3.6. $\sigma_{\pm} = \frac{1}{2}(\sigma_x \pm i\sigma_y)$ $\quad$ $\sigma_+ = |0\rangle\langle 1| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$\sigma_- = \sigma_+^\dagger = |1\rangle\langle 0| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_+\sigma_- = |0\rangle\langle 0| = \frac{1}{2}(1 + \sigma_z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\sigma_-\sigma_+ = |1\rangle\langle 1| = \frac{1}{2}(1 - \sigma_z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

(a) $$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \end{pmatrix}$$

Eigenvalues: $0 = \det\begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix} = \lambda^2 \Rightarrow \lambda = 0$

Eigenvectors: $b = \lambda a = 0$, a arbitrary, choose $a = 1$

σ_+ has one eigenvalue $\lambda = 0$ and one eigenvector $|e\rangle = |0\rangle$. A vector $a|0\rangle + b|1\rangle$ is mapped to $b|0\rangle$ by σ_+ and so can be an eigenvector only if $b = 0$.

(b) The singular values of σ_+ are the eigenvalues of $\sigma_-\sigma_+ = |1\rangle\langle 1|$ or $\sigma_-\sigma_+ = |0\rangle\langle 0|$. Thus the singular values are 0 and 1. The polar decomposition is

$$\sigma_+ = U\underbrace{\sqrt{\sigma_-\sigma_+}}_{|1\rangle\langle 1|} = \underbrace{\sqrt{\sigma_+\sigma_-}}_{|0\rangle\langle 0|}\, U$$

U is the unitary matrix that maps the eigenvectors of $\sigma_-\sigma_+$ to the eigenvectors of $\sigma_+\sigma_-$. This is equivalent to saying that $U\sigma_-\sigma_+U^\dagger = \sigma_+\sigma_-$. Thus U is the unitary that exchanges $|0\rangle$ and $|1\rangle$, i.e.,

$$U = |0\rangle\langle 1| + |1\rangle\langle 0| = \sigma_x$$

The polar decomposition of σ_+ is thus

$$\boxed{\sigma_+ = \sigma_x \sqrt{\sigma_- \sigma_+} = \sqrt{\sigma_+ \sigma_-}\, \sigma_x}$$

$$\sigma_x \underbrace{\sqrt{\sigma_- \sigma_+}}_{|1\rangle\langle 1|} |0\rangle = 0 \quad , \quad \sigma_x \underbrace{\sqrt{\sigma_- \sigma_+}}_{|1\rangle\langle 1|} |1\rangle = \sigma_x |1\rangle = |0\rangle$$

null eigenvector of $\sqrt{\sigma_- \sigma_+}$ and σ_+

+1 eigenvector of $\sqrt{\sigma_- \sigma_+}$ is mapped by $U = \sigma_x$ to null eigenvector of σ_+.