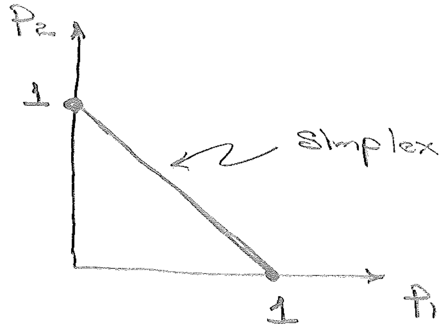
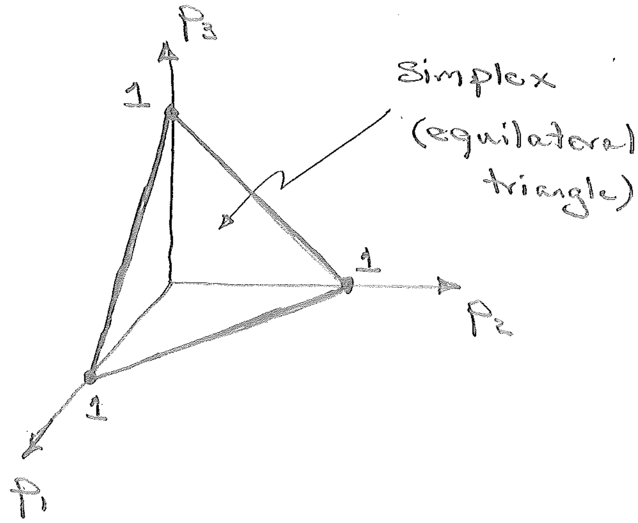
1.3. Probability simplex: $1 = \sum_{j=1}^{D} p_j$ ← hyperplane in $\mathbb{R}^D$

$p_j \geq 0, \; j = 1, \ldots, D$ ← restricts to part of hyperplane that is in the all positive 2^D-ant.

(a) D = 2

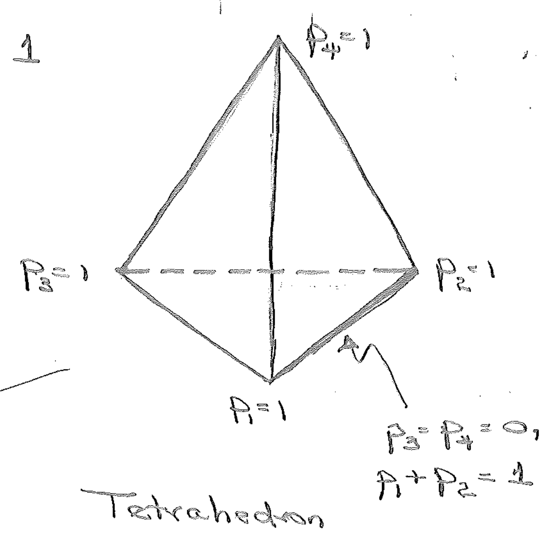


D = 3



D = 4

$p_4 = 0$
$p_1 + p_2 + p_3 = 1$



Tetrahedron

Regular polyhedra with edges of "length" $\sqrt{2}$. The boundaries are probability simplexes for all numbers of alternatives between 2 and D-1, ranging from the edges (2 alternatives) to the faces (D-1 alternatives

Vertices: one unit probability, rest zero
Edges: two probabilities zero, simplex for other two
Faces: one probability zero, simplex for other three

(b) $q_j = p_j + \delta p_j$

$$1 = \sum_j q_j = \underbrace{\sum_j p_j}_{1} + \sum_j \delta p_j \Longrightarrow \sum_j \delta p_j = 0$$

Relative information:

$$H(\vec{q}\,\|\,\vec{p}) = \sum_j q_j \log \frac{q_j}{p_j} = -H(\vec{q}) - \sum_j q_j \log p_j$$

Expand: $$H(\vec{q}\,\|\,\vec{p}) = \sum_j (p_j + \delta p_j) \underbrace{\log\left(1 + \frac{\delta p_j}{p_j}\right)}$$

$$= \frac{\ln(1 + \delta p_j / p_j)}{\ln 2}$$

$$= \frac{1}{\ln 2}\left(\frac{\delta p_j}{p_j} - \frac{1}{2}\left(\frac{\delta p_j}{p_j}\right)^2 + \cdots\right)$$

$$= \frac{1}{\ln 2} \sum_j \delta p_j - \frac{1}{2}\frac{\delta p_j^2}{p_j} + \frac{\delta p_j^2}{p_j} + \cdots$$

$$\boxed{H(\vec{q}\,\|\,\vec{p}) = \frac{1}{\ln 2}\sum_j \frac{\delta p_j^2}{2 p_j} = \frac{2}{\ln 2}\sum_j \frac{\delta p_j^2}{4 p_j}}$$

(c) $r_j = \sqrt{p_j}$

Probability simplex: $\sum_j p_j = \sum_j r_j^2 = 1 \longleftarrow$ sphere of unit radius

$p_j \geq 0 \Rightarrow r_j \geq 0 \longleftarrow$ restricts to portion of sphere in all positive 2^D-ant

$$dr_j = \frac{dp_j}{2\sqrt{p_j}} \Longrightarrow \boxed{\sum_j \frac{dp_j^2}{4 p_j} = \sum_j dr_j^2}$$

$\longleftarrow$ The metric on the sphere is the standard round metric induced by the flat metric on Euclidean space.