2.3. $U_R = e^{-i\vec{n}\cdot\vec{\sigma}\,\theta/2}$

(a) $\vec{n}\cdot\vec{\sigma} = |\vec{n}\rangle\langle\vec{n}| - |-\vec{n}\rangle\langle-\vec{n}|$

$$\boxed{U_R = e^{-i\theta/2}|\vec{n}\rangle\langle\vec{n}| + e^{i\theta/2}|-\vec{n}\rangle\langle-\vec{n}|}$$

Eigenvalues: $e^{-i\theta/2}$, $e^{+i\theta/2}$

Eigenvectors: $|\vec{n}\rangle$ $|-\vec{n}\rangle$

(b) Method 1: Use $|\vec{n}\rangle\langle\vec{n}| = \frac{1}{2}(1+\vec{n}\cdot\vec{\sigma})$

$$U_R = e^{-i\theta/2}\frac{1}{2}(1+\vec{n}\cdot\vec{\sigma}) + e^{+i\theta/2}\frac{1}{2}(1-\vec{n}\cdot\vec{\sigma})$$

$$= \underbrace{\frac{1}{2}\left(e^{-i\theta/2}+e^{i\theta/2}\right)}_{\cos(\theta/2)} + \vec{n}\cdot\vec{\sigma}\underbrace{\frac{1}{2}\left(e^{-i\theta/2}-e^{i\theta/2}\right)}_{-i\sin(\theta/2)}$$

$$\boxed{U_R = 1\cos(\theta/2) - i\vec{n}\cdot\vec{\sigma}\sin(\theta/2)}$$

Method 2 (traditional):

$$\boxed{\begin{aligned}&(\vec{n}\cdot\vec{\sigma})^2 = 1\\ &(\vec{n}\cdot\vec{\sigma})^k = \begin{cases}1, & k \text{ even}\\ \vec{n}\cdot\vec{\sigma}, & k \text{ odd}\end{cases}\end{aligned}}$$

$$U_R = \sum_{k=0}^{\infty}\frac{1}{k!}\left(-i\vec{n}\cdot\vec{\sigma}\,\theta/2\right)^k$$

$$= \sum_{k\text{ even}}\frac{1}{k!}(-i)^k(\theta/2)^k + \sum_{k\text{ odd}}\frac{1}{k!}(-i)^k(\vec{n}\cdot\vec{\sigma})(\theta/2)^k$$

$$= \underbrace{\sum_{l=0}^{\infty}\frac{(-1)^l(\theta/2)^{2l}}{(2l)!}}_{\cos(\theta/2)} - i\vec{n}\cdot\vec{\sigma}\underbrace{\sum_{l=0}^{\infty}\frac{(-1)^l(\theta/2)^{2l+1}}{(2l+1)!}}_{\sin(\theta/2)}$$

$k=2l$ $\qquad$ $k=2l+1$

$$\boxed{U_R = 1\cos(\theta/2) - i\vec{n}\cdot\vec{\sigma}\sin(\theta/2)}$$

(c) See the lecture notes for a picture of why $R_{\vec{n}}(\theta)$, as defined, is a rotation.

<u>Method 1:</u>

$$U_R^\dagger \vec{\sigma} U_R = e^{\overbrace{+i\vec{n}\cdot\vec{\sigma}\,\theta/2}^{A}} \vec{\sigma}\, e^{-i\vec{n}\cdot\vec{\sigma}\,\theta/2}$$

Use $$e^{A} B e^{-A} = B + [A,B] + \tfrac{1}{2}[A,[A,B]] + \dots$$
$$= B + \sum_{k=1}^{\infty} \frac{1}{k!} \underbrace{[A,[A,\dots[A,B]\dots]]}_{\equiv [A,B]^{(k)}}$$

$k$ nested commutators

$$[A,\sigma_j] = i\frac{\theta}{2} n_k \underbrace{[\sigma_k,\sigma_j]}_{2i\epsilon_{kjl}\sigma_l} = +\theta\,\epsilon_{jkl} n_k \sigma_l = \theta(\vec{n}\times\vec{\sigma})_j$$

$$\Rightarrow [A,\vec{\sigma}] = \vec{e}_j [A,\sigma_j] = \theta\,\vec{n}\times\vec{\sigma}$$

$$[A,\vec{\sigma}]^{(2)} = [A,[A,\vec{\sigma}]] = [A,\theta\vec{n}\times\vec{\sigma}] = \theta\vec{n}\times\underbrace{[A,\vec{\sigma}]}_{\theta\vec{n}\times\vec{\sigma}}$$

$$\Rightarrow [A,\vec{\sigma}]^{(2)} = \theta^2\,\vec{n}\times(\vec{n}\times\vec{\sigma})$$

$$[A,\vec{\sigma}]^{(3)} = [A,[A,\vec{\sigma}]^{(2)}] = \theta^3\,\vec{n}\times(\underbrace{\vec{n}\times(\vec{n}\times\vec{\sigma})}_{\vec{n}(\vec{n}\cdot\vec{\sigma})-\vec{\sigma}}) = -\theta^3\,\vec{n}\times\vec{\sigma}$$

The pattern is now established:

$$k>0:\ [A,\vec{\sigma}]^{(k)} = \begin{cases} (-1)^{(k-1)/2}\,\theta^k\,\vec{n}\times\vec{\sigma}, & k \text{ odd} \\ (-1)^{(k-2)/2}\,\theta^k\,\vec{n}\times(\vec{n}\times\vec{\sigma}), & k \text{ even} \end{cases}$$

$$U_R^{\dagger}\vec{\sigma}\,U_R = \vec{\sigma} + \sum_{k=1}^{\infty}\frac{1}{k!}\,[A,\vec{\sigma}]^{(k)}$$

$$= \vec{\sigma} + \vec{n}\times\vec{\sigma}\sum_{k\ \text{odd}}\frac{(-1)^{(k-1)/2}}{k!}\,\theta^{k}$$

$$+\ \vec{n}\times(\vec{n}\times\vec{\sigma})\sum_{\substack{k\ \text{even}\\ k\geq 2}}\frac{(-1)^{(k-2)/2}}{k!}\,\theta^{k}$$

$$k=2l+1:\quad \sum_{l=0}^{\infty}\frac{(-1)^{l}}{(2l+1)!}\,\theta^{2l+1} = \sin\theta$$

$$k=2l:\quad \sum_{l=1}^{\infty}\frac{(-1)^{l-1}}{(2l)!}\,\theta^{2l} = 1-\sum_{l=0}^{\infty}\frac{(-1)^{l}}{(2l)!}\,\theta^{2l}$$

$$= 1-\cos\theta$$

$$\therefore\ \boxed{\begin{aligned} U_R^{\dagger}\vec{\sigma}\,U_R &= \vec{\sigma} + \underbrace{\vec{n}\times(\vec{n}\times\vec{\sigma})}_{\vec{n}(\vec{n}\cdot\vec{\sigma})-\vec{\sigma}}(1-\cos\theta) + \vec{n}\times\vec{\sigma}\,\sin\theta \\ &= \vec{n}(\vec{n}\cdot\vec{\sigma}) - \vec{n}\times(\vec{n}\times\vec{\sigma})\cos\theta + \vec{n}\times\vec{\sigma}\,\sin\theta \end{aligned}}$$

This method only uses the commutators, $[\sigma_j,\sigma_k] = 2i\,\epsilon_{jkl}\,\sigma_l$, so rewritten in terms of the spin vector $\vec{S} = \frac{1}{2}\hbar\vec{\sigma}$, which has commutators $[S_j,S_k] = i\hbar\,\epsilon_{jkl}\,S_l$, it applies to any angular momentum operator $\vec{J}$, which has commutators $[J_j,J_k] = i\hbar\,\epsilon_{jkl}\,S_l$.

Method 2: Use $U_R = \mathbb{1}\cos(\theta/2) - i\vec{n}\cdot\vec{\sigma}\sin(\theta/2)$

$$\sigma_j U_R = \sigma_j c - i \underbrace{\sigma_j\sigma_k}_{=\delta_{jk}\mathbb{1} + i\epsilon_{jkl}\sigma_l} n_k s = \sigma_j c - i n_j s + \epsilon_{jkl} n_k \sigma_l s$$

$$\vec{\sigma} U_R = \vec{\sigma} c - i\vec{n} s + \vec{n}\times\vec{\sigma} s$$

$$U_R^\dagger \sigma_j U_R = (\mathbb{1}c + i n_m \sigma_m s)(\sigma_j c - i n_j s + \epsilon_{jkl} n_k \sigma_l s)$$

$$= \sigma_j c^2 - i n_j c s + \epsilon_{jkl} n_k \sigma_l s c$$

$$+ i s c n_m \underbrace{\sigma_m \sigma_j}_{=\delta_{mj} + i\epsilon_{mjl}\sigma_l} + s^2 n_j n_m \sigma_m + i s^2 \epsilon_{jkl} n_k \underbrace{\sigma_m \sigma_l}_{=\delta_{ml} + i\epsilon_{mln}\sigma_n}$$

$$= \sigma_j c^2 - \cancel{i n_j c s} + \epsilon_{jkl} n_k \sigma_l s c$$

$$+ \cancel{i n_j s c} + \epsilon_{jml} n_m \sigma_l s c$$

$$+ n_j n_m \sigma_m s^2$$

$$+ \cancel{i s^2 \epsilon_{jkl} n_k n_l s^2}^{\,0} + \epsilon_{jkl}\epsilon_{lmn} n_k n_m \sigma_n s^2$$

$$= \frac{1}{2}\sigma_j(1+\cos\theta) + \epsilon_{jkl} n_k \sigma_l \sin\theta$$

$$+ n_j n_k \sigma_k \frac{1}{2}(1-\cos\theta)$$

$$+ \epsilon_{jkl} n_k \epsilon_{lmn} n_m \sigma_n \frac{1}{2}(1-\cos\theta)$$

$$U_R^\dagger \vec{\sigma} U_R = \frac{1}{2}\vec{\sigma}(1+\cos\theta) + \vec{n}\times\vec{\sigma}\sin\theta$$

$$+ \vec{n}(\vec{n}\cdot\vec{\sigma})\frac{1}{2}(1-\cos\theta)$$

$$+ \underbrace{\vec{n}\times(\vec{n}\times\vec{\sigma})}\frac{1}{2}(1-\cos\theta) \rightarrow \vec{n}(\vec{n}\cdot\vec{\sigma}) - \vec{\sigma}$$

$$= \frac{1}{2}\vec{\sigma}(1+\cos\theta) + \vec{n}\times\vec{\sigma}\sin\theta$$

$$+ \vec{n}(\vec{n}\cdot\vec{\sigma})(1-\cos\theta) - \vec{\sigma}\frac{1}{2}(1-\cos\theta)$$

$$\boxed{U_R^\dagger\vec{\sigma}U_R = \vec{\sigma}\cos\theta + \vec{n}(\vec{n}\cdot\vec{\sigma})(1-\cos\theta) + \vec{n}\times\vec{\sigma}\sin\theta}$$

(d) $\vec{\sigma}\cdot R\vec{m}\, U_R = U_R U_R^\dagger \vec{\sigma} U_R \cdot R\vec{m}$ , $R = R_{\vec{n}}(\theta)$

$= U_R\, R\vec{\sigma}\cdot R\vec{m}$ ⟵ part (c)

$= U_R\, \vec{\sigma}\cdot\vec{m}$ ⟵ Rotation R preserves inner products

$$\Rightarrow \vec{\sigma}\cdot R\vec{m}\, U_R|\vec{m}\rangle = U_R \underbrace{\vec{\sigma}\cdot\vec{m}\,|\vec{m}\rangle}_{|\vec{m}\rangle} = U_R|\vec{m}\rangle$$

This means $U_R|\vec{m}\rangle$ is an eigenstate of $\vec{\sigma}\cdot R\vec{m}$ with eigenvalue $+1$, so

$$U_R|\vec{m}\rangle = e^{i\phi(R,\vec{m})}|R\vec{m}\rangle$$

↑ phase that depends on $R$ and $\vec{m}$.

(e) $$\begin{pmatrix}\text{rotation by}\\ 180^\circ \text{ about } \vec{n}\end{pmatrix} = e^{-i\vec{n}\cdot\vec{\sigma}\pi/2} = \cos\frac{\pi}{2} - i\vec{n}\cdot\vec{\sigma}\sin\frac{\pi}{2} = -i\vec{n}\cdot\vec{\sigma}$$

$$\boxed{\vec{n}\cdot\vec{\sigma} = i\begin{pmatrix}\text{rotation by}\\ 180^\circ \text{ about } \vec{n}\end{pmatrix}}$$

(f) Hadamard transform $H = i\, e^{-i\vec{n}\cdot\vec{\sigma}\,\pi/2}$

$$= \vec{n}\cdot\vec{\sigma}$$

$$= \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z)$$

Matrix representation

$$H \longleftrightarrow \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$