Homework Problem 3.1

Ensemble decompositions for qubit states. For qubits the convex set of density operators can be represented on the Bloch sphere. An ensemble decomposition of a density operator,

$$\rho = \sum_{\mathbf{n}} p_{\mathbf{n}} |\mathbf{n}\rangle\langle\mathbf{n}| = \frac{1}{2}\left[1 + \left(\sum_{\mathbf{n}} p_{\mathbf{n}}\mathbf{n}\right)\cdot\boldsymbol{\sigma}\right] ,$$

is a way of writing the Bloch vector,

$$\mathbf{S} = \sum_{\mathbf{n}} p_{\mathbf{n}}\mathbf{n} ,$$

as a convex combination (average) of unit vectors.

Consider the state

$$\rho = \frac{3}{4}|\mathbf{e}_z\rangle\langle\mathbf{e}_z| + \frac{1}{4}|-\mathbf{e}_z\rangle\langle-\mathbf{e}_z| = \frac{1}{2}\left(1 + \frac{1}{2}\sigma_z\right) ,$$

which has Bloch vector $\mathbf{S} = \frac{1}{2}\mathbf{e}_z$.

(a) *Give* ensemble decompositions for ρ in which all the pure states in the decomposition are selected from those with $\langle\sigma_z\rangle = 1/2$: (i) two such states; (ii) three such states; (iii) all such states.

(b) *Give* an ensemble decomposition for ρ that includes all pure states, i.e., is an integral over the surface of the Bloch sphere.