4.1. $E_\alpha = |\bar{\psi}_\alpha\rangle\langle\bar{\psi}_\alpha| = \mu_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|$

Completeness relation: $P = \sum_{\alpha=1}^{N} E_\alpha = \sum_{\alpha=1}^{N} |\bar{\psi}_\alpha\rangle\langle\bar{\psi}_\alpha|$

$\uparrow$ identity operator on D-dimensional space

(a) Expand $|\bar{\psi}_\alpha\rangle = \sum_{j=1}^{D} c_{\alpha j} |e_j\rangle, \quad c_{\alpha j} = \langle e_j | \bar{\psi}_\alpha\rangle$

$$\sum_{\alpha=1}^{N} c_{\alpha j} c_{\alpha k}^{*} = \sum_{\alpha=1}^{N} \langle e_j | \bar{\psi}_\alpha\rangle\langle\bar{\psi}_\alpha | e_k\rangle$$

$$= \langle e_j | \underbrace{\left(\sum_{\alpha=1}^{N} |\bar{\psi}_\alpha\rangle\langle\bar{\psi}_\alpha|\right)}_{P \,=\, \left(\substack{\text{identity operator on}\\ \text{D-dimensional space}}\right) \,=\, \sum_{j=1}^{D} |e_j\rangle\langle e_j| \,=\, \left(\substack{\text{projector onto}\\ \text{D-dimensional}\\ \text{subspace}}\right)} | e_k\rangle$$

$$= \langle e_j | e_k\rangle$$

$$= \delta_{jk}$$

$$\| c_{\alpha j} \| = \left(\begin{array}{cccc|ccc} c_{11} & c_{12} & \cdots & c_{1D} & c_{1,D+1} & \cdots & c_{1N} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{N1} & c_{N2} & \cdots & c_{ND} & c_{N,D+1} & \cdots & c_{NN} \end{array}\right)$$

D orthonormal columns

Tack on an additional N−D orthonormal columns

N×N unitary matrix

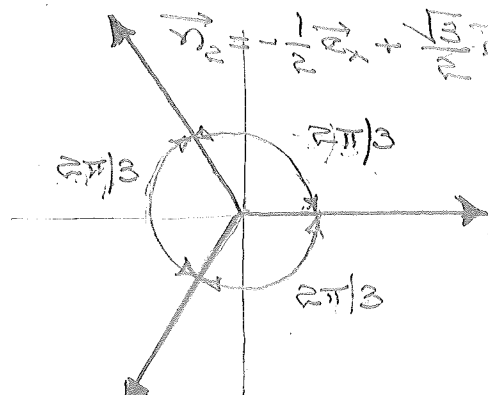
Add N-D dimensions spanned by orthonormal vectors $|e_{D+1}\rangle, \ldots, |e_N\rangle$. Notice that $P|e_\alpha\rangle = 0$, $\alpha = D+1, \ldots, N$; i.e., P projects onto the original D-dimensional space. Define an orthonormal basis

$$\boxed{|\hat{\psi}_\alpha\rangle = \sum_{\beta=1}^{N} c_{\alpha\beta}|e_\beta\rangle, \quad \alpha = 1, \ldots, N.}$$

This is a <u>Neumark extension</u> of the POVM because

$$\boxed{P|\hat{\psi}_\alpha\rangle = \sum_{\beta=1}^{N} c_{\alpha\beta} P|e_\beta\rangle = \sum_{j=1}^{D} c_{\alpha j}|e_j\rangle = |\psi_\alpha\rangle}$$

(b) <u>Trine</u>



$\vec{n}_2 = -\frac{1}{2}\vec{e}_x + \frac{\sqrt{3}}{2}\vec{e}_y$, $\theta = \pi/2$, $\phi = \pi/3$

$\vec{n}_1 = \vec{e}_x$, $\theta = \pi/2$, $\phi = 0$

$\vec{n}_3 = -\frac{1}{2}\vec{e}_x - \frac{\sqrt{3}}{2}\vec{e}_y$, $\theta = \pi/2$, $\phi = 2\pi/3$

$$|\vec{n}\rangle = \cos(\theta/2)|0\rangle + e^{+i\phi}\sin(\theta/2)|1\rangle$$

$$\boxed{\begin{aligned} |\vec{n}_1\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |\psi_1\rangle \\ |\vec{n}_2\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi/3}|1\rangle) = |\psi_2\rangle \\ |\vec{n}_3\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + e^{-i\pi/3}|1\rangle) = |\psi_3\rangle \end{aligned}} \longleftarrow \text{Normalized vectors}$$

$$E_1 = |\tilde{\psi}_1\rangle\langle\tilde{\psi}_1| = \frac{2}{3}|\vec{n}_1\rangle\langle\vec{n}_1| = \frac{1}{3}(I + \vec{n}_1\cdot\vec{\sigma})$$

$$E_2 = |\tilde{\psi}_2\rangle\langle\tilde{\psi}_2| = \frac{2}{3}|\vec{n}_2\rangle\langle\vec{n}_2| = \frac{1}{3}(I + \vec{n}_2\cdot\vec{\sigma})$$

$$E_3 = |\tilde{\psi}_3\rangle\langle\tilde{\psi}_3| = \frac{2}{3}|\vec{n}_3\rangle\langle\vec{n}_3| = \frac{1}{3}(I + \vec{n}_3\cdot\vec{\sigma})$$

$$\sum \vec{n}_\alpha = 0 \Rightarrow \sum E_\alpha = I$$

$$|\tilde{\psi}_1\rangle = \sqrt{\tfrac{2}{3}}\,|\vec{n}_1\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + |1\rangle\right)$$

$$|\tilde{\psi}_2\rangle = \sqrt{\tfrac{2}{3}}\,|\vec{n}_2\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + e^{i\pi/3}|1\rangle\right)$$

$$|\tilde{\psi}_3\rangle = \sqrt{\tfrac{2}{3}}\,|\vec{n}_3\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + e^{-i\pi/3}|1\rangle\right)$$

$$\|c_{\alpha\beta}\| = \left(\begin{array}{cc|c} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{3} & e^{i\pi/3}/\sqrt{3} & e^{-i\pi/3}/\sqrt{3} \\ 1/\sqrt{3} & e^{-i\pi/3}/\sqrt{3} & e^{+i\pi/3}/\sqrt{3} \end{array}\right)$$

Added row is unique up to a phase

$$|\hat{\psi}_1\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + |1\rangle + |2\rangle\right)$$

$$|\hat{\psi}_2\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + e^{i\pi/3}|1\rangle + e^{-i\pi/3}|2\rangle\right)$$

$$|\hat{\psi}_3\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + e^{-i\pi/3}|1\rangle + e^{i\pi/3}|2\rangle\right)$$

Neumark extension

(c) Tetrahedron

$\vec{n}_1 = \vec{e}_3$, $\quad \theta = 0$,

$\vec{n}_2 = \sqrt{\frac{8}{9}}\,\vec{e}_x - \frac{1}{3}\vec{e}_z$, $\quad \phi = 0$

$\vec{n}_3 = -\sqrt{\frac{2}{9}}\,\vec{e}_x + \sqrt{\frac{2}{3}}\,\vec{e}_y - \frac{1}{3}\vec{e}_z$, $\quad \phi = \pi/3$

$\vec{n}_4 = -\sqrt{\frac{2}{9}}\,\vec{e}_x - \sqrt{\frac{2}{3}}\,\vec{e}_y - \frac{1}{3}\vec{e}_z$, $\quad \phi = 2\pi/3$

$\cos\theta = -1/3$

$\cos(\theta/2) = 1/\sqrt{3}$

$\sin(\theta/2) = \sqrt{2/3}$

$$|\vec{n}\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$$

$$|\vec{n}_1\rangle = |0\rangle = |\psi_1\rangle$$

$$|\vec{n}_2\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle = |\psi_2\rangle$$

$$|\vec{n}_3\rangle = \frac{1}{\sqrt{3}}|0\rangle + e^{i\pi/3}\sqrt{\frac{2}{3}}|1\rangle = |\psi_3\rangle$$

$$|\vec{n}_4\rangle = \frac{1}{\sqrt{3}}|0\rangle + e^{-i\pi/3}\sqrt{\frac{2}{3}}|1\rangle = |\psi_4\rangle$$

Normalized vectors

$$E_\alpha = |\bar{\psi}_\alpha\rangle\langle\bar{\psi}_\alpha| = \frac{1}{2}|\psi_\alpha\rangle\langle\psi_\alpha| = \frac{1}{4}\left(I + \vec{n}_\alpha\cdot\vec{\sigma}\right)$$

$\sum_\alpha \vec{n}_\alpha = 0 \Rightarrow \sum_\alpha E_\alpha = I$

$$|\bar{\psi}_1\rangle = \frac{1}{\sqrt{2}}|\vec{n}_1\rangle = \frac{1}{\sqrt{2}}|0\rangle$$

$$|\bar{\psi}_2\rangle = \frac{1}{\sqrt{2}}|\vec{n}_2\rangle = \frac{1}{\sqrt{6}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle$$

$$|\bar{\psi}_3\rangle = \frac{1}{\sqrt{2}}|\vec{n}_3\rangle = \frac{1}{\sqrt{6}}|0\rangle + \frac{e^{i\pi/3}}{\sqrt{3}}|1\rangle$$

$$|\bar{\psi}_4\rangle = \frac{1}{\sqrt{2}}|\vec{n}_4\rangle = \frac{1}{\sqrt{6}}|0\rangle + \frac{e^{-i\pi/3}}{\sqrt{3}}|1\rangle$$

$$\|c_{\alpha\beta}\| = \left(\begin{array}{cc|cc} 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{6} & e^{i\pi/3}/\sqrt{3} & 1/\sqrt{6} & e^{-i\pi/3}/\sqrt{3} \\ 1/\sqrt{6} & e^{-i\pi/3}/\sqrt{3} & 1/\sqrt{6} & e^{i\pi/3}/\sqrt{3} \end{array}\right)$$

$$\boxed{\begin{aligned} |\hat{\psi}_1\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle - |2\rangle\right) \\ |\hat{\psi}_2\rangle &= \frac{1}{\sqrt{6}}\left(|0\rangle + |2\rangle\right) + \frac{1}{\sqrt{3}}\left(|1\rangle + |3\rangle\right) \\ |\hat{\psi}_3\rangle &= \frac{1}{\sqrt{6}}\left(|0\rangle + |2\rangle\right) + \frac{1}{\sqrt{3}}\left(e^{i\pi/3}|1\rangle + e^{-i\pi/3}|3\rangle\right) \\ |\hat{\psi}_4\rangle &= \frac{1}{\sqrt{6}}\left(|0\rangle + |2\rangle\right) + \frac{1}{\sqrt{3}}\left(e^{-i\pi/3}|1\rangle + e^{i\pi/3}|3\rangle\right) \end{aligned}}$$

Neumark extension