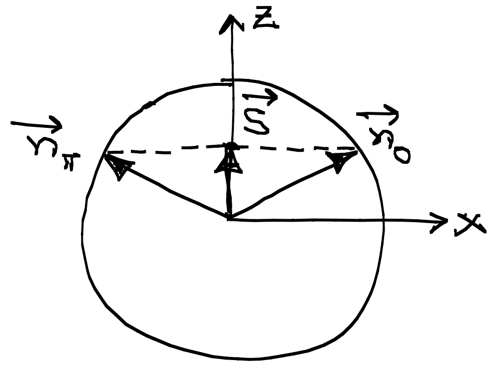
3.1.

(a) We can use any distribution of vectors of the form

$$\vec{n}_\phi = \frac{1}{2}\hat{e}_z + \frac{\sqrt{3}}{2}(\hat{e}_x\cos\phi + \hat{e}_y \sin\phi),$$



provided the equatorial components average to zero.

(i) two vectors: $\left\{\begin{matrix} \vec{n}_0, & p = 1/2 \\ \vec{n}_\pi, & p = 1/2 \end{matrix}\right\}$

Here you have to choose $\vec{n}_\phi$ and $\vec{n}_{\phi+\pi}$ with equal probabilities

(ii) three vectors: $\left\{\begin{matrix} \vec{n}_0, & p = 1/3 \\ \vec{n}_{2\pi/3}, & p = 1/3 \\ \vec{n}_{4\pi/3}, & p = 1/3 \end{matrix}\right\}$

With three vectors you have a lot of freedom. This is a symmetric example.

(iii) all vectors: $p_{\vec{n}} = p(\phi) = \frac{1}{2\pi}$

normalized probability distribution

(b) We need $p_{\vec{n}} = p(\theta,\phi)$ such that

$$\int d\Omega\, p(\theta,\phi) = 1$$

$$\int d\Omega\, \vec{n}\, p(\theta,\phi) = \vec{S} = \frac{1}{2}\hat{e}_z$$

Generally, if we expand $p(\theta,\phi)$ in terms of spherical harmonics,

$$p(\theta,\phi) = \sum_{l,m} c_{lm} Y_{lm}(\theta,\phi), \quad c_{lm} = \int d\Omega\, Y^*_{lm}(\theta,\phi)\, p(\theta,\phi),$$

$$Y_{l,-m} = (-1)^m Y^*_{lm}$$

What the two constraints do is to fix the $l=0$ and $l=1$ components, i.e.,

$$C_{00} = \int d\Omega\, Y_{00}^{*}\, p = \frac{1}{\sqrt{4\pi}}$$

$$\uparrow$$

$$\frac{1}{\sqrt{4\pi}}$$

$$C_{10} = \int d\Omega\, Y_{10}^{*}\, p = \sqrt{\frac{3}{4\pi}} \int d\Omega\, n_z\, p = \sqrt{\frac{3}{4\pi}}\, S_z$$

$$\uparrow$$

$$\sqrt{\frac{3}{4\pi}} \cos\theta = \sqrt{\frac{3}{4\pi}}\, n_z$$

$$C_{1,\pm1} = \int d\Omega\, Y_{1,\pm1}^{*}\, p = \mp\sqrt{\frac{3}{8\pi}} \int d\Omega\, (n_x \mp i n_y)\, p = \mp\sqrt{\frac{3}{8\pi}}\,(S_x \mp i S_y)$$

$$\uparrow$$

$$\mp\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{\mp i\phi} = \mp\sqrt{\frac{3}{8\pi}}\,(n_x \mp i n_y)$$

The $l \ge 2$ components can be anything, provided only that $p(\theta,\phi)$ is real $[C_{l,-m} = (-1)^m C_{lm}^{*}]$ and (harder to arrange) nonnegative. So the general form is

$$\begin{aligned} p(\theta,\phi) &= C_{00} Y_{00} + C_{10} Y_{10} + C_{11} Y_{11} + C_{1,-1} Y_{1,-1} + (\text{higher terms}) \\ &= \frac{1}{4\pi} + \frac{3}{4\pi}\left(S_z n_z + S_x n_x + S_y n_y\right) + (\text{higher terms}) \\ &= \frac{1}{4\pi}\left(1 + 3\vec{S}\cdot\vec{n}\right) + (\text{higher terms}) \end{aligned}$$

If $|\vec{S}| > \frac{1}{3}$, we have to have some higher terms to make $p(\theta,\phi)$ nonnegative. In our case, $\vec{S} = \frac{1}{2}\vec{e}_z$, and a Y_{20} term suffices:

$$p(\theta,\phi) = \frac{1}{4\pi}\left(1 + \frac{3}{2} n_z + a\left(3n_z^2 - 1\right)\right) \qquad Y_{20} = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Choosing $a = \frac{1}{4}$ makes $p(\theta,\phi)$ zero at the south pole and positive everywhere else:

$$p(\theta,\phi) = \frac{1}{4\pi}\left(1 + \frac{3}{2} n_z + \frac{1}{4}\left(3n_z^2 - 1\right)\right) = \boxed{\frac{3}{16\pi}\left(1 + n_z\right)^2 = p(\theta,\phi)}$$