2.6. $$A = A_\alpha \sigma_\alpha = A_0 1 + \vec{A}\cdot\vec{\sigma} = A_0 1 + \vec{B}\cdot\vec{\sigma} + i\vec{C}\cdot\vec{\sigma}$$

$$\vec{A} = \underset{\uparrow}{\vec{B}} + i\underset{\uparrow}{\vec{C}}$$

real vectors

(a) Normal $\Leftrightarrow [M, M^\dagger] = 0$

$$A^\dagger = A_0^* 1 + \vec{A}^* \cdot \vec{\sigma}$$

$$[A, A^\dagger] = [\vec{A}\cdot\vec{\sigma}, \vec{A}^*\cdot\vec{\sigma}] = A_j A_k^* \underbrace{[\sigma_j, \sigma_k]}_{2i\epsilon_{jkl}\sigma_l} = 2i\sigma_l \epsilon_{ljk} A_j A_k^*$$

$$[A, A^\dagger] = 2i\vec{\sigma}\cdot \underbrace{\vec{A}\times\vec{A}^*}_{} = 4\vec{\sigma}\cdot\vec{B}\times\vec{C}$$

$$(\vec{B}+i\vec{C})\times(\vec{B}-i\vec{C}) = -2i\vec{B}\times\vec{C}$$

A is normal $\Leftrightarrow [A, A^\dagger] = 0$

$$\Leftrightarrow 0 = \vec{A}\times\vec{A}^* \Leftrightarrow 0 = \vec{B}\times\vec{C} \Leftrightarrow \vec{C} = \underset{\uparrow}{\mu}\vec{B}$$

real constant

(b) ① A Hermitian $\Leftrightarrow A = A^\dagger \Leftrightarrow A_\alpha = A_\alpha^*$

② A unitary $\Leftrightarrow AA^\dagger = 1 = A^\dagger A$

$$AA^\dagger = (A_0 1 + \vec{A}\cdot\vec{\sigma})(A_0^* 1 + \vec{A}^*\cdot\vec{\sigma})$$

$$= |A_0|^2 1 + A_0^*\vec{A}\cdot\vec{\sigma} + A_0\vec{A}^*\cdot\vec{\sigma} + \underbrace{(\vec{A}\cdot\vec{\sigma})(\vec{A}^*\cdot\vec{\sigma})}_{}$$

$$A_j A_k^* \sigma_j \sigma_k = A_j A_k^* (\delta_{jk} + i\epsilon_{jkl}\sigma_l)$$

$$= \vec{A}\cdot\vec{A}^* + i\vec{\sigma}\cdot\vec{A}\times\vec{A}^*$$

$$= (|A_0|^2 + \vec{A}\cdot\vec{A}^*)1 + (A_0^*\vec{A} + A_0\vec{A}^* + i\vec{A}\times\vec{A}^*)\cdot\vec{\sigma}$$

We know that if A is unitary, it is normal, so $\vec{A}\times\vec{A}^* = 0$.

So A unitary $\Leftrightarrow |A_0|^2 + \vec{A}\cdot\vec{A}^* = 1$ and $A_0\vec{A}^* = -A_0^*\vec{A}$

Let $A_0 = |A_0| e^{i\delta}$, $\vec{A} = e^{i\delta}\vec{A}'$. Then we have

$$|A_0|^2 + \vec{A}'\cdot\vec{A}'^{*} = 1 \quad \text{and} \quad \vec{A}' = \vec{A}'^{*},$$

so we can write

$$\vec{A}' = \sqrt{1-|A_0|^2}\,\vec{n}$$

unit vector

We end up with

$$A \text{ unitary} \iff A = e^{i\delta}\left(|A_0|\,\mathbb{1} + \sqrt{1-|A_0|^2}\,\vec{n}\cdot\vec{\sigma}\right)$$

(c)

① A has a s.d. $\Rightarrow$ $A = a_+|\vec{n}\rangle\langle\vec{n}| + a_-|-\vec{n}\rangle\langle-\vec{n}|$

$$= \frac{1}{2}(a_+ + a_-)\,\mathbb{1} + \underbrace{\frac{1}{2}(a_+ - a_-)\vec{n}}_{\vec{A}}\cdot\vec{\sigma}$$

$$\Rightarrow \vec{A}\times\vec{A}^{*} = 0$$

$\Rightarrow$ A is normal

② A is normal $\Rightarrow$ $\vec{C} = \mu\vec{B}$

$$\Rightarrow A = A_0\mathbb{1} + (1+i\mu)\vec{B}\cdot\vec{\sigma}$$

$$= A_0\mathbb{1} + (1+i\mu)B\,\vec{n}\cdot\vec{\sigma}$$

$$= \left(A_0 + (1+i\mu)B\right)|\vec{n}\rangle\langle\vec{n}| + \left(A_0 - (1+i\mu)B\right)|-\vec{n}\rangle\langle-\vec{n}|$$

spectral decomposition