2.4.

Two-level atom dynamical evolution

$$\hat{H} = \frac{1}{2}\hbar\Omega \underbrace{\vec{n}\cdot\vec{\sigma}}_{n_3\sigma_3 + (n_1 - in_2)\sigma_+ + (n_1+in_2)\sigma_-}$$

(a) State vectors (pure states):

$$\sigma_+ = \frac{1}{2}(\sigma_1 + i\sigma_2) = |0\rangle\langle 1|$$
$$\sigma_- = \frac{1}{2}(\sigma_1 - i\sigma_2) = |1\rangle\langle 0|$$

$\sigma_+\|0\rangle = 0$	$\langle 0\|\sigma_- = 0$
$\sigma_+\|1\rangle = \|0\rangle$	$\langle 1\|\sigma_- = \langle 0\|$
$\sigma_-\|0\rangle = \|1\rangle$	$\langle 0\|\sigma_+ = \langle 1\|$
$\sigma_-\|1\rangle = 0$	$\langle 1\|\sigma_+ = 0$

$$i\hbar \frac{d|\psi\rangle}{dt} = \hat{H}|\psi\rangle$$

$$i\hbar \frac{d\langle 0|\psi\rangle}{dt} = \langle 0|H|\psi\rangle$$

$$= \frac{1}{2}\hbar\Omega\left[n_3\langle 0|\psi\rangle + (n_1 - in_2)\langle 1|\psi\rangle\right]$$

$$i\hbar \frac{d\langle 1|\psi\rangle}{dt} = \frac{1}{2}\hbar\Omega\left[-n_3\langle 1|\psi\rangle + (n_1 + in_2)\langle 0|\psi\rangle\right]$$

$$\boxed{\begin{aligned} \frac{dc_0}{dt} &= -\frac{i}{2}\Omega\left[n_3 c_0 + (n_1 - in_2)c_1\right] \\ \frac{dc_1}{dt} &= -\frac{i}{2}\Omega\left[-n_3 c_1 + (n_1 + in_2)c_0\right] \end{aligned}}$$

(b) Solution: $|\psi(t)\rangle = U(t,0)|\psi(0)\rangle = \exp\left(-\frac{i}{2}\Omega t\, \vec{n}\cdot\vec{\sigma}\right)|\psi(0)\rangle$

$$|\psi(t)\rangle = \left[\cos(\Omega t/2)\mathbb{1} - i\sin(\Omega t/2)\,\vec{n}\cdot\vec{\sigma}\right]|\psi(0)\rangle$$

$$c_0(t) = \langle 0|\psi(t)\rangle = \cos(\Omega t/2)\langle 0|\psi(0)\rangle$$
$$- i\sin(\Omega t/2)\left[n_3\langle 0|\psi(0)\rangle + (n_1 - in_2)\langle 1|\psi(0)\rangle\right]$$

(2)

$$c_1(t) = \langle 1 | \psi(t) \rangle = \cos(\Omega t/2) \langle 1 | \psi(0) \rangle$$
$$- i \sin(\Omega t/2) \left[-n_3 \langle 1 | \psi(0) \rangle + (n_1 + i n_2) \langle 0 | \psi(0) \rangle \right]$$

$$\boxed{\begin{aligned} c_0(t) &= \left[\cos(\Omega t/2) - i n_3 \sin(\Omega t/2) \right] c_0(0) \\ &\quad - i(n_1 - i n_2) \sin(\Omega t/2)\, c_1(0) \\ c_1(t) &= \left[\cos(\Omega t/2) + i n_3 \sin(\Omega t/2) \right] c_1(0) \\ &\quad - i(n_1 + i n_2) \sin(\Omega t/2)\, c_0(t) \end{aligned}}$$

This is a rotation with angular velocity Ω about $\vec{n}$

( ) Verification of solution:

$$\frac{dc_0}{dt} = \frac{1}{2}\Omega \left[-\sin(\Omega t/2) - i n_3 \cos(\Omega t/2) \right] c_0(0)$$
$$- \frac{i}{2}\Omega (n_1 - i n_2) \cos(\Omega t/2)\, c_1(0)$$
$$= -\frac{i}{2}\Omega \left[n_3 \cos(\Omega t/2) - i \sin(\Omega t/2) \right] c_0(0)$$
$$- \frac{i}{2}\Omega (n_1 - i n_2) \cos(\Omega t/2)\, c_1(0)$$

$$n_3 c_0(t) + (n_1 - i n_2)\, c_1(t)$$

$$= n_3 \left[\cos(\Omega t/2) - i n_3 \sin(\Omega t/2) \right] c_0(0)$$

$$- i n_3 (n_1 - i n_2) \sin(\Omega t/2)\, c_1(0)$$

$$+ (n_1 - i n_2) \left[\cos(\Omega t/2) + i n_3 \sin(\Omega t/2) \right] c_1(0)$$

$$- i \underbrace{(n_1 - i n_2)(n_1 + i n_2)}_{n_1^2 + n_2^2 = 1 - n_3^2} \sin(\Omega t/2)\, c_0(0)$$

$$= \left[n_3 \cos(\Omega t/2) - i \sin(\Omega t/2) \right] c_0(0)$$

$$+ (n_1 - i n_2) \cos(\Omega t/2)\, c_1(0)$$

$$\therefore \quad \frac{dc_0}{dt} = -\frac{i}{2}\Omega \left[n_3 c_0 + (n_1 - i n_2) c_1 \right]$$

Equation for $c_1(t)$ follows from exchanges $1 \longleftrightarrow 0$, $n_3 \longleftrightarrow -n_3$, $n_2 \longleftrightarrow -n_2$.

You can also solve the differential equations directly, by decoupling them, as for any coupled linear d.e.'s.

(c) $\vec{n} = \vec{e}_z \Rightarrow n_x = n_y = 0,\ n_z = 1$

$$\boxed{\begin{aligned} c_0(t) &= e^{-i\Omega t/2} c_0(0) \\ c_1(t) &= e^{+i\Omega t/2} c_1(0) \end{aligned}}$$