{\psi}_\alpha|) &= \mathrm{tr}(\rho P|\hat{\psi}_\alpha\rangle\langle\hat{\psi}_\alpha|P) \\ &= \mathrm{tr}(\rho|\overline{\psi}_\alpha\rangle\langle\overline{\psi}_\alpha|) = \mathrm{tr}(\rho E_\alpha) = p_\alpha \ .\end{aligned}$$

We can thus regard a measurement of a rank-one POVM as an ODOP measurement on a higher-dimensional space.

(b) Consider the three Bloch vectors,

$$\begin{aligned}\mathbf{n}_1 &= \mathbf{e}_x \ , \\ \mathbf{n}_2 &= -\frac{1}{2}\mathbf{e}_x + \frac{\sqrt{3}}{2}\mathbf{e}_y \ , \\ \mathbf{n}_3 &= -\frac{1}{2}\mathbf{e}_x - \frac{\sqrt{3}}{2}\mathbf{e}_y \ ,\end{aligned}$$

which satisfy $\mathbf{n}_\alpha \cdot \mathbf{n}_\beta = -1/2$ for any pair $\alpha \neq \beta$ and which point to the vertices of an equilateral triangle in the equatorial plane. The corresponding Hilbert-space vectors, $|\mathbf{n}_1\rangle$,

$|\mathbf{n}_2\rangle$, and $|\mathbf{n}_3\rangle$, can be used to form a three-outcome, rank-one POVM called the *trine*. *Write* out the trine POVM elements, and *construct* a Neumark extension for it.

(c) The four Bloch vectors,

$$\begin{aligned}
\mathbf{n}_1 &= \mathbf{e}_3\ ,\\
\mathbf{n}_2 &= \sqrt{\frac{8}{9}}\mathbf{e}_1 - \frac{1}{3}\mathbf{e}_3\ ,\\
\mathbf{n}_3 &= -\sqrt{\frac{2}{9}}\mathbf{e}_1 + \sqrt{\frac{2}{3}}\mathbf{e}_2 - \frac{1}{3}\mathbf{e}_3\ ,\\
\mathbf{n}_4 &= -\sqrt{\frac{2}{9}}\mathbf{e}_1 - \sqrt{\frac{2}{3}}\mathbf{e}_2 - \frac{1}{3}\mathbf{e}_3\ ,
\end{aligned}$$

satisfy $\mathbf{n}_\alpha \cdot \mathbf{n}_\beta = -1/3$ for any pair $\alpha \neq \beta$ and point to the vertices of a tetrahedron. The corresponding Hilbert-space vectors, $|\mathbf{n}_1\rangle$, $|\mathbf{n}_2\rangle$, $|\mathbf{n}_3\rangle$, and $|\mathbf{n}_4\rangle$, can be used to form a four-outcome POVM called, not surprisingly, the *tetrahedron*. *Write* out the tetrahedron POVM elements, and *construct* a Neumark extension for it.

4.2 (10 points) **Convex set of POVM elements.** The POVM elements for a D-dimensional quantum system consist of the positive operators E that satisfy $0 \le E \le 1$. This is equivalent to saying that the POVM elements are the Hermitian operators whose eigenvalues lie between 0 and 1, inclusive. The POVM elements form a convex set; i.e., if E and F are POVM elements, then for $0 \le \lambda \le 1$, so is $\lambda E + (1-\lambda)F$.

(a) The POVM elements for a qubit can be written in the Pauli representation as

$$E = r1 + s\mathbf{n}\cdot\boldsymbol{\sigma}\ ,$$

where r and s are real numbers. *Find* the allowed values of r and s, and *describe* the convex set of qubit POVM elements. By suppressing one of the "spatial" dimensions in the Pauli representation, *draw* a picture of the set of qubit POVM elements. From the picture, *identify* the extreme points of the convex set.

(b) For a D-dimensional system, *show* that the extreme points of the convex set of POVM elements are the projection operators of all ranks (the rank is the dimension of the support, i.e., the dimension of the subspace onto which the projection operator projects), including the zero operator (projector onto the zero-dimensional subspace) and the unit operator (projector onto the entire Hilbert space). (Hint: It's easy to show that a projector can't be written as a convex combination of other POVM elements. To show that any POVM element E can be written as a convex combination of projectors, start with the eigendecomposition of E, and cleverly rewrite it as a convex combination of projectors. You will need to use the zero operator in the convex combination.)

4.3 (10 points)

(a) Suppose that you make a von Neumann measurement of the spin component of a spin-1/2 particle along one of the three Cartesian axes, tossing a fair three-sided die to determine which of the three measurements to make. *Determine* the Kraus operators, quantum operations, and POVM elements for this six-outcome measurement.

(b) Suppose now that all you know after the measurement of part (a) is whether the the result is $+1$ or -1 , but not which axis was used. *Determine* the Kraus operators, quantum operations, and POVM elements for this two-outcome measurement. This is a model of a very noisy measurement, the noise a consequence of not knowing how the measuring apparatus is oriented.

(c) *Find* at least one other pair of quantum operations that corresponds to the POVM of part (b).