

Quantum information theory

Lectures 26-28

Clonability and distinguishability

Introduction.

	Clonability	Distinguishability	
Classical	✓	✓	
Quantum	X	X	← much richer set of possibilities

Clonability. Set of pure states, $\{|\psi_j\rangle, j=1, \dots, N\}$

Cloning: $|\psi_j\rangle \otimes |S\rangle \longrightarrow |\psi_j\rangle \otimes |\psi_j\rangle$

$\uparrow$
 Standard state of
 system receiving copy

① Is there a unitary U such that

$$U|\psi_j\rangle \otimes |S\rangle = |\psi_j\rangle \otimes |\psi_j\rangle?$$

$$\langle\psi_j|\psi_k\rangle = \langle\psi_j| \otimes \langle S| U^\dagger U |\psi_k\rangle \otimes |S\rangle = \langle\psi_j|\psi_k\rangle^2 \Rightarrow \langle\psi_j|\psi_k\rangle = 0 \text{ or } 1$$

$\Rightarrow$ States in list that are not identical must be orthogonal.

Nonorthogonal states cannot be cloned.

$\uparrow$
Classical
info

② Is there a trace-preserving operation $\mathcal{Q}$ such that

$$\mathcal{Q}(|\psi\rangle\langle\psi| \otimes |S\rangle\langle S|) = |\psi\rangle\langle\psi| \otimes |\psi\rangle\langle\psi|$$

$\mathcal{Q}$ has a measurement model: $\mathcal{Q}(\rho) = \text{tr}_A(U \rho \otimes |\phi\rangle\langle\phi| U^\dagger)$

$$\begin{aligned} \mathcal{Q}(|\psi\rangle\langle\psi| \otimes |S\rangle\langle S|) &= \text{tr}_A(U |\psi\rangle\langle\psi| \otimes |S\rangle\langle S| \otimes |\phi\rangle\langle\phi| U^\dagger) \\ &= |\psi\rangle\langle\psi| \otimes |\psi\rangle\langle\psi| \end{aligned}$$

$$U|\psi_j\rangle \otimes |S\rangle \otimes |\phi\rangle = |\psi_j\rangle \otimes |\psi_j\rangle \otimes |\phi_j\rangle$$

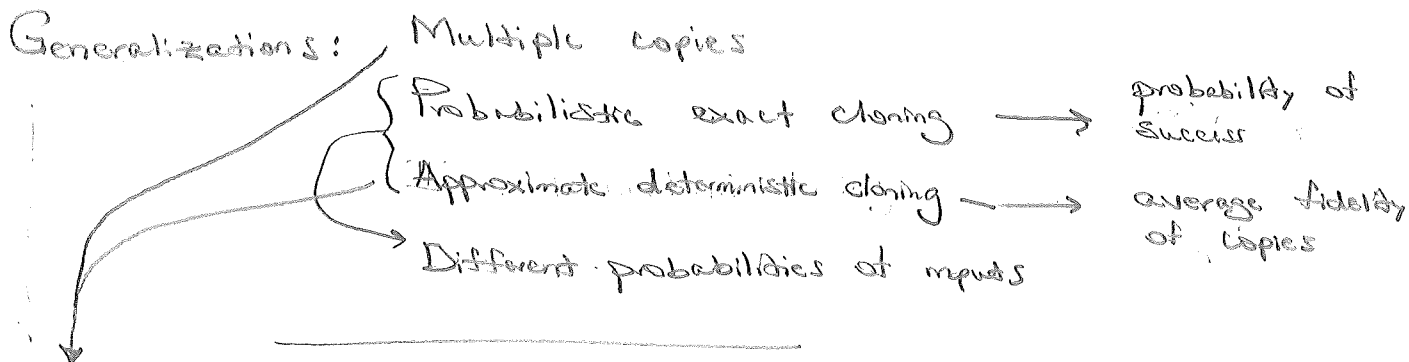
Why? (2)
State inside tr_A is a pure state; the only way the partial trace can be pure is for it to be a product state.

$$\begin{aligned}\langle \psi_j | \psi_k \rangle &= \langle \psi_j | \otimes \langle S | \otimes \langle \phi | U^\dagger U | \psi_k \rangle \otimes | S \rangle \otimes | \phi \rangle \\ &= \langle \psi_j | \psi_k \rangle \langle \phi_j | \phi_k \rangle\end{aligned}$$

$$\Rightarrow \langle \psi_j | \psi_k \rangle = 0 \text{ or } \langle \psi_j | \psi_k \rangle = 1 = \langle \phi_j | \phi_k \rangle$$

$\Rightarrow$ States in the list that are not identical must be orthogonal.

Generalizations: Multiple copies



One comment: Qubits

Measure the isotropic POVM $E_{\vec{n}} = \frac{1}{2\pi} |\vec{n}\rangle \langle \vec{n}| = \frac{1}{4\pi} (1 + \vec{\sigma} \cdot \vec{n})$

Input state $|\vec{m}\rangle$.

$$\int d\Omega E_{\vec{n}} = 1$$

Get result $\vec{n}$; make as many copies as you want of $|\vec{n}\rangle$.

$$\left(\text{squared fidelity of output } |\vec{n}\rangle \text{ with input } |\vec{m}\rangle \right) = |\langle \vec{n} | \vec{m} \rangle|^2$$

$$\left(\begin{array}{c} \text{average (fidelity)}^2 \\ \text{of copies} \end{array} \right) = \int \frac{d\Omega_{\vec{n}}}{4\pi} \int d\Omega_{\vec{m}} \underbrace{p(\vec{n}|\vec{m})}_{\substack{\text{probability for result:} \\ \vec{n} \text{ given input state } |\vec{m}\rangle \\ = \frac{1}{4\pi} |\langle \vec{n} | \vec{m} \rangle|^2}} |\langle \vec{n} | \vec{m} \rangle|^2$$

$\uparrow$
 average over input state $|\vec{m}\rangle$

average over measurement results

$\downarrow$
 (fidelity)² of copy $|\vec{n}\rangle$ given input $\vec{m}$

$$|\langle \vec{n} | \vec{m} \rangle|^2 = \frac{1}{2} (1 + \vec{n} \cdot \vec{m})$$

$$p(\vec{n}|\vec{m}) = \langle \vec{m} | E_{\vec{n}} | \vec{m} \rangle = \frac{1}{4\pi} (1 + \vec{m} \cdot \vec{n})$$

$$\begin{aligned}
 \int d\Omega_{\vec{n}} p(\vec{n}|\vec{m}) |\langle \vec{n} | \vec{m} \rangle|^2 &= \frac{1}{8\pi} \int d\Omega_{\vec{n}} \underbrace{(1 + \vec{n} \cdot \vec{m})^2}_{1 + 2\vec{n} \cdot \vec{m} + (\vec{n} \cdot \vec{m})^2} \\
 &= \frac{1}{8\pi} \left(4\pi + \underbrace{\int d\Omega_{\vec{n}} (\vec{n} \cdot \vec{m})^2}_{\frac{4\pi}{3}} \right) \\
 &= \frac{2}{3}
 \end{aligned}$$

Average over $\vec{m}$ is irrelevant

$$\left(\begin{array}{c} \text{average (fidelity)}^2 \\ \text{of copies} \end{array} \right) = \frac{2}{3}$$

Generalization to D dimensions?

Distinguishability.

Relation to clonability.

Error probability.

Distinguish two pure states, $|\psi_1\rangle$ and $|\psi_2\rangle$, by measuring a POVM $\{F_a, a=1, \dots, N\}$

$F_1, \dots, F_n$
decide on $|\psi_1\rangle$

$F_{n+1}, \dots, F_N$
decide on $|\psi_2\rangle$

$$E_1 = \sum_{a=1}^n F_a$$

$$E_2 = \sum_{a=n+1}^N F_a$$

← always use a 2-outcome measurement for the decision

$$E_1 + E_2 = 1$$

$$\begin{aligned} \text{(error probability)} = P_e &= P_1|\psi_2\rangle P_{\psi_2} + P_2|\psi_1\rangle P_{\psi_1} \\ &= \langle \psi_2 | E_1 | \psi_2 \rangle = \text{tr}(E_1 P_2) \\ &= \langle \psi_1 | E_2 | \psi_1 \rangle = \text{tr}(E_2 P_1) \end{aligned}$$

$$= \frac{1}{2} \text{tr}(E_1 P_2 + E_2 P_1)$$

$$\uparrow 1 - E_1$$

$$P_e = \frac{1}{2} - \frac{1}{2} \text{tr}(E_1 (P_1 - P_2))$$

Minimizing P_e means maximizing $\text{tr}(E_1 (P_1 - P_2))$ over positive $E_1 \leq 1$.

Go to a Bloch-sphere description in the 2-d subspace spanned by $|\psi_1\rangle$ and $|\psi_2\rangle$:

$$P_j = |\psi_j\rangle\langle\psi_j| = |\vec{n}_j\rangle\langle\vec{n}_j| = \frac{1}{2}(1 + \vec{\sigma} \cdot \vec{n}_j)$$

$$\begin{aligned}
 P_1 - P_2 &= \frac{1}{2} \vec{\sigma} \cdot (\vec{n}_1 - \vec{n}_2) = \frac{1}{2} |\vec{n}_1 - \vec{n}_2| \vec{\sigma} \cdot \underbrace{\frac{\vec{n}_1 - \vec{n}_2}{|\vec{n}_1 - \vec{n}_2|}}_{\substack{\uparrow \\ \text{unit vector}}} \\
 &= \frac{1}{2} |\vec{n}_1 - \vec{n}_2| (|\vec{m}\rangle\langle\vec{m}| - |-\vec{m}\rangle\langle-\vec{m}|)
 \end{aligned}$$

$$\begin{aligned}
 \text{tr}(E_1(P_1 - P_2)) &= \frac{1}{2} |\vec{n}_1 - \vec{n}_2| (\langle\vec{m}|E_1|\vec{m}\rangle - \langle-\vec{m}|E_1|-\vec{m}\rangle) \\
 &\leq \frac{1}{2} |\vec{n}_1 - \vec{n}_2| \\
 &\quad \uparrow \text{Equality iff } E_1 = |\vec{m}\rangle\langle\vec{m}| = \frac{1}{2}(1 + \vec{\sigma} \cdot \vec{m})
 \end{aligned}$$

$$(P_2)_{\min} = \frac{1}{2} - \frac{1}{4} |\vec{n}_1 - \vec{n}_2|$$

$$\begin{aligned}
 \cos^2 \theta &= |\langle\psi_1|\psi_2\rangle|^2 = \text{tr}(P_1 P_2) = \frac{1}{2}(1 + \vec{n}_1 \cdot \vec{n}_2) \\
 \uparrow \\
 \text{Hilbert-space angle} &= \theta
 \end{aligned}$$

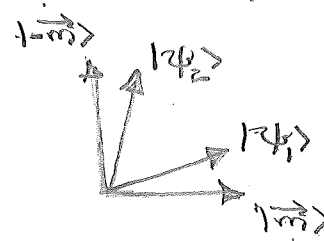
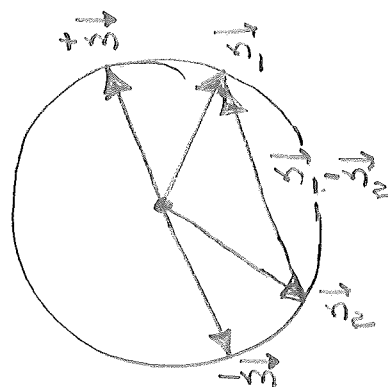
$$\sqrt{2 - 2\vec{n}_1 \cdot \vec{n}_2} = \sqrt{2} \sqrt{1 - \vec{n}_1 \cdot \vec{n}_2}$$

$$= \frac{1}{2} - \frac{\sqrt{2}}{4} \sqrt{1 - \vec{n}_1 \cdot \vec{n}_2}$$

$$= \frac{1}{2} - \frac{1}{2} \sqrt{1 - |\langle\psi_1|\psi_2\rangle|^2}$$

$$= \frac{1}{2}(1 - \sin \theta)$$

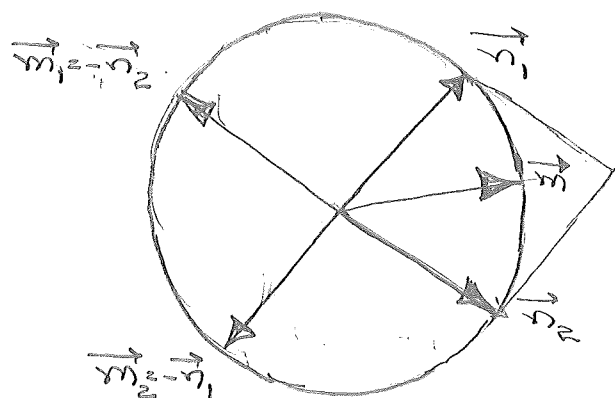
minimum error
determined by
inner product



Nonorthogonal states
cannot be reliably
distinguished.

Unambiguous state discrimination.

What if you can't afford a mistake, but can afford not to decide?



$$E_1 = \frac{\alpha}{2} (1 - \vec{\sigma} \cdot \vec{n}_2) \quad \begin{array}{l} \text{falsifies } |\psi_2\rangle \\ \text{confirms } |\psi_1\rangle \end{array}$$

$$E_2 = \frac{\alpha}{2} (1 - \vec{\sigma} \cdot \vec{n}_1) \quad \begin{array}{l} \text{falsifies } |\psi_1\rangle \\ \text{confirms } |\psi_2\rangle \end{array}$$

$$E_3 = 1 - E_1 - E_2$$

How big can α be?

$$1 \geq E_1 + E_2 = \alpha \left(1 - \frac{1}{2} (\vec{n}_1 + \vec{n}_2) \cdot \vec{\sigma} \right)$$

$$= \alpha \left(1 - \frac{1}{2} |\vec{n}_1 + \vec{n}_2| \frac{(\vec{n}_1 + \vec{n}_2) \cdot \vec{\sigma}}{|\vec{n}_1 + \vec{n}_2|} \right)$$

$$\uparrow \frac{|\vec{n}_1 + \vec{n}_2|}{|\vec{n}_1 + \vec{n}_2|}$$

$$= \alpha \left(1 - \frac{1}{2} |\vec{n}_1 + \vec{n}_2| |\vec{m}\rangle\langle\vec{m}| \right) + \alpha \left(1 + \frac{1}{2} |\vec{n}_1 + \vec{n}_2| |\vec{-m}\rangle\langle\vec{-m}| \right) \leq 1$$

$$\alpha \leq \frac{1}{1 + \frac{1}{2} |\vec{n}_1 + \vec{n}_2|}$$

$$\frac{1}{2} |\vec{n}_1 + \vec{n}_2| = \frac{1}{\sqrt{2}} (1 + \vec{n}_1 \cdot \vec{n}_2)^{1/2} = |\langle\psi_1|\psi_2\rangle|$$

$$\alpha \leq \frac{1}{1 + |\langle\psi_1|\psi_2\rangle|}$$

$$\left(\begin{array}{l} \text{probability of} \\ \text{no decision} \end{array} \right) = \frac{1}{2} \text{tr}(E_3 P_1) + \frac{1}{2} \text{tr}(E_3 P_2)$$

$$= 1 - \text{tr}((E_1 + E_2) \frac{1}{2} (P_1 + P_2)) = \frac{1}{2} (1 + \frac{1}{2} (\vec{n}_1 + \vec{n}_2) \cdot \vec{\sigma}) = \frac{1}{2} (1 + \frac{1}{2} |\vec{n}_1 + \vec{n}_2| \vec{m} \cdot \vec{\sigma})$$

$$= 1 - \alpha \left(1 - \frac{1}{4} |\vec{n}_1 + \vec{n}_2|^2 \right)$$

$$\geq 1 - \left(1 - \frac{1}{2} |\vec{n}_1 + \vec{n}_2| \right)$$

$$\left(\begin{array}{l} \text{probability of} \\ \text{no decision} \end{array} \right) \geq \frac{1}{2} |\vec{n}_1 + \vec{n}_2| = |\langle\psi_1|\psi_2\rangle|$$

cos θ

Nonorthogonal states cannot be reliably distinguished.

Mixed states and distance measures

The only distinguishability measure for pure states is the inner product.
What about mixed states?

① Trace distance generalizes $(P_e)_{\min}$

② Fidelity generalizes $|\langle \psi_1 | \psi_2 \rangle|$ (to square or not to square)

Trace distance

$$P_e = \frac{1}{2} \text{tr}(E_1 \rho_2 + E_2 \rho_1) = \frac{1}{2} - \frac{1}{2} \text{tr}(E_1 (\rho_1 - \rho_2))$$

Why not the more obvious Euclidean distance

Lemma: $\max_{0 \leq E \leq 1} \text{tr}(E(\rho_1 - \rho_2)) = \frac{1}{2} \text{tr}(|\rho_1 - \rho_2|) \equiv D(\rho_1, \rho_2)$

$$\sqrt{\text{tr}((\rho_1 - \rho_2)^2)}$$

↑
trace distance

$$|A| \equiv \sqrt{A^\dagger A}$$

$$\text{tr}(|A|) = \text{tr}(\sqrt{A^\dagger A}) = \left(\begin{array}{l} \text{Sum of the} \\ \text{singular values of } A \end{array} \right)$$

$$\rho_1 - \rho_2 = \sum_j \lambda_j |e_j\rangle\langle e_j| = \underbrace{\sum_{\lambda_j > 0} |\lambda_j| |e_j\rangle\langle e_j|}_{P_+ (\rho_1 - \rho_2) P_+} - \underbrace{\sum_{\lambda_j < 0} |\lambda_j| |e_j\rangle\langle e_j|}_{P_- |\rho_1 - \rho_2| P_-}$$

$$P_+ = \sum_{\lambda_j > 0} |e_j\rangle\langle e_j|$$

$$P_- = \sum_{\lambda_j < 0} |e_j\rangle\langle e_j|$$

$$P_0 = I - P_+ - P_- = \sum_{\lambda_j = 0} |e_j\rangle\langle e_j|$$

$$P_+ (\rho_1 - \rho_2) P_+$$

$$P_- |\rho_1 - \rho_2| P_-$$

Q

S

N&C

$$|\rho_1 - \rho_2| = \sum_j |\lambda_j| |e_j\rangle\langle e_j| = \sum_{\lambda_j > 0} |\lambda_j| |e_j\rangle\langle e_j| + \sum_{\lambda_j < 0} |\lambda_j| |e_j\rangle\langle e_j|$$

$$0 = \text{tr}(\rho_1 - \rho_2) = \sum_{\lambda_j > 0} \lambda_j + \sum_{\lambda_j < 0} \lambda_j = \sum_{\lambda_j > 0} |\lambda_j| - \sum_{\lambda_j < 0} |\lambda_j|$$

$$\Rightarrow \text{tr}(P_+ (\rho_1 - \rho_2)) = \sum_{\lambda_j > 0} |\lambda_j| = \sum_{\lambda_j < 0} |\lambda_j| = \text{tr}(P_- |\rho_1 - \rho_2|)$$

$$D(\rho_1, \rho_2) = \frac{1}{2} \text{tr}(|\rho_1 - \rho_2|) = \frac{1}{2} \sum_{\lambda_j > 0} |\lambda_j| + \frac{1}{2} \sum_{\lambda_j < 0} |\lambda_j| = \sum_{\lambda_j > 0} |\lambda_j| = \sum_{\lambda_j < 0} |\lambda_j|$$

$\text{tr}(P_+ (\rho_1 - \rho_2))$
 $\text{tr}(P_- |\rho_1 - \rho_2|)$

Proof: $\text{tr}(E(p_1 - p_2)) = \sum_j \lambda_j \langle e_j | E | e_j \rangle$

$$= \underbrace{\sum_{\lambda_j \geq 0} \lambda_j \langle e_j | E | e_j \rangle}_{\geq 0} + \underbrace{\sum_{\lambda_j < 0} \lambda_j \langle e_j | E | e_j \rangle}_{\leq 0}$$

iff $E = P_+ + E'$
 $P_0 E' P_0 = E'$

$$\leq \sum_{\lambda_j \geq 0} \lambda_j \underbrace{\langle e_j | E | e_j \rangle}_{\leq 1}$$

$$\leq \sum_{\lambda_j \geq 0} |\lambda_j|$$

$$= D(p_1, p_2)$$

■ QED.

$$(P_2)_{\text{min}} = \frac{1}{2} - \frac{1}{4} \text{tr}(P_1 - P_2)$$

$$= \frac{1}{2} - \frac{1}{2} D(p_1, p_2)$$

$$E_1 = P_+ + E'$$

$$E_2 = P_- + E'_2$$

$$E'_1 + E'_2 = P_0$$

$P_1 = P_2$ in the P_0 subspace

Properties of trace distance:

① Pure states: $|\psi_1\rangle, |\psi_2\rangle$

$$\text{tr}(P_1 - P_2) = |\vec{n}_1 - \vec{n}_2|$$

$$\frac{1}{2} \vec{\sigma} \cdot (\vec{n}_1 - \vec{n}_2) = \frac{1}{2} |\vec{n}_1 - \vec{n}_2| (|\vec{n}_1\rangle \langle \vec{n}_1| - |\vec{n}_2\rangle \langle \vec{n}_2|)$$

$$D(p_1, p_2) = \frac{1}{2} \text{tr}(P_1 - P_2) = \frac{1}{2} |\vec{n}_1 - \vec{n}_2| = \left(1 - \underbrace{|\langle \psi_1 | \psi_2 \rangle|^2}_{\cos^2 \theta}\right)^{1/2} = \sin \theta$$

② Qubits: P_1, P_2

$$\text{tr}(P_1 - P_2) = |\vec{S}_1 - \vec{S}_2|$$

$$\frac{1}{2} (\vec{S}_1 - \vec{S}_2) \cdot \vec{\sigma} = \frac{1}{2} |\vec{S}_1 - \vec{S}_2| (|\vec{n}_1\rangle \langle \vec{n}_1| - |\vec{n}_2\rangle \langle \vec{n}_2|)$$

$$D(p_1, p_2) = \frac{1}{2} \text{tr}(p_1 - p_2) = \frac{1}{2} |\vec{S}_1 - \vec{S}_2| \leftarrow \begin{array}{l} \text{half the} \\ \text{Euclidean distance} \\ \text{on Bloch ball} \end{array} \quad (9)$$

③ Classical distance: $[p_1, p_2] = 0 \Rightarrow$

$$p_1 = \sum_j p_j |e_j\rangle\langle e_j|$$

$$p_2 = \sum_j q_j |e_j\rangle\langle e_j|$$

$$\text{tr}(p_1 - p_2) = \sum_j |p_j - q_j|$$

$$D(p_1, p_2) = \frac{1}{2} \text{tr}(p_1 - p_2) = \frac{1}{2} \sum_j |p_j - q_j| = D(\vec{p}, \vec{q})$$

$$= \sum_{p_j - q_j > 0} |p_j - q_j| = \sum_{p_j - q_j < 0} |p_j - q_j|$$

$$= \max_S [p(S) - q(S)]$$

④ Invariance under unitaries:

$$U(p_1 - p_2)U^\dagger = \sum_j |\lambda_j| U|e_j\rangle\langle e_j|U^\dagger$$

$$|U(p_1 - p_2)U^\dagger| = \sum_j |\lambda_j| U|e_j\rangle\langle e_j|U^\dagger$$

$$= U\left(\sum_j |\lambda_j| |e_j\rangle\langle e_j|\right)U^\dagger$$

$$= U|p_1 - p_2|U^\dagger$$

$$D(U p_1 U^\dagger, U p_2 U^\dagger) = \frac{1}{2} \text{tr}|U(p_1 - p_2)U^\dagger| = \frac{1}{2} \text{tr}(p_1 - p_2) = D(p_1, p_2)$$

⑤ Trace distance is a distance:

① $D(p_1, p_2) \geq 0$, equality iff $p_1 = p_2$.

② $D(p_1, p_2) = D(p_2, p_1)$

③ Triangle inequality: $D(p, \sigma) \leq D(p, \tau) + D(\tau, \sigma)$

$$D(p, \sigma) = \text{tr}(P_+(p - \sigma)) = \underbrace{\text{tr}(P_+(p - \tau))}_{\leq D(p, \tau)} + \underbrace{\text{tr}(P_+(\tau - \sigma))}_{\leq D(\tau, \sigma)} \leq D(p, \tau) + D(\tau, \sigma)$$

(10)

⑥ $D(\rho_1, \rho_2) \leq 1$ iff $\rho_1 \perp \rho_2$, i.e., $\text{tr}(\rho_1 \rho_2) = 0 \iff \rho_1 \rho_2 = 0$

$$D(\rho_1, \rho_2) = \text{tr}(P_+(\rho_1 - \rho_2)) \leq \text{tr}(P_+ \rho_1) \leq 1$$

iff support of ρ_1 lies in subspace of P_+ .

iff $\text{tr}(\rho_2 P_+) = 0$, i.e., $\rho_2 \perp P_+$

⑦ Relation to measurements:

$$D(\rho, \sigma) = \max_{\{E_\alpha\}} D(\vec{\rho}, \vec{\sigma})$$

$\uparrow \quad \nwarrow$
 $P_\alpha = \text{tr}(\rho E_\alpha) \quad g_\alpha = \text{tr}(\sigma E_\alpha)$

maximum achieved by POVMs such that P_+ is the sum of some subset of POVM elements, and P_- is the sum of the remaining POVM elements.

$$D(\vec{\rho}, \vec{\sigma}) = \frac{1}{2} \sum_{\alpha} |\text{tr}(\rho E_\alpha) - \text{tr}(\sigma E_\alpha)|$$

$$= \sum_{P_\alpha - g_\alpha \geq 0} \text{tr}((\rho - \sigma) E_\alpha)$$

$$= \text{tr}\left((\rho - \sigma) \sum_{P_\alpha - g_\alpha \geq 0} E_\alpha\right)$$

$$\leq \max_E \text{tr}((\rho - \sigma) E) = \text{tr}((\rho - \sigma) P_+)$$

$$= D(\rho, \sigma)$$

⑧ Marginalization: $D(\rho_A, \sigma_A) \leq D(\rho_{AB}, \sigma_{AB})$

$$D(\rho_A, \sigma_A) = \text{tr}(P_+(\rho_A - \sigma_A)) = \text{tr}(P_+ \otimes I_B (\rho_{AB} - \sigma_{AB})) \leq D(\rho_{AB}, \sigma_{AB})$$

$$+ \text{tr}(P_+ \otimes I_B (\rho_{AB} - \sigma_{AB})) = \frac{1}{2} \text{tr}(|\rho_{AB} - \sigma_{AB}|)$$

⑨ Contractivity: $D(R(\rho), R(\sigma)) \leq D(\rho, \sigma)$, R trace-preserving quantum operation

Measurement model

$$R(\rho) = \text{tr}_A(U \rho \otimes |0\rangle\langle 0| U^\dagger)$$

$$R(\sigma) = \text{tr}_A(U \sigma \otimes |0\rangle\langle 0| U^\dagger)$$

$$D(R(\rho), R(\sigma)) \leq D(U \rho \otimes |0\rangle\langle 0| U^\dagger, U \sigma \otimes |0\rangle\langle 0| U^\dagger)$$

$$= D(\rho \otimes |0\rangle\langle 0|, \sigma \otimes |0\rangle\langle 0|)$$

$$= D(\rho, \sigma)$$

$$R(\rho) - R(\sigma) = R(\rho - \sigma)$$

Let Π_+ project onto the positive eigensubspace of $R(\rho - \sigma)$

$$D(R(\rho), R(\sigma)) = \text{tr}(\Pi_+ R(\rho - \sigma))$$

$$R(\rho - \sigma) = R(P_+(\rho - \sigma) P_+)$$

$$= R(P_+ (\rho - \sigma) P_+)$$

These are positive operators because R maps positive operators to positive operators

$$D(R(\rho), R(\sigma)) = \text{tr}(\Pi_+ R(\rho - \sigma))$$

$$\leq \text{tr}(\Pi_+ R(P_+(\rho - \sigma) P_+))$$

$$\leq \text{tr}(R(P_+(\rho - \sigma) P_+))$$

$$= \text{tr}(P_+(\rho - \sigma))$$

$$= D(\rho, \sigma)$$

⑩ Convexity properties

Strong convexity: $D(\sum_i P_i \rho_i, \sum_i P_i \sigma_i) \leq D(\vec{\rho}, \vec{\sigma}) + \sum_i P_i D(\rho_i, \sigma_i)$

$$D(\sum_i P_i \rho_i, \sum_i P_i \sigma_i) = \text{tr}(P_+(\sum_i P_i \rho_i - \sum_i P_i \sigma_i))$$

$$= \sum_i P_i \underbrace{\text{tr}(P_+(\rho_i - \sigma_i))}_{\leq D(\rho_i, \sigma_i)} + \sum_i (P_i - P_i) \underbrace{\text{tr}(P_+ \sigma_i)}_{\leq \sum_{P_i - P_i \geq 0} (P_i - P_i)}$$

$$\leq D(\vec{\rho}, \vec{\sigma}) + \sum_i P_i D(\rho_i, \sigma_i) = D(\vec{\rho}, \vec{\sigma})$$

Double convexity: $D(\sum_j p_j \rho_j, \sum_j p_j \sigma_j) \leq \sum_j p_j D(\rho_j, \sigma_j)$

Convexity: $D(\sum_j p_j \rho_j, \sigma) \leq \sum_j p_j D(\rho_j, \sigma)$

Fidelity

Properties we want:

- ① $F(|\psi_1\rangle, |\psi_2\rangle) = |\langle \psi_1 | \psi_2 \rangle| = \cos \theta$ Hilbert space angle
↓
To square or not to square?
- ② $F(\rho, |\psi\rangle) = \sqrt{\langle \psi | \rho | \psi \rangle} = \left(\text{probability to find } |\psi\rangle \text{ in an ODP measurement on } \rho \right)^{1/2}$
- ③ $F(\rho, \rho) = 1$

Guesses:

① $F(\rho, \rho_2) = \sqrt{\text{tr}(\rho \rho_2)}$, but then $F(\rho, \rho) = \sqrt{\text{tr}(\rho^2)} = (\text{purity})^{1/2} < 1$ except for pure states.

② We might think $\sqrt{\rho}$ is analogous to an amplitude, and so guess $F(\rho, \rho_2) = \text{tr}(\sqrt{\rho} \sqrt{\rho_2})$, but this gives $F(\rho, |\psi\rangle) = \langle \psi | \sqrt{\rho} | \psi \rangle$.

Definition: Let $|p\rangle$ denote a purification of ρ , i.e., $\text{tr}_R(|p\rangle\langle p|) = \rho$. Define the fidelity as

$$F(\rho, \rho_2) \equiv \max_{|p_1\rangle, |p_2\rangle} |\langle p_1 | p_2 \rangle|$$

We put this in a different form using the following:

① $|\text{tr}(GU)| \leq \text{tr} G$ iff U is $\pm$ identity on the support of G
 $G \geq 0$
 U unitary

$$|\text{tr}(GU)| = \left| \sum_j G_j \langle e_j | U | e_j \rangle \right| \leq \sum_j G_j |\langle e_j | U | e_j \rangle| \leq \sum_j G_j = \text{tr}(G)$$

iff $\langle e_j | U | e_j \rangle \geq 0$ or $\leq 0 \quad \forall G_j > 0$ iff $|\langle e_j | U | e_j \rangle| = 1 \quad \forall G_j > 0$

$$\textcircled{2} \quad |tr(Au)| \leq tr(\sqrt{A^T A}) = tr(\sqrt{AA^T})$$

$$A = V \sqrt{A^T A} = \sqrt{A A^T} V \quad \left(\begin{array}{l} \text{polar decomposition} \\ \text{if } A \text{ is invertible, } V = A(A^T A)^{-1/2} = (A A^T)^{-1/2} A \end{array} \right)$$

$$|tr(Au)| = |tr(\sqrt{A^T A} u)| = |tr(\sqrt{A^T A} u v)| \leq tr(\sqrt{A^T A})$$

↔ $U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} A^T = \frac{1}{2} A^T (A A^T)^{-1/2}$

Purifications: $\rho = \sum_j \lambda_j |e_j\rangle\langle e_j|$; any purification has the

form $|p\rangle = \sum_j \sqrt{\lambda_j} |f_j\rangle \otimes |g_j\rangle = \sum_j |f_j\rangle \otimes \sqrt{\lambda_j} |g_j\rangle$. The freedom is

$\uparrow$
 orthonormal basis in reference system R

$|p\rangle$ is that we can use any reference-system basis $|f\rangle$, i.e., we can multiply $|p\rangle$ by any reference-system unitary U .

There is another, more useful way to write an arbitrary purification. Let $|f_j\rangle$ and $|e_j\rangle$ be arbitrary orthonormal bases. Then

$$|\Phi_{\sqrt{p}}\rangle \xrightarrow{\text{vec'ing}} I \otimes \sqrt{p} |\Phi\rangle = \sum_j |f_j\rangle \otimes \sqrt{p} |e_j\rangle$$

is a purification, i.e., $\text{tr}_R(|\Phi_p\rangle\langle\Phi_p|) = \sum_j |\sqrt{p}|e_j\rangle\langle e_j|\sqrt{p}| = \rho$.

The apparent freedom in $|\Phi_{\sqrt{\rho}}\rangle$ is to choose both bases arbitrarily, i.e., to multiply both by unitaries. So an arbitrary purification can be written as

$$|p\rangle = U \otimes \sqrt{p} V |\Phi\rangle = (I \otimes \sqrt{p}) \underbrace{(U \otimes V)} |\Phi\rangle,$$

Special
property
of $|\Psi\rangle$

$$\rightarrow \mathbb{I} \otimes \sqrt{u}^T |\Psi\rangle = u \sqrt{v}^T \otimes \mathbb{I} |\Psi\rangle$$

which means we can get arbitrary purifications using either of the two unitaries. So an arbitrary purification can be written as

$$|p\rangle = U \otimes I |\Phi_{\sqrt{p}}\rangle = U \otimes \sqrt{p} |\Phi\rangle = I \otimes \sqrt{p} U^T |\Phi\rangle.$$

Now we're set.

$$\max_{|p\rangle, |\sigma\rangle} |\langle p | \sigma \rangle| = \max_{U, V} |\langle \Phi | (I \otimes U^T \sqrt{p}) (I \otimes \sqrt{\sigma} V) | \Phi \rangle|$$

$$= \max_{U, V} \left| \sum_j \langle e_j | U^T \sqrt{p} \sqrt{\sigma} V | e_j \rangle \right|$$

$$= \max_{U, V} |\text{tr}(U^T \sqrt{p} \sqrt{\sigma} V)|$$

$$= \max_{U, V} |\text{tr}(\sqrt{p} \sqrt{\sigma} V U^T)|$$

We only need to maximize over one unitary, which means we only need to maximize over one purification, either $|p\rangle$ or $|\sigma\rangle$.

$$\rightarrow = \max_U |\text{tr}(\sqrt{p} \sqrt{\sigma} U)|$$

$$= \text{tr}(\sqrt{\sqrt{p} \sigma \sqrt{p}})$$

If p and σ are invertible, the U that achieves the maximum is

$$U = \pm (\sqrt{\sigma} p \sqrt{\sigma})^{-1/2} \sqrt{\sigma} \sqrt{p} \\ = \pm \sqrt{\sigma} \sqrt{p} (\sqrt{p} \sigma \sqrt{p})^{-1/2}$$

So we have

$$F(p_1, p_2) = \max_U |\text{tr}(\sqrt{p_1} \sqrt{p_2} U)|$$

$$= \text{tr}(\sqrt{\sqrt{p_1} p_2 \sqrt{p_1}})$$

$$= \text{tr}(\sqrt{\sqrt{p_2} p_1 \sqrt{p_2}}) = F(p_2, p_1)$$

Properties of fidelity

$$\begin{aligned} \sqrt{\rho_1} &= |\psi\rangle\langle\psi| \\ \sqrt{\rho_1} \rho_2 \sqrt{\rho_1} &= |\psi\rangle\langle\psi| \rho_2 |\psi\rangle\langle\psi| \\ \textcircled{1} \quad \rho_1 &= |\psi\rangle\langle\psi|, \quad \rho_2 = \rho: \quad \sqrt{\rho_1} \rho_2 \sqrt{\rho_1} = |\psi\rangle\langle\psi| \sqrt{\langle\psi| \rho_2 |\psi\rangle} \end{aligned}$$

$$F(|\psi\rangle\langle\psi|, \rho) = \sqrt{\langle\psi| \rho_2 |\psi\rangle}$$

$$\textcircled{2} \text{ Qubits: } \rho_j = \frac{1}{2} (I + \vec{\sigma} \cdot \vec{S}_j)$$

$$F(\rho_1, \rho_2) = \frac{1}{\sqrt{2}} \left(1 + \underbrace{\vec{S}_1 \cdot \vec{S}_2}_{\vec{N}_1 \cdot \vec{N}_2} + \sqrt{(1-S_1^2)(1-S_2^2)} \right)^{1/2}$$

$$\vec{N}_j = \vec{S}_j + \sqrt{1-S_j^2} \vec{e}_0$$

Unit vector on northern hemisphere of a sphere in 4 dimensions

$$= \cos \left(\underbrace{\frac{1}{2} \cos^{-1}(\vec{N}_1 \cdot \vec{N}_2)}_{\theta} \right)$$

$$\textcircled{3} \text{ Classical fidelity: } [\rho_1, \rho_2] = 0$$

$$\rho_1 = \sum_j p_j |e_j\rangle\langle e_j|$$

$$\rho_2 = \sum_j q_j |e_j\rangle\langle e_j|$$

$$F(\rho_1, \rho_2) = \text{tr}(\sqrt{\rho_1 \rho_2}) = \sum_j \sqrt{p_j q_j} = F(\vec{p}, \vec{q}) = \cos \left(\begin{array}{l} \text{Bhattacharyya-} \\ \text{Womers} \\ \text{distance} \end{array} \right)$$

$$\textcircled{4} \text{ Invariance under unitaries:}$$

$$F(U \rho_1 U^\dagger, U \rho_2 U^\dagger) = F(\rho_1, \rho_2)$$

$$\textcircled{5} \quad 0 \leq F(\rho_1, \rho_2) \leq 1$$

$\left\{ \begin{array}{l} \text{iff } \rho_1 \perp \rho_2 \\ \text{iff } \rho_1 = \rho_2 \end{array} \right.$

⑥ Fidelity is not a distance, but $\Theta(\rho, \rho_2) = \cos^{-1} F(\rho, \rho_2)$ is a Riemannian distance that generalizes Hilbert-space angle to mixed states

① $\Theta(\rho, \rho_2) \geq 0$, with equality iff $\rho = \rho_2$

② $\Theta(\rho, \rho_2) = \Theta(\rho_2, \rho)$

③ Triangle inequality: $\Theta(\rho, \sigma) \leq \Theta(\rho, \tau) + \Theta(\tau, \sigma)$.

Let $|\rho\rangle$, $|\tau\rangle$, and $|\sigma\rangle$ be such that $F(\rho, \tau) = |\langle \rho | \tau \rangle|$ and $F(\sigma, \tau) = |\langle \sigma | \tau \rangle|$. Adjust the overall phases of $|\rho\rangle$, $|\tau\rangle$, and $|\sigma\rangle$ so that $\langle \rho | \sigma \rangle$, $\langle \rho | \tau \rangle$, and $\langle \sigma | \tau \rangle$ are real and positive. Now we're effectively in a real vector space, where it is obvious that

$$\cos^{-1} \langle \rho | \sigma \rangle \leq \cos^{-1} \langle \rho | \tau \rangle + \cos^{-1} \langle \tau | \sigma \rangle = \Theta(\rho, \tau) + \Theta(\tau, \sigma).$$

But we have $F(\rho, \sigma) \geq \langle \rho | \sigma \rangle$, which implies

$$\Theta(\rho, \sigma) \leq \cos^{-1} \langle \rho | \sigma \rangle \leq \Theta(\rho, \tau) + \Theta(\tau, \sigma).$$

⑦ Relation to measurements: $F(\rho, \sigma) = \min_{\{E_a\}} F(\vec{p}, \vec{q})$

$\uparrow \quad \uparrow$
 $\vec{p} = \text{tr}(\rho E_a) \quad \vec{q} = \text{tr}(\sigma E_a)$

if ρ, σ invertible,

$$F(\rho, \sigma) = \text{tr}(\sqrt{\rho} \sqrt{\sigma} U) \quad U = \sqrt{\sigma} \sqrt{\rho} (\sqrt{\rho} \sigma \sqrt{\rho})^{-1/2} = (\sqrt{\sigma} \rho \sqrt{\sigma})^{-1/2} \sqrt{\sigma} \sqrt{\rho}$$

$$= \sum_a \text{tr}(\sqrt{\rho} E_a \sqrt{\sigma} U) \leftarrow \text{completeness of POVM}$$

$$= \sum_a \text{tr}((\sqrt{E_a} \sqrt{\rho})^\dagger \sqrt{E_a} \sqrt{\sigma} U)$$

$$\leq \sum_{\alpha} \text{tr}((\sqrt{E_{\alpha}} \sqrt{\rho})^{\dagger} \sqrt{E_{\alpha}} \sqrt{\rho})^{1/2} \text{tr}((\sqrt{E_{\alpha}} \sqrt{\sigma} u)^{\dagger} (\sqrt{E_{\alpha}} \sqrt{\sigma} u))^{1/2}$$

Schwarz inequality for operator inner product

iff $\sqrt{E_{\alpha}} \sqrt{\rho} = \lambda_{\alpha} \sqrt{E_{\alpha}} \sqrt{\sigma} u$

$$= \sum_{\alpha} \sqrt{\text{tr}(\rho E_{\alpha}) \text{tr}(\sigma E_{\alpha})}$$

$$= F(\vec{p}, \vec{q})$$

We get equality by choosing a POVM that satisfies

$$\sqrt{E_{\alpha}} \sqrt{\rho} = \lambda_{\alpha} \sqrt{E_{\alpha}} \sqrt{\sigma} u$$

Assuming ρ and σ are invertible (noninvertible ρ and σ follow by continuous limits to the boundary), we have

$$\begin{aligned} \sqrt{E_{\alpha}} &= \lambda_{\alpha} \sqrt{E_{\alpha}} \sqrt{\sigma} u \rho^{-1/2} = \lambda_{\alpha} \sqrt{E_{\alpha}} \sqrt{\sigma} (\sqrt{\sigma} \rho \sqrt{\sigma})^{-1/2} \sqrt{\sigma} \\ &= \sum_j u_j |e_j\rangle \langle e_j| \end{aligned}$$

spectral decomposition of Hermitian operator

Choose $E_j = |e_j\rangle \langle e_j|$ (projection-valued measurement),

and it works with $\lambda_j = u_j^{-1}$.

⑧ Marginalization: $F(\rho_A, \sigma_A) \geq F(\rho_{AB}, \sigma_{AB})$

Let $|p\rangle, |\sigma\rangle$ be optimal purifications of ρ_{AB}, σ_{AB} , i.e.,

$F(\rho_{AB}, \sigma_{AB}) = \langle p | \sigma \rangle$. Since they are also purifications of ρ_A and σ_A , we have

$$F(\rho_A, \sigma_A) \geq \langle p | \sigma \rangle = F(\rho_{AB}, \sigma_{AB}).$$

For probabilities, this is the property

$$\sum_j \sqrt{p_j q_j} \geq \sum_{j,k} \sqrt{p_{jk} q_{jk}},$$

which is a trivial consequence of

$$\sum_k \sqrt{p_{jk} q_{jk}} = \sqrt{p_j q_j} \sum_k \sqrt{p_{jk} q_{jk} / (p_j q_j)} \leq \sqrt{p_j q_j} \leq 1$$

⑨ Monotonicity: $F(Q(\rho), Q(\sigma)) \geq F(\rho, \sigma)$.

$$F(Q(\rho), Q(\sigma)) = F(\text{tr}_A(u \rho u^\dagger), \text{tr}_A(u \sigma u^\dagger))$$

$$\geq F(u \rho u^\dagger, u \sigma u^\dagger)$$

$$= F(\rho \otimes |0\rangle\langle 0|, \sigma \otimes |0\rangle\langle 0|)$$

$$= F(\rho, \sigma)$$

⑩ Convexity properties

Strong concavity: $F(\sum_j p_j \rho_j, \sum_j q_j \sigma_j) \geq \sum_j \sqrt{p_j q_j} F(\rho_j, \sigma_j)$

Let $|p_j\rangle$ and $|\sigma_j\rangle$ be optimal purifications of ρ_j and σ_j , i.e.,

$F(\rho_j, \sigma_j) = \langle p_j | \sigma_j \rangle$. Define

$$|p\rangle = \sum_j \sqrt{p_j} |p_j\rangle \otimes |j\rangle,$$

$$|\sigma\rangle = \sum_j \sqrt{q_j} |\sigma_j\rangle \otimes |j\rangle,$$

which are purifications of

$$\rho = \sum_j p_j \rho_j \text{ and } \sigma = \sum_j q_j \sigma_j,$$

so

First show strong concavity for probabilities: $\vec{p}_j$ and $\vec{q}_j$ are vectors of conditional probabilities, p_{kj} and q_{kj} , with joint probabilities $p_{kj} = p_{kj} p_j$ and $q_{kj} = q_{kj} q_j$ and unconditioned probabilities $p_k = \sum_j p_{kj} p_j$ and $q_k = \sum_j q_{kj} q_j$.

$$F(\sum_j p_j \vec{p}_j, \sum_j q_j \vec{q}_j) = \sum_k \sqrt{p_k q_k}$$

$$F(p, \sigma) \geq |\langle p | \sigma \rangle|$$

$$= \left| \sum_j \sqrt{p_j q_j} \langle p_j | \sigma_j \rangle \right|$$

$$= \sum_j \sqrt{p_j q_j} F(p_j, \sigma_j)$$

$$\sum_j \sqrt{p_j q_j} F(p_j, \sigma_j) = \sum_{j \in \mathcal{A}} \sqrt{p_j q_j} \sqrt{q_j} \sqrt{p_j} \quad (19)$$

Thus strong concavity for probabilities is just the marginalization property.

Now assume E_d is an optimal POVM for $p = \sum_j p_j p_j$ and $\sigma = \sum_j q_j \sigma_j$, i.e.,

$$F(p, \sigma) = F(\vec{p}, \vec{q}) \quad \begin{aligned} p_d &= \text{tr}(\sigma E_d) \\ &= \sum_j q_j \underbrace{\text{tr}(\sigma_j E_d)}_{p_{d|j}} \end{aligned}$$

$$p_d = \text{tr}(p E_d) = \sum_j p_j \underbrace{\text{tr}(p_j E_d)}_{p_{d|j}}$$

We have

$$F(p, \sigma) = F(\vec{p}, \vec{q})$$

$$\geq \sum_j \sqrt{p_j q_j} \underbrace{F(p_j, \sigma_j)}_{\geq F(p_j, \sigma_j)}$$

$$\geq F(p_j, \sigma_j)$$

$$\geq \sum_j \sqrt{p_j q_j} F(p_j, \sigma_j)$$

Double concavity: $F(\sum_j p_j p_j, \sum_j p_j \sigma_j) \geq \sum_j p_j F(p_j, \sigma_j)$

Concavity: $F(\sum_j p_j p_j, \sigma) \geq \sum_j p_j F(p_j, \sigma)$

Relations between trace distance and fidelity:

① Pure states: $D(|\psi\rangle, |\phi\rangle) = \sqrt{1 - |\langle\psi|\phi\rangle|^2} = \sqrt{1 - F^2(|\psi\rangle, |\phi\rangle)}$

② Mixed states:

③ Let $|\rho\rangle, |\sigma\rangle$ be purifications such that $F(\rho, \sigma) = \langle\rho|\sigma\rangle$

$$D(\rho, \sigma) \leq D(|\rho\rangle, |\sigma\rangle) = \sqrt{1 - |\langle\rho|\sigma\rangle|^2} = \sqrt{1 - F^2(\rho, \sigma)}$$

↑
marginalization

④ Let E_a be a POVM such that

$$F(\rho, \sigma) = F(\vec{p}, \vec{q}) = \sum_a \sqrt{p_a q_a}$$

We know that

$$\begin{aligned} D(\rho, \sigma) &\geq \sum_a D(\vec{p}, \vec{q}) = \frac{1}{2} \sum_a |p_a - q_a| \\ &= \frac{1}{2} \sum_a |\sqrt{p_a} - \sqrt{q_a}| |\sqrt{p_a} + \sqrt{q_a}| \\ &\geq \frac{1}{2} \sum_a (\sqrt{p_a} - \sqrt{q_a})^2 \quad \rightarrow \geq |\sqrt{p_a} - \sqrt{q_a}| \\ &= \frac{1}{2} \sum_a p_a + q_a - 2\sqrt{p_a q_a} \\ &= 1 - F(\vec{p}, \vec{q}) \\ &= 1 - F(\rho, \sigma) \end{aligned}$$

$$1 - F(\rho, \sigma) \leq D(\rho, \sigma) \leq \sqrt{1 - F^2(\rho, \sigma)}$$

Entanglement fidelity: (A)

State ρ ; operation $\mathcal{Q} = \sum_{\alpha} A_{\alpha} \otimes A_{\alpha}^{\dagger}$ arbitrary Kraus decomposition

Purify ρ to $|\Phi_{RA}\rangle$.

$$F_e(\rho, \mathcal{Q}) = F^2(|\Phi_{RA}\rangle, \mathcal{Q} \otimes \mathcal{Q}(|\Phi_{RA}\rangle\langle\Phi_{RA}|))$$

Squared fidelity for transmitting purification through operation $\mathcal{Q}$

$$= \langle\Phi_{RA}| \mathcal{Q} \otimes \mathcal{Q}(|\Phi_{RA}\rangle\langle\Phi_{RA}|) |\Phi_{RA}\rangle$$

$$= \sum_{\alpha} |\langle\Phi_{RA}| I \otimes A_{\alpha} |\Phi_{RA}\rangle|^2$$

$$= \sum_{\alpha} |\text{tr}(\rho A_{\alpha})|^2$$

$$= (\rho | \mathcal{Q} | \rho) \quad \leftarrow \text{independent of Kraus decomposition and purification}$$

Suppose $|\Phi_{RA}\rangle = |\Phi_{\sqrt{\rho}}\rangle = I \otimes \sqrt{\rho} |\Psi\rangle$

Then

$$F_e(\rho, \mathcal{Q}) = \langle\Phi_{\sqrt{\rho}}| \mathcal{Q} \otimes \mathcal{Q} \cdot \sqrt{\rho} \otimes \sqrt{\rho} (|\Psi\rangle\langle\Psi|) |\Phi_{\sqrt{\rho}}\rangle$$

$$= \langle\sqrt{\rho}| \mathcal{Q} \cdot \sqrt{\rho} \otimes \sqrt{\rho} |\sqrt{\rho}\rangle$$

$$= \sum_{\alpha} |\text{tr}(\sqrt{\rho} A_{\alpha} \sqrt{\rho})|^2$$

$$= \sum_{\alpha} |\text{tr}(\rho A_{\alpha})|^2$$

$$= (\rho | \mathcal{Q} | \rho)$$

Properties of entanglement fidelity

$$\begin{aligned} \textcircled{1} \quad \rho = |\psi\rangle\langle\psi| = \pi \quad \text{pure: } F_e(\pi, \mathcal{A}) &= (\pi | \mathcal{A} | \pi) \\ &= \langle\psi | \mathcal{A}(\pi) | \psi\rangle \\ &= F^2(\pi, \mathcal{A}(\pi)) \end{aligned}$$

$$\textcircled{2} \quad \text{Convexity of } F_e: F_e(\rho, \mathcal{A}) \leq \sum_j p_j F_e(\rho_j, \mathcal{A})$$

Proof is simply the proof for x^2 .

$$\begin{aligned} \sum_j p_j F_e(\rho_j, \mathcal{A}) &= \sum_j p_j (\rho_j | \mathcal{A} | \rho_j) \\ &= \sum_j p_j (\rho + \Delta \rho_j | \mathcal{A} | \rho + \Delta \rho_j) \\ &= (\rho | \mathcal{A} | \rho) + \sum_j p_j (\Delta \rho_j | \mathcal{A} | \Delta \rho_j) \\ &\quad + 2\mathcal{R}\left(\underbrace{\sum_j p_j (\rho | \mathcal{A} | \Delta \rho_j)}_{(\rho | \mathcal{A} | \sum_j p_j \Delta \rho_j) = 0}\right) \\ &\geq (\rho | \mathcal{A} | \rho) \\ &= F_e(\rho, \mathcal{A}) \end{aligned}$$

$$p_j = |\psi_j\rangle\langle\psi_j| = \pi_j \quad \text{pure: } F_e(\rho, \mathcal{A}) \leq \sum_j p_j F^2(\pi_j, \mathcal{A}(\pi_j))$$

$$\textcircled{8} \quad p = \sum_j p_j \pi_j : F_e(p, Q) \leq \left(\sum_j p_j F(\pi_j, Q(\pi_j)) \right)^2 = \overline{F^2} \quad \textcircled{c}$$

$$\leq \sum_j p_j F^2(\pi_j, Q(\pi_j)) = \overline{F^2}$$

$$F_e(p, Q) = (p | Q | p) = \sum_{j,k} p_j p_k (\pi_j | Q | \pi_k)$$

$$\leq \sum_{j,k} p_j p_k \underbrace{|(\pi_j | Q | \pi_k)|}_{\leq (\pi_j | Q | \pi_j)^{1/2} (\pi_k | Q | \pi_k)^{1/2}}$$

$$= \left(\sum_j p_j \underbrace{(\pi_j | Q | \pi_j)^{1/2}}_{F(\pi_j, Q(\pi_j))} \right)^2$$

$$= \left(\sum_j p_j F(\pi_j, Q(\pi_j)) \right)^2$$