4.3.

Von Neumann measurement of spin component along $\vec{n}$:

Operations: $\mathcal{A}_{\pm} = \underbrace{P_{\pm\vec{n}}}_{\text{Kraus operator}} \odot P_{\pm\vec{n}}$, $P_{\pm\vec{n}} = |\pm\vec{n}\rangle\langle\pm\vec{n}| = \frac{1}{2}(I + \vec{\sigma}\cdot\vec{n})$

POVM: $E_{\pm} = P_{\pm\vec{n}}$

(a) In our case, each of the 3 pairs of operations occurs with probability $\frac{1}{3}$, so

Operations	Kraus operators	POVM elements
$\mathcal{A}_{\pm\vec{e}_x} = \frac{1}{3} P_{\pm\vec{e}_x} \odot P_{\pm\vec{e}_x}$	$A_{\pm\vec{e}_x} = \frac{1}{\sqrt{3}} P_{\pm\vec{e}_x}$	$E_{\pm\vec{e}_x} = \frac{1}{3} P_{\pm\vec{e}_x}$
$\mathcal{A}_{\pm\vec{e}_y} = \frac{1}{3} P_{\pm\vec{e}_y} \odot P_{\pm\vec{e}_y}$	$A_{\pm\vec{e}_y} = \frac{1}{\sqrt{3}} P_{\pm\vec{e}_y}$	$E_{\pm\vec{e}_y} = \frac{1}{3} P_{\pm\vec{e}_y}$
$\mathcal{A}_{\pm\vec{e}_z} = \frac{1}{3} P_{\pm\vec{e}_z} \odot P_{\pm\vec{e}_z}$	$A_{\pm\vec{e}_z} = \frac{1}{\sqrt{3}} P_{\pm\vec{e}_z}$	$E_{\pm\vec{e}_z} = \frac{1}{3} P_{\pm\vec{e}_z}$

(b)

$$\frac{\mathcal{A}_{\pm}(\rho)}{p_{\pm}} = \rho_{\pm} = \sum_j \underbrace{p_{j|\pm}}_{p_{\pm j}/p_{\pm}} \underbrace{\rho_{\pm j}}_{\mathcal{A}_{\pm\vec{e}_j}(\rho)/p_{\pm j}}$$

$$= \frac{1}{p_{\pm}} \sum_j \mathcal{A}_{\pm\vec{e}_j}(\rho)$$

$$\Rightarrow \mathcal{A}_{\pm} = \sum_j \mathcal{A}_{\pm\vec{e}_j}$$

Completeness relation

$$\sum_{j,s=\pm} E_{s\vec{e}_j} = \sum_{j,s} \frac{1}{6}(I + s\vec{\sigma}\cdot\vec{n}_j) = I$$

$$\boxed{\mathcal{A}_{\pm} = \sum_j \mathcal{A}_{\pm\vec{e}_j} = \frac{1}{3}\left(P_{\pm\vec{e}_x} \odot P_{\pm\vec{e}_x} + P_{\pm\vec{e}_y} \odot P_{\pm\vec{e}_y} + P_{\pm\vec{e}_z} \odot P_{\pm\vec{e}_z}\right)}$$

Kraus operators:

$A_{+\vec{e}_j} = \frac{1}{\sqrt{3}} P_{\vec{e}_j} = \frac{1}{2\sqrt{3}}(I + \sigma_j)$, $j = 1,2,3$

$A_{-\vec{e}_j} = \frac{1}{\sqrt{3}} P_{-\vec{e}_j} = \frac{1}{2\sqrt{3}}(I - \sigma_j)$, $j = 1,2,3$

POVM elements: $E_+ = \frac{1}{3}\sum_j P_{\vec{e}_j} = \frac{1}{2}\left(I + \vec{\sigma}\cdot\frac{1}{3}(\vec{e}_x+\vec{e}_y+\vec{e}_z)\right)$

$E_- = \frac{1}{3}\sum_j P_{-\vec{e}_j} = \frac{1}{2}\left(I - \vec{\sigma}\cdot\frac{1}{3}(\vec{e}_x+\vec{e}_y+\vec{e}_z)\right)$

(c) 3 examples of <u>different</u> quantum operations that have POVM E_j

① $\mathcal{Q}_\pm = \sqrt{E_\pm}\odot\sqrt{E_\pm}$

↑ Kraus operator

This is some sort of an imprecise measurement of the spin component along $\vec{n}$.

Unit vector $\vec{n} = \frac{1}{\sqrt{3}}(\vec{e}_x+\vec{e}_y+\vec{e}_z)$

$$E_\pm = \frac{1}{2}\left(I \pm \frac{1}{\sqrt{3}}\vec{\sigma}\cdot\vec{n}\right) = \frac{1}{2}\left(1\pm\frac{1}{\sqrt{3}}\right)|\vec{n}\rangle\langle\vec{n}| + \frac{1}{2}\left(1\mp\frac{1}{\sqrt{3}}\right)|-\vec{n}\rangle\langle-\vec{n}|$$

$$\sqrt{E_\pm} = \frac{1}{\sqrt{2}}\left(1\pm\frac{1}{\sqrt{3}}\right)^{1/2}|\vec{n}\rangle\langle\vec{n}| + \frac{1}{2}\left(1\mp\frac{1}{\sqrt{3}}\right)|-\vec{n}\rangle\langle-\vec{n}|$$

② $\mathcal{Q}_\pm = \frac{1}{2}\left(1\pm\frac{1}{\sqrt{3}}\right)P_{\vec{n}}\odot P_{\vec{n}} + \frac{1}{2}\left(1\mp\frac{1}{\sqrt{3}}\right)P_{-\vec{n}}\odot P_{-\vec{n}}$

This is the measurement where one measures the spin component along $\vec{n}$, using the "right" orientation, i.e., along $\vec{n}$, with probability $\frac{1}{2}(1+1/\sqrt{3})$, and "wrong" orientation, i.e., along $-\vec{n}$, with probability $\frac{1}{2}(1-1/\sqrt{3})$, but recording only the result $\pm$ and not the orientation of the apparatus.

Kraus operators: $\mathcal{Q}_+ = A_{++}\odot A_{++}^\dagger + A_{+-}\odot A_{+-}^\dagger$

$$A_{++} = \frac{1}{\sqrt{2}}(1+1/\sqrt{3})^{1/2}P_{\vec{n}}$$

$$A_{+-} = \frac{1}{\sqrt{2}}(1-1/\sqrt{3})^{1/2}P_{-\vec{n}}$$

$$A_{++}^\dagger A_{++} + A_{+-}^\dagger A_{+-} = E_+$$

$$Q_- = A_{-+} \odot A_{-+}^\dagger + A_{--} \odot A_{--}^\dagger$$

$$A_{-+} = \frac{1}{\sqrt{2}}(1 - 1/\sqrt{3})^{1/2} P_{\vec{n}}$$

$$A_{--} = \frac{1}{\sqrt{2}}(1 + 1/\sqrt{3})^{1/2} P_{-\vec{n}}$$

$$A_{-+}^\dagger A_{-+} + A_{--}^\dagger A_{--} = E_-$$

(a)

$$A_{+\vec{e}_j} = \frac{1}{\sqrt{3}} |0\rangle\langle \vec{e}_j|$$

$$A_{-\vec{e}_j} = \frac{1}{\sqrt{3}} |0\rangle\langle -\vec{e}_j|$$

Kraus operators

$$Q_+ = \sum_j A_{+\vec{e}_j} \odot A_{+\vec{e}_j}^\dagger = |0\rangle\left(\frac{1}{3}\sum_j \langle \vec{e}_j | \odot | \vec{e}_j\rangle\right)\langle 0|$$

$$Q_- = \sum_j A_{-\vec{e}_j} \odot A_{-\vec{e}_j}^\dagger = |0\rangle\left(\frac{1}{3}\sum_j \langle -\vec{e}_j | \odot | -\vec{e}_j\rangle\right)\langle 0|$$

Post-measurement state is always $|0\rangle$

$$\sum_j A_{\pm\vec{e}_j}^\dagger A_{\pm\vec{e}_j} = \frac{1}{3}\sum_j |\pm\vec{e}_j\rangle\langle \pm\vec{e}_j| = E_\pm$$