4.2.

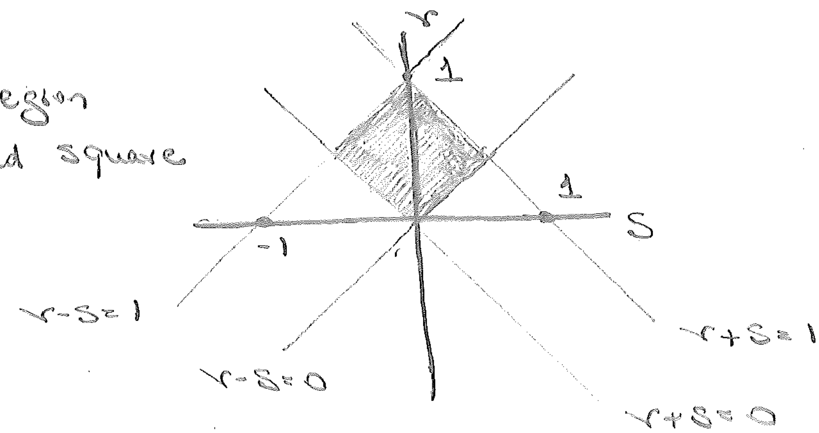
(a) $E = r\underbrace{\mathbb{1}}_{|\vec{n}\rangle\langle\vec{n}| + |-\vec{n}\rangle\langle-\vec{n}|} + s\underbrace{\vec{n}\cdot\vec{\sigma}}_{|\vec{n}\rangle\langle\vec{n}| - |-\vec{n}\rangle\langle-\vec{n}|}$, r and s real since E is Hermitian

$$= (r+s)|\vec{n}\rangle\langle\vec{n}| + (r-s)|-\vec{n}\rangle\langle-\vec{n}|$$

$E \geq 0 \Rightarrow r+s \geq 0$ and $r-s \geq 0$

$E \leq 1 \Rightarrow r+s \leq 1$ and $r-s \leq 1$

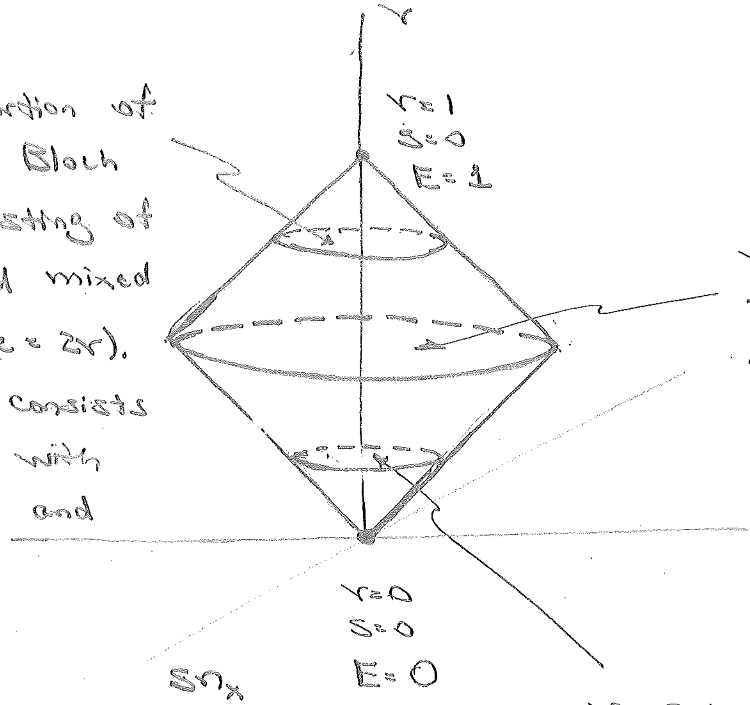
Allowed region is the shaded square



For each value of r, $s\vec{n}$ describes a ball of radius r for $0 \leq r \leq \frac{1}{2}$ and radius $1-r$ for $\frac{1}{2} \leq r \leq 1$. The convex set is a sequence of balls as r increases from 0 to 1. By suppressing one dimension of the balls and thus drawing them as disks, we can draw the convex set as a pair of cones:

r, $S \le 1-r$

This is a portion of an expanded Bloch Sphere consisting of supernormalized mixed states (trace = $2r$). The surface consists of operators with eigenvalues 1 and $2r-1$.



$r = \frac{1}{2}$, $0 \le S \le \frac{1}{2}$

This is the Bloch Sphere with 1-d projectors on the surface

r, $S \le r$

This is a contracted Bloch Sphere, with subnormalized mixed states in the interior (trace = $2r$) and multiples of 1-d projectors,

$$r(1+\vec{n}\cdot\vec{\sigma}) = 2r|\vec{n}\rangle\langle\vec{n}|,$$

on the surface.

It should be obvious that the extreme points are 0, 1, and all 1-d projectors.

(b) Write E in its eigendecomposition

$$E = \sum_{j=1}^{D} \lambda_j |e_j\rangle\langle e_j| = \sum_{j=1}^{D} \lambda_j P_j ,$$

where the eigenvalues are in decreasing order, i.e., $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_D$. Now write

$$E = \lambda_1 P_1 + \lambda_2 P_2 + \dots + \lambda_{D-1} P_{D-1} + \lambda_D P_D$$

$$= (\lambda_1 - \lambda_2) P_1 + (\lambda_2 - \lambda_3) P_1 + \dots + (\lambda_{D-1} - \lambda_D) P_1 + \lambda_D P_1$$

$$+ (\lambda_2 - \lambda_3) P_2 + \dots + (\lambda_{D-1} - \lambda_D) P_2 + \lambda_D P_2$$

$$\vdots \qquad\qquad \vdots$$

$$+ (\lambda_{D-1} - \lambda_D) P_{D-1} + \lambda_D P_{D-1}$$

$$+ \lambda_D P_D$$

$$= (\lambda_1 - \lambda_2) P_1 + (\lambda_2 - \lambda_3)(P_1 + P_2) + \dots + (\lambda_{D-1} - \lambda_D)(P_1 + \dots + P_{D-1})$$
$$+ \lambda_D (P_1 + \dots + P_D)$$

$$= \sum_{j=1}^{D} p_j \Pi_j \, , \qquad p_j = \lambda_j - \lambda_{j+1} \geq 0 \quad (\lambda_{D+1} = 0)$$

$$\Pi_j = \sum_{k=1}^{j} P_j = \begin{pmatrix} \text{rank-}j \text{ projector onto subspace} \\ \text{spanned by } |e_1\rangle, \dots, |e_j\rangle \end{pmatrix}$$

$$\sum_{j=1}^{D} p_j = \sum_{j=1}^{D} \lambda_j - \lambda_{j+1} = \sum_{j=1}^{D} \lambda_j - \sum_{j=2}^{D+1} \lambda_j = \lambda_1 - \lambda_{D+1} = \lambda_1 \leq 1$$

Now define $p_0 = 1 - \lambda_1 \geq 0$, so that

$$E = p_0 \, 0 + \sum_{j=1}^{D} p_j \Pi_j \quad \text{and} \quad \sum_{j=0}^{D} p_j = 1$$

E is a convex combination of projectors.

Can a projector be written as a proper combination? Suppose

$$\Pi = \lambda E + (1-\lambda) F , \quad 0 < \lambda < 1 .$$

$|\psi\rangle$ in null subspace of Π:

$$0 = \langle\psi|\Pi|\psi\rangle = \lambda \underbrace{\langle\psi|E|\psi\rangle}_{\geq 0} + (1-\lambda) \underbrace{\langle\psi|F|\psi\rangle}_{\geq 0} \geq 0$$

$$\Longrightarrow \quad 0 = \langle\psi|E|\psi\rangle = \langle\psi|F|\psi\rangle$$

$\Longrightarrow$ E and F have same support as Π.

$|\psi\rangle$ in support of Π:

$$1 = \langle\psi|\Pi|\psi\rangle = \lambda \underbrace{\langle\psi|E|\psi\rangle}_{\leq 1} + (1-\lambda) \underbrace{\langle\psi|F|\psi\rangle}_{\leq 1} \leq 1$$

$$\Longrightarrow \langle\psi|E|\psi\rangle = \langle\psi|F|\psi\rangle = 1$$

$$\Longrightarrow \quad E = F = \Pi .$$

$\therefore$ All projectors are extreme points.