2.5. Two-level atom in a classical single-mode field

$$\hat{H} = \frac{1}{2}\hbar\Omega\hat{\sigma}_3 + \underset{\uparrow\,\text{real}}{\hbar g_0}\left(\hat{\sigma}_- e^{i(\overbrace{\Omega+\underset{\uparrow\,\text{detuning}}{\Delta}}^{\omega})t} + \hat{\sigma}_+ e^{-i(\overbrace{\Omega+\Delta}^{\omega})t}\right)$$

(a) Evolution operator: $i\hbar \dfrac{d\hat{U}(t,0)}{dt} = \hat{H}(t).\hat{U}(t,0)$

Write $\hat{U}(t,0) = \underbrace{\exp\left(-\frac{i}{2}\omega t\hat{\sigma}_3\right)}\hat{U}_I(t,0)$

This is an interaction picture (IP) wrt the atomic Hamiltonian $\hat{H}_0 = \frac{1}{2}\hbar\omega\hat{\sigma}_3$.

$$\hat{U}_{\underbrace{R_{\vec{e}_3}(\omega t)}_{R}} = \hat{U}_R$$

$$i\hbar\frac{d\hat{U}_R}{dt} = \frac{1}{2}\hbar(\overbrace{\Omega+\Delta}^{\omega})\hat{\sigma}_3\hat{U}_R$$

$$i\hbar\frac{d\hat{U}_R^\dagger}{dt} = -\hat{U}_R^\dagger\frac{1}{2}\hbar\underbrace{(\Omega+\Delta)}_{\omega}\hat{\sigma}_3$$

$$\hat{U}_I(t,0) = \hat{U}_R^\dagger\hat{U}(t,0)$$

$$i\hbar\frac{d\hat{U}_I(t,0)}{dt} = -\hat{U}_R^\dagger\frac{1}{2}\hbar(\overbrace{\Omega+\Delta}^{\omega})\hat{\sigma}_3\hat{U}_R\hat{U}_I(t,0) + \hat{U}_R^\dagger\hat{H}\hat{U}_R\hat{U}_I(t,0)$$

$$= \hat{U}_R^\dagger\underbrace{\left(\hat{H} - \frac{1}{2}\hbar(\overbrace{\Omega+\Delta}^{\omega})\hat{\sigma}_3\right)}_{-\frac{1}{2}\hbar\Delta\hat{\sigma}_3 + \hbar g_0\left(\hat{\sigma}_- e^{+i\omega t} + \hat{\sigma}_+ e^{-i\omega t}\right)}\hat{U}_R\hat{U}_I(t,0)$$

Use $\hat{U}_R^\dagger\hat{\sigma}_\pm\hat{U}_R = \hat{\sigma}_\pm e^{\pm i\omega t}$

$\hat{U}_R^\dagger\hat{\sigma}_3\hat{U}_R = \hat{\sigma}_3$

$$i\hbar \frac{d\hat{U}_I(t,0)}{dt} = \Big[-\tfrac{1}{2}\hbar\Delta\hat{\sigma}_3 + \hbar g_0 \underbrace{(\hat{\sigma}_- + \hat{\sigma}_+)}_{\hat{\sigma}_1}\Big]\hat{U}_I(t,0)$$

$$\boxed{i\hbar \frac{d\hat{U}_I(t,0)}{dt} = \tfrac{1}{2}\hbar\left(-\Delta\hat{\sigma}_3 + 2g_0\hat{\sigma}_1\right)\hat{U}_I(t,0)}$$

Introduce unit vector $\vec{n} = \dfrac{2g_0\vec{e}_1 - \Delta\vec{e}_3}{r}$,

$r \equiv \sqrt{\Delta^2 + 4g_0^2}$

$$i\hbar \frac{d\hat{U}_I(t,0)}{dt} = \tfrac{1}{2}\hbar r\,\vec{n}\cdot\hat{\vec{\sigma}}\,\hat{U}_I(t,0)$$

$$\Longrightarrow \hat{U}_I(t,0) = \exp\left(-\tfrac{i}{2}rt\,\vec{n}\cdot\hat{\vec{\sigma}}\right) = \hat{U}_{R'}$$

$$R' = R_{\vec{n}}(rt)$$

$$\Longrightarrow \boxed{\hat{U}(t,0) = \hat{U}_R\hat{U}_{R'}}$$

$$\boxed{\begin{aligned}\hat{\vec{\sigma}}(t) &= \hat{U}^\dagger_{R'}\underbrace{\hat{U}^\dagger_R\hat{\vec{\sigma}}(0)\hat{U}_R}_{R\hat{\vec{\sigma}}(0)}\hat{U}_{R'}\\ &= R\underbrace{(\hat{U}^\dagger_{R'}\hat{\vec{\sigma}}(0)\hat{U}_{R'})}_{R'\hat{\vec{\sigma}}(0)}\\ &= RR'\hat{\vec{\sigma}}(0)\end{aligned}}$$

(b) $\hat{\vec{\sigma}}(t) = \hat{U}^\dagger(t,0)\hat{\vec{\sigma}}(0)\hat{U}(t,0) = RR'\hat{\vec{\sigma}}(0)$

$\vec{S}(t) = RR'\vec{S}(0)$

Component form

$$\boxed{\begin{aligned}\hat{\sigma}_j(t) &= \hat{U}^\dagger_{R'}\underbrace{\hat{U}^\dagger_R\hat{\sigma}_j(0)\hat{U}_R}_{R_{jk}\hat{\sigma}_k(0)}\hat{U}_{R'}\\ &= R_{jk}\underbrace{(\hat{U}^\dagger_{R'}\hat{\sigma}_k(0)\hat{U}_{R'})}_{R'_{kl}\hat{\sigma}_l(0)}\\ &= R_{jk}R'_{kl}\hat{\sigma}_l(0)\\ &= (RR')_{jl}\hat{\sigma}_l(0)\end{aligned}}$$

(c) If $|\psi(0)\rangle = |0\rangle$, then $\vec{S}(0) = \vec{e}_3$

$$\Longrightarrow \vec{S}(t) = R_{\vec{e}_3}(\omega t)R_{\vec{n}}(rt)\vec{e}_3$$

Polarization analogue of this rotation

$$R_{\vec n}(rt)\,\vec e_3 = \vec n\,\overbrace{(\vec n\cdot\vec e_3)}^{-\Delta/r} - \vec n\times(\vec n\times\vec e_3)\cos rt + \vec n\times\vec e_3 \sin rt$$

$$+\ \frac{4g_0^2}{r^2}\vec e_3 + \frac{2g_0\Delta}{r^2}\vec e_1 \qquad - \frac{2g_0}{r}\vec e_2$$

$$-\left(\frac{2g_0\Delta}{r^2}\vec e_1 + \frac{\Delta^2}{r^2}\vec e_3\right.$$

$$= \vec e_1\,\frac{2g_0\Delta}{r^2}(-1+\cos rt) - \frac{2g_0}{r}\sin rt\,\vec e_2$$

$$+\ \vec e_3\,\frac{1}{r^2}\left(\Delta^2 + 4g_0^2\cos rt\right)$$

$$\vec S(t) = R_{\vec e_3}(\omega t)\,R_{\vec n}(rt)\,\vec e_3$$

$$= \frac{2g_0\Delta}{r^2}(-1+\cos rt)\left(\vec e_1\cos\omega t + \vec e_2\sin\omega t\right)$$

$$-\ \frac{2g_0}{r}\sin rt\left(-\vec e_1\sin\omega t + \vec e_2\cos\omega t\right)$$

$$+\ \frac{\Delta^2 + 4g_0^2\cos rt}{r^2}\,\vec e_3$$

$$\boxed{\begin{aligned}\vec S(t) = {} & \underbrace{\frac{2g_0}{r}\left[\frac{\Delta}{r}(-1+\cos rt)\cos\omega t + \sin rt\,\sin\omega t\right]}_{\langle\hat\sigma_1(t)\rangle}\vec e_1 \\ & + \underbrace{\frac{2g_0}{r}\left[\frac{\Delta}{r}(-1+\cos rt)\sin\omega t - \sin rt\,\cos\omega t\right]}_{\langle\hat\sigma_2(t)\rangle}\vec e_2 \\ & + \underbrace{\frac{\Delta^2 + 4g_0^2\cos rt}{r^2}}_{\langle\hat\sigma_3(t)\rangle}\vec e_3\end{aligned}}$$

(c) $\Delta = 0$: $r = 2g_0$, $\vec{n} = \vec{e}_1$

$$\vec{S}(t) = R_{\vec{e}_3}(\Omega t)\, R_{\vec{e}_1}(2g_0 t)\, \vec{e}_1$$

$$= -\sin 2g_0 t \left(-\sin \Omega t\, \vec{e}_1 + \cos \Omega t\, \vec{e}_2\right) + \cos 2g_0 t\, \vec{e}_3$$

(d) $$\hat{U}_{R'}|0\rangle = \left[\hat{1}\cos(rt/2) - i\sin(rt/2)\left(\frac{2g_0}{r}\hat{\sigma}_1 - \frac{\Delta}{r}\hat{\sigma}_3\right)\right]|0\rangle$$

$$= \left[\cos(rt/2) + i\sin(rt/2)\frac{\Delta}{r}\right]|0\rangle - i\sin(rt/2)\frac{2g_0}{r}|1\rangle$$

$$\hat{U}_R\hat{U}_{R'}|0\rangle = \left[\hat{1}\cos(\omega t/2) - i\sin(\omega t/2)\hat{\sigma}_3\right]\hat{U}_{R'}|0\rangle$$

$$= \cos(\omega t/2)\Big(\left[\cos(rt/2) + i\frac{\Delta}{r}\sin(rt/2)\right]|0\rangle - i\frac{2g_0}{r}\sin(rt/2)|1\rangle\Big)$$

$$- i\sin(\omega t/2)\Big(\left[\cos(rt/2) + i\frac{\Delta}{r}\sin(rt/2)\right]|0\rangle + i\frac{2g_0}{r}\sin(rt/2)|1\rangle\Big)$$

$$\hat{U}_R\hat{U}_{R'}|0\rangle = \underbrace{e^{-i\omega t/2}\left[\cos(rt/2) + i\frac{\Delta}{r}\sin(rt/2)\right]}_{\langle 0|\hat{U}_R\hat{U}_{R'}|0\rangle}|0\rangle$$

$$\underbrace{-i e^{+i\omega t/2}\frac{2g_0}{r}\sin(rt/2)}_{\langle 1|\hat{U}_R\hat{U}_{R'}|0\rangle}|1\rangle$$

$$\langle\hat{\sigma}_3\rangle = \left|\cos(rt/2) + i\frac{\Delta}{r}\sin(rt/2)\right|^2 - \frac{4g_0^2}{r^2}\sin^2(rt/2)$$

$$= \cos^2(rt/2) + \left(\frac{\Delta^2}{r^2} - \frac{4g_0^2}{r^2}\right)\sin^2(rt/2)$$

$$\langle\hat{\sigma}_3\rangle = \frac{1}{2}(1+\cos rt) + \frac{\Delta^2 - 4g_0^2}{r^2}\,\frac{1}{2}(1-\cos rt)$$

$$= \frac{\Delta^2}{r^2} + \frac{4g_0^2}{r^2}\cos rt \qquad \checkmark$$

$$\langle\hat{\sigma}_+\rangle = e^{+i\omega t/2}\left[\cos(rt/2) - i\frac{\Delta}{r}\sin(rt/2)\right]$$

$$\times\ -i\,e^{+i\omega t/2}\,\frac{2g_0}{r}\sin(rt/2)$$

$$= \underbrace{-i\,e^{+i\omega t}}_{-i\cos\omega t + \sin\omega t}\,\frac{2g_0}{r}\left[\frac{1}{2}\sin rt + i\frac{\Delta}{r}\frac{1}{2}(-1+\cos rt)\right]$$

$$\langle\hat{\sigma}_+\rangle = \frac{1}{2}\frac{2g_0}{r}\Big[\sin\omega t\,\sin rt + \frac{\Delta}{r}\cos\omega t\,(-1+\cos rt)$$

$$+\,i\Big(-\cos\omega t\,\sin rt + \frac{\Delta}{r}\sin\omega t\,(-1+\cos rt)\Big)\Big] \qquad \checkmark$$