

Homework Assignment #6
(30 points)

Due Thursday, November 17
(at lecture)

6.1 (10 points) **Cloning and the isotropic POVM.** The isotropic POVM for a D -dimensional Hilbert space has a POVM element, $dE_{|\phi\rangle} = \alpha|\phi\rangle\langle\phi| d\Gamma_{|\phi\rangle}$, for every ray $|\phi\rangle$ in projective Hilbert space, where $d\Gamma_{|\phi\rangle}$ is the unitarily invariant measure on projective Hilbert space and α is a positive constant, thus implying that all outcomes are equally weighted. The POVM satisfies a completeness relation

$$I = \int dE_{|\phi\rangle} = \alpha \int d\Gamma_{|\phi\rangle} |\phi\rangle\langle\phi| .$$

If the system is in state ρ , the probability to get outcome $|\phi\rangle$ in a measurement of the isotropic POVM is

$$dp(|\phi\rangle|\rho) = \text{tr}(\rho dE_{|\phi\rangle}) = \alpha d\Gamma_{|\phi\rangle} \langle\phi|\rho|\phi\rangle .$$

To do this problem, we need to be able to do at least one integral over the invariant measure. The unitarily invariant line element on projective Hilbert space, called the *Fubini-Studi metric*, measures lengths in terms of Hilbert-space angle: the distance $d\gamma$ between a normalized vector $|\phi\rangle$ and a nearby normalized vector $|\phi'\rangle = |\phi\rangle + |d\phi\rangle$ is given by $\cos d\gamma = |\langle\phi|\phi'\rangle| = |1 + \langle\phi|d\phi\rangle|$. Hilbert-space angle is not changed by global phase changes, confirming that we are dealing with rays in projective Hilbert space. The Fubini-Studi line element is given by

$$d\gamma^2 = \sin^2 d\gamma = 1 - \cos^2 d\gamma = -2\text{Re}(\langle\phi|d\phi\rangle) - |\langle\phi|d\phi\rangle|^2 .$$

Normalization of $|\phi'\rangle$ requires that

$$0 = \langle\phi'|\phi'\rangle - 1 = 2\text{Re}(\langle\phi|d\phi\rangle) + \langle d\phi|d\phi\rangle ,$$

which gives

$$d\gamma^2 = \langle d\phi|d\phi\rangle - |\langle\phi|d\phi\rangle|^2 = \langle d\phi|d\phi\rangle - (\text{Im}(\langle\phi|d\phi\rangle))^2 = \langle d\phi_{\perp}|d\phi_{\perp}\rangle ,$$

where $|d\phi_{\perp}\rangle = |d\phi\rangle - |\phi\rangle\langle\phi|d\phi\rangle$ is the projection of the small displacement $|d\phi\rangle$ orthogonal to $|\phi\rangle$.

The normalized vectors in a D -dimensional Hilbert space make up the sphere of unit radius, \mathcal{S}_{2D-1} , in $2D$ real dimensions. The contribution $\langle d\phi|d\phi\rangle$ to the Fubini-Studi line element is the standard metric on this unit sphere. The quantity $\langle\phi|d\phi\rangle$ is the component of the small displacement along $|\phi\rangle$: the real part describes changes in normalization, and the imaginary part describes changes in phase. The real part disappears from the line element because of the normalization constraint; the square of the imaginary part is subtracted away to remove the contribution of phase changes, because a global phase change does not change the Hilbert-space angle between two vectors.

Given a particular normalized vector $|\psi\rangle$, any other normalized vector can be written as

$$|\phi\rangle = e^{i\delta}(\cos\theta|\psi\rangle + \sin\theta|\chi\rangle) .$$

where δ is a global phase, θ is a “polar angle” in the range $0 \leq \theta \leq \pi/2$, and $|\chi\rangle$ is a normalized vector orthogonal to $|\psi\rangle$. We can get rid of the global phase freedom by choosing $\delta = 0$, thus working with rays in projective Hilbert-space. A small change in $|\phi\rangle$ takes the form

$$|d\phi\rangle = d\theta(-\sin\theta|\psi\rangle + \cos\theta|\chi\rangle) + \sin\theta|d\chi\rangle ,$$

which gives

$$\begin{aligned} \langle\phi|d\phi\rangle &= \sin^2\theta\langle\chi|d\chi\rangle , \\ \langle d\phi|d\phi\rangle &= d\theta^2 + \sin^2\theta\langle d\chi|d\chi\rangle . \end{aligned}$$

The resulting line element is

$$\begin{aligned} d\gamma^2 &= d\theta^2 + \sin^2\theta(\langle d\chi|d\chi\rangle - \sin^2\theta|\langle\chi|d\chi\rangle|^2) \\ &= d\theta^2 + \sin^2\theta(\langle d\chi_\perp|d\chi_\perp\rangle + \cos^2\theta|\langle\chi|d\chi\rangle|^2) , \end{aligned}$$

where $d\chi_\perp\rangle = |d\chi\rangle - |\chi\rangle\langle\chi|d\chi\rangle$ is the projection of $|d\chi\rangle$ orthogonal to $|\chi\rangle$.

The line element $\langle d\chi|d\chi\rangle - \sin^2\theta|\langle\chi|d\chi\rangle|^2 = \langle d\chi_\perp|d\chi_\perp\rangle + \cos^2\theta|\langle\chi|d\chi\rangle|^2$ is the standard metric on the unit sphere, \mathcal{S}_{2D-3} , in $2D - 2$ real dimensions, except that along one real dimension, corresponding to phase changes of $|\chi\rangle$, lengths are scaled by a factor $\cos\theta$. The $\sin^2\theta$ in the line element means that lengths along all $2D - 3$ real dimensions of \mathcal{S}_{2D-3} are scaled by a factor of $\sin\theta$. Thus the integration measure that goes with the Fubini-Study metric is

$$d\Gamma_{|\phi\rangle} = \sin^{2D-3}\theta \cos\theta d\theta dS_{2D-3} ,$$

where dS_{2D-3} is the standard measure on the unit sphere \mathcal{S}_{2D-3} . This form of the integration measure is useful for doing integrals over functions of $|\langle\phi|\psi\rangle| = \cos\theta$.

In doing this problem, you should never have to calculate an explicit form for the area of a sphere. Instead, the area S_{2D-3} of \mathcal{S}_{2D-3} can be left as a normalization constant, whose value cancels out of the ultimate answer to part (c).

- (a) Using the completeness relation, determine the value of the positive constant α .
- (b) Find the value of the integral

$$\int d\Gamma_{|\phi\rangle} |\langle\phi|\psi\rangle|^2 .$$

(c) One strategy for approximate cloning of an arbitrary state $|\psi\rangle$ is to measure the isotropic POVM and then make copies of the result $|\phi\rangle$. Find the average squared fidelity of the copies.

6.2 (10 points) **Minimum error probability for two mixed states.** Consider two mixed states, ρ_1 and ρ_2 , occurring with probabilities q_1 and q_2 . Let E_1 and E_2 make up a two-outcome POVM such that on occurrence of outcome 1, you decide the state was ρ_1 , and on outcome 2, you decide on ρ_2 .

(a) Show that the error probability is

$$P_e = q_1 - \text{tr}(E_1(q_1\rho_1 - q_2\rho_2)) .$$

(b) Show that

$$\max_{0 \leq E \leq 1} \text{tr}(E(q_1\rho_1 - q_2\rho_2)) = \frac{1}{2} \text{tr}(|q_1\rho_1 - q_2\rho_2|) + \frac{1}{2}(q_1 - q_2) .$$

(c) Use the results of parts (a) and (b) to show that the minimum error probability is

$$(P_e)_{\min} = \frac{1}{2} - \frac{1}{2} \text{tr}(|q_1\rho_1 - q_2\rho_2|) ,$$

and find a POVM that gives the minimum error probability.

(d) Determine the minimum error probability when the two states are pure, i.e., $\rho_1 = |\psi_1\rangle\langle\psi_1|$ and $\rho_2 = |\psi_2\rangle\langle\psi_2|$.

6.3 (10 points) **Pretty good measurements.** Consider a density operator ρ and a pure-state ensemble decomposition

$$\rho = \sum_{\alpha=1}^N p_{\alpha} |\psi_{\alpha}\rangle\langle\psi_{\alpha}| = \sum_{\alpha} |\bar{\psi}_{\alpha}\rangle\langle\bar{\psi}_{\alpha}| .$$

The support of ρ is the subspace spanned by the eigenvectors with nonzero eigenvalues. The HJW theorem for ensemble decompositions tells us that the decomposition vectors $|\bar{\psi}_{\alpha}\rangle$ lie in and span the support. In this problem, we restrict attention to the support, which we assume to be D -dimensional, and forget about the rest of Hilbert space. This means that ρ is invertible.

We can define a POVM that has POVM elements

$$E_{\alpha} = \rho^{-1/2} |\bar{\psi}_{\alpha}\rangle\langle\bar{\psi}_{\alpha}| \rho^{-1/2} = |\bar{\phi}_{\alpha}\rangle\langle\bar{\phi}_{\alpha}| .$$

The POVM elements are clearly positive operators and the POVM satisfies the completeness relation

$$\sum_{\alpha} E_{\alpha} = \rho^{-1/2} \underbrace{\left(\sum_{\alpha} |\bar{\psi}_{\alpha}\rangle\langle\bar{\psi}_{\alpha}| \right)}_{=\rho} \rho^{-1/2} = I .$$

This measurement is called the *pretty good measurement*.

(a) *Show* that the outcome probabilities $q_\alpha = \text{tr}(\rho E_\alpha)$, when the state is ρ , are the same as the ensemble probabilities p_α . This is the unique property of the pretty good measurement.

Since $p_\alpha = q_\alpha$ for the pretty good measurement, the preparation information inequality,

$$H(\mathbf{p}) \geq S(\rho) ,$$

and the POVM inequality,

$$H(\mathbf{q}) + \sum_{\alpha} q_{\alpha} \log(\text{tr}(E_{\alpha})) \geq S(\rho) ,$$

are constraints on the same probability distribution $\mathbf{p} = \mathbf{q}$.

(b) Which of these inequalities provides the tighter constraint on $H(\mathbf{p})$?

(c) *Show* that the POVM inequality can be rewritten as

$$\sum_{\alpha} p_{\alpha} \log(\langle \psi_{\alpha} | \rho^{-1} | \psi_{\alpha} \rangle) \geq S(\rho) .$$

(d) *Show* that

$$H(\mathbf{p}) \geq \sum_{\alpha} p_{\alpha} \log(\langle \psi_{\alpha} | \rho^{-1} | \psi_{\alpha} \rangle) .$$