PHYSICAL REVIEW A, VOLUME 65, 012320

From classical state swapping to quantum teleportation

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(Received 29 May 2001; published 14 December 2001)

The quantum teleportation protocol is extracted directly out of a standard classical circuit that exchanges the states of two qubits using only controlled-NOT gates. This construction of teleportation from a classically transparent circuit generalizes straightforwardly to d-state systems.

DOI: 10.1103/PhysRevA.65.012320 PACS number(s): 03.67.Hk, 03.67.Lx

Quantum teleportation [1] transfers the quantum state of a two-state system (Alice's qubit, the source) to another remote two-state system (Bob's qubit, the destination) without any direct dynamical coupling between the two qubits. To do this trick Alice, who in general does not herself know the form of the state to be transferred, must possess a third qubit (the ancilla) that initially is maximally entangled with Bob's qubit in the two-qubit state

$$\frac{1}{\sqrt{2}}(|0\rangle|0\rangle+|1\rangle|1\rangle). \tag{1}$$

Depending on the outcomes of appropriate measurements on the source and ancilla, Alice can send Bob instructions that enable him to transform the state of the destination into that originally possessed by the source. The term "teleportation" is apt because the measurements that provide the information to recreate the state at the destination obliterate all traces of it from the source.

If two qubits are allowed to interact, however, then their states can be exchanged in a much less subtle way, with the help of three controlled-NOT (cNOT) gates [2]. The action of these gates can be understood in entirely classical terms. This is illustrated in Fig. 1.

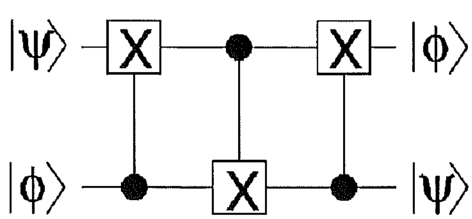

FIG. 1.

That the classical [3] circuit in Fig. 1 does indeed exchange states is readily confirmed by letting it act on a general computational basis state $|x\rangle|y\rangle$. If x is the value (0 or 1) of the control bit and y is the value of the target bit, then the action of a single controlled-NOT gate can be compactly summarized as

$$|x\rangle|y\rangle\rightarrow|x\rangle|y\oplus x\rangle, \tag{2}$$

where $\oplus$ denotes addition modulo 2. If $|\psi\rangle=|x\rangle$ and $|\phi\rangle=|y\rangle$, then the action of the three successive gates in Fig. 1 is (reading the figure from left to right)

$$|x\rangle|y\rangle\rightarrow|x\oplus y\rangle|y\rangle\rightarrow|x\oplus y\rangle|x\rangle\rightarrow|y\rangle|x\rangle. \tag{3}$$

This process makes perfect sense for classical bits, as well as for quantum superpositions of classical bits, to which it extends by linearity.

If the state $|\phi\rangle$ in Fig. 1 is taken to be $|0\rangle$, then the controlled-NOT gate on the left acts as the identity, so the classical state-swapping circuit simplifies to

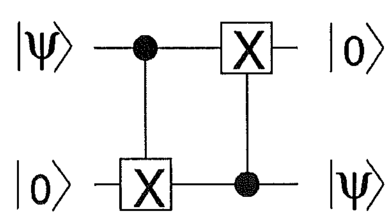

FIG. 2.

If the upper qubit (source) in Fig. 2 belongs to Alice and the lower qubit (destination) to Bob, then this special case of the general classical state-swapping circuit provides a considerably simpler version of what happens in quantum teleportation. But the classical circuit in Fig. 2 is not teleportation, because it requires direct dynamical couplings between the qubits—couplings that teleportation manages to avoid by the use of an entangled pair of qubits and the classical communication of quantum measurement outcomes.

This paper illuminates the way in which quantum mechanics obviates the need for the direct dynamical couplings in Fig. 2, showing explicitly how this intuitive classical state-swapping circuit leads directly to the transference of a state between uncoupled qubits that constitute quantum teleportation. It is possible to eliminate all direct couplings between the source and the destination because quantum qubits have a richer range of logical capabilities than do classical bits. Only one indirect dynamical coupling between Alice and Bob survives this process of elimination as the initial interaction necessary to entangle Alice's ancilla with the Bob's destination qubit. All other direct dynamical coupling is replaced by classical communication.

The key to relate quantum teleportation to the apparently quite different way of exchanging a general state in Fig. 2 is to replace the controlled-NOT gate on the left of Fig. 2 with an elementary classical circuit, only slightly more elaborate than that of Fig. 1, that changes the direct coupling of the controlled-NOT into four couplings, all acting only through the intermediary of an unaltered ancillary qubit.

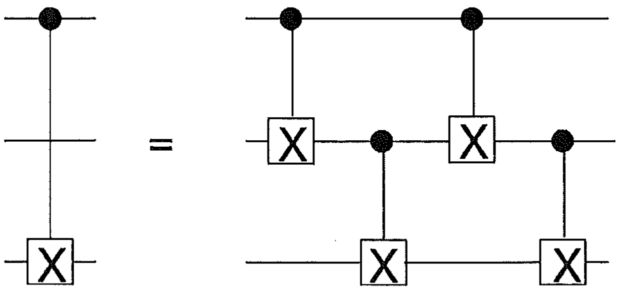

FIG. 3.

To confirm this identity note that the four gates on the right act on the eight computational basis states $|x\rangle|y\rangle|z\rangle$ (with $|x\rangle$ the input state on the top left, $|z\rangle$ on the bottom, and $|y\rangle$ in the middle) as follows [4]:

$$|x\rangle|y\rangle|z\rangle \rightarrow |x\rangle|y\oplus x\rangle|z\rangle \rightarrow |x\rangle|y\oplus x\rangle|z\oplus y\oplus x\rangle$$
$$\rightarrow |x\rangle|y\rangle|z\oplus y\oplus x\rangle \rightarrow |x\rangle|y\rangle|z\oplus x\rangle. \quad (4)$$

Thus the circuit on the right of Fig. 3 does indeed act as indicated on the left, performing a controlled-NOT operation on the qubits associated with the top (control) and bottom (target) wires, while acting as the identity on the qubit associated with the middle wire.

Quantum mechanics first appears when we interchange control and target in the controlled-NOT gate on the right of Fig. 2, using the quantum circuit identity

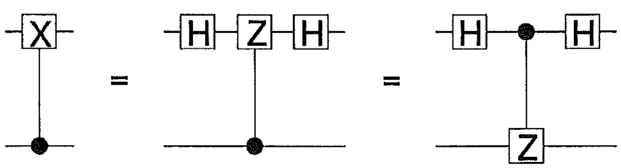

FIG. 4.

This follows from the fact that the unitary, self-inverse, Hadamard operator $\mathsf{H}=1/\sqrt{2}(\sigma_x+\sigma_z)$ takes eigenstates of $\mathsf{X}=\sigma_x$ into eigenstates of $\mathsf{Z}=\sigma_z$ with corresponding eigenvalues, and vice versa

$$\mathsf{H}:\quad |0\rangle \leftrightarrow (|0\rangle+|1\rangle), \quad |1\rangle \leftrightarrow \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle), \quad (5)$$

together with the fact that controlled-Z gate has the same action regardless of which qubit is the target and which the control [5]. The utility of this interchange emerges below.

So if we introduce an ancilla in a state $|\chi\rangle$, to be specified in a moment, we can replace the two gates in Fig. 2, with the equivalent circuits of Figs. 3 and 4, to get

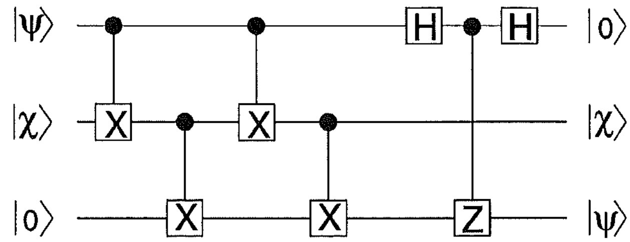

FIG. 5.

I emphasize that Fig. 5 is merely a cumbersome way of constructing the classical circuit of Fig. 2, with the direct coupling on the left of Fig. 2 replaced by the four gates on the left, mediated by an ancillary qubit whose state is unaltered, and the direct coupling on the right replaced by the three gates on the right, which by exploiting the quantum-mechanical H gates make it possible to interchange control and target qubits.

To further convert the circuit of Fig. 5 into teleportation, we must first eliminate the unacceptable leftmost coupling between the source and the ancilla. This can be done by taking the state $|\chi\rangle$ of the ancilla to be $\mathsf{H}|0\rangle$, which the magic of quantum mechanics—this is the second place where it appears—allows to be invariant under the NOT operation. Because

$$\mathsf{XH}|0\rangle=\mathsf{H}|0\rangle, \quad (6)$$

the leftmost controlled-X gate in Fig. 5 always acts as the identity, and can be removed from the circuit. So Fig. 5 becomes

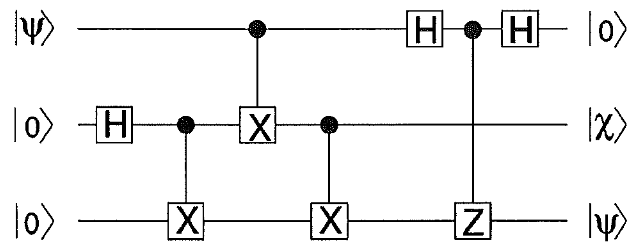

FIG. 6.

To see that Fig. 6 represents quantum-teleportation note that we can also remove the final Hadamard transformation on the upper wire in Fig. 6, provided we change the final state of the qubit associated with that wire from $|0\rangle$ to $\mathsf{H}^{-1}|0\rangle=\mathsf{H}|0\rangle=|\chi\rangle$. Because the remaining Hadamard gate on the upper wire commutes with the controlled-NOT gate that immediately precedes it on the lower two wires, we may also exchange the order of these two gates. The result is

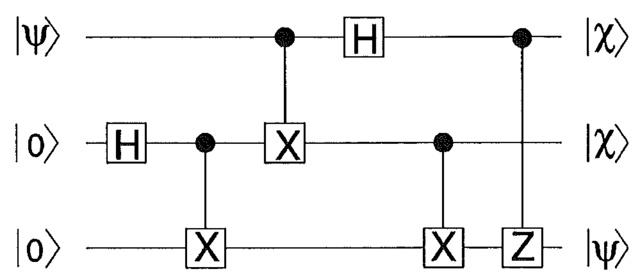

FIG. 7.

This is precisely the reversible quantum-teleportation circuit described by Brassard, Braunstein, and Cleve (BBC) [6]. We have thus made a direct passage from the classical circuit of Fig. 2, which requires coupling between source and destination to swap their states, to the BBC quantum-teleportation circuit of Fig. 7, which, as reviewed below, can be further modified to remove all remaining coupling.

I repeat BBC's description of the connection between the circuit of Fig. 7 and teleportation, to indicate what has become of the couplings originally present in Fig. 2 and to show that the four controlled-NOT gates arising from the classical expansion in Fig. 3 of the first controlled-NOT gate in Fig. 2 now play roles in three distinct stages of the quantum teleportation process [7].

The controlled-NOT gate on the left in Fig. 7, along with the Hadamard gate immediately to its left, used to eliminate the fourth controlled-NOT gate from Fig. 3, serve to turn the state of the ancilla and destination into the maximally entangled state $1/\sqrt{2}(|0\rangle|0\rangle+|1\rangle|1\rangle)$. After these two gates have acted, Alice keeps the ancilla and Bob takes the destination to a faraway place. Only after this does Alice need to acquire the source, in the state $|\Psi\rangle$, which may or may not be known to her.

The effect of the next controlled-NOT gate and Hadamard gate of Fig. 7 on the source and ancilla, both in Alice's possession, is to transform unitarily the four mutually orthogonal maximally entangled states of the Bell basis [8] into the four computational basis states $|x\rangle|y\rangle$. If Alice's two qubits were to be measured in the computational basis after the action of the first four gates, the measurement could, therefore, be viewed as a coherent two-qubit measurement in the Bell basis, taking place immediately after the first two gates [9].

Such measurements in the computational basis, which are the third and final place where quantum mechanics enters the process, can be introduced, though initially at the wrong stage of the process, by noting that in the final state on the right of Fig. 7. Alice's two qubits are each in the pure state $|\chi\rangle$, completely disentangled from Bob's. As a result, the state of Bob's qubit is entirely unaffected if Alice measures each of her qubits. So we can safely add two measurements to Fig. 7 without disrupting the transfer of $|\psi\rangle$ from Alice's qubit to Bob's,

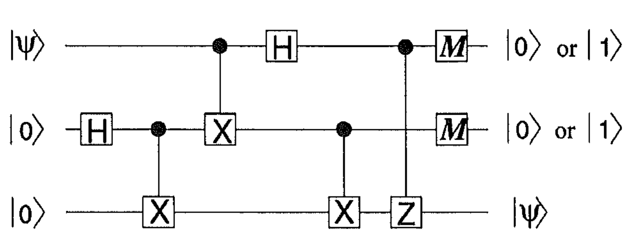

FIG. 8.

Not only do these measurements occur too late in the process, but there also remain in Fig. 8 two other interactions between Alice's qubit or her ancilla and Bob's, besides the controlled-NOT gate that originally entangles her ancilla with his destination. The controlled-Z gate on the right comes directly from the controlled-X gate on the right of Fig. 2, and the controlled-X gate immediately preceding it comes from the last of the four controlled-X gates on the right of Fig. 3. Both these interactions can be replaced by classical communication of measurement results from Alice to Bob, by moving the measurements to the earlier stage of the process mentioned above, which it is possible to do for the following reason.

Quite generally the effect of a controlled unitary operation on any number of qubits followed by a measurement of the control qubit is unaltered if the measurement of the control qubit precedes the controlled operation [10]

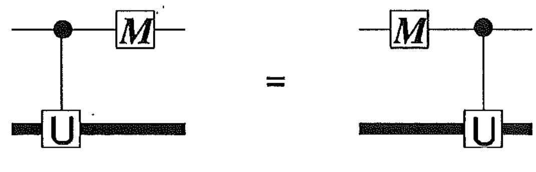

FIG. 9.

Here the heavy horizontal wire represents N additional qubits, and U represents a unitary transformation acting on any or all of those qubits, controlled by the single qubit represented by the light wire.

The measurement and the controlled-unitary operation commute because an arbitrary input state $|\Psi\rangle$ of the $N+1$ qubits is necessarily of the form

$$|\Psi\rangle=a|0\rangle|\Phi_0\rangle+b|1\rangle|\Phi_1\rangle, \tag{7}$$

where $|a|^2+|b|^2=1$, $|0\rangle$ and $|1\rangle$ are computational basis states of the control qubit, and $|\Phi_0\rangle$ and $|\Phi_1\rangle$ are normalized (but in general nonorthogonal) states of the other N qubits. An immediate measurement on the control qubit takes $|\Psi\rangle$ into $|0\rangle|\Phi_0\rangle$ with probability $|a|^2$, or into $|1\rangle|\Phi_1\rangle$ with probability $|b|^2$ [11]. In the first case, subsequent application of a controlled-U gate has no further effect; in the second case it produces the state $|1\rangle\mathsf{U}|\Phi_1\rangle$.

On the other hand an immediate application of the controlled-U operation takes $|\Psi\rangle$ into

$$a|0\rangle|\Phi_0\rangle+b|1\rangle\mathsf{U}|\Phi_1\rangle \tag{8}$$

and a subsequent measurement of the control qubit takes this state into $|0\rangle|\Phi_0\rangle$ with probability $|a|^2$, or $|1\rangle\mathsf{U}|\Phi_1\rangle$ with probability $|b|^2$. Thus the two output states are the same and occur with the same probabilities, regardless of the order in which the measurement and controlled-U operations are performed.

Figure 9 allows Fig. 8 to be rewritten as

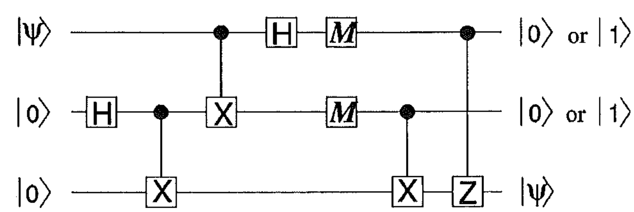

FIG. 10.

which shifts the actual measurements to the position of the hypothetical measurements mentioned above. Since the controlled-X or controlled-Z gates in Fig. 10 now follow a measurement of the control bit, their action is identical to applying the X or Z to the target qubit if and only if the outcome of the corresponding measurement is 1; i.e., the controlled operation can be executed locally by Bob depending on what Alice tells him about the outcomes of the two measurements she made on her own qubits.

To summarize, we can look at the teleportation protocol of Fig. 10, and ask what became of the original three couplings in the general classical state-swapping protocol of Fig. 1. The coupling on the left of Fig. 1 vanished by virtue of the initial choice $|0\rangle$ for the state of the destination (bottom wire of Fig. 10). The middle coupling of Fig. 1 survives in the three controlled-NOT gates coupled to the ancilla (middle wire) in Fig. 10 [12]. Two of the three controlled-NOT gates that remain do indeed provide links from Alice's qubits to the destination. But one gate (on the left of Fig. 10) operates only to create the initial entanglement of the ancilla with the destination, while the other gate (on the right) operates only through Alice's telling Bob, depending on the result of her measurement on the ancilla, whether or not to apply the transformation X to the destination [13]. The coupling on the right of Fig. 1 survives as the transformation Z applied to the destination or not by Bob depending on what Alice tells him about the result of her measurement on the source.

So you can take the BBC circuit of Fig. 7 and look back to its classical ancestry (Fig. 1) or forward to conventional teleportation (Fig. 10), seeing the same controlled-NOT gates play entirely different roles, depending on which way you want to view the circuit, rather like an optical illusion or a piece of kinetic sculpture. Depending on how you put the punctuation marks into a sequence of operations, you can get a process that is either entirely classical or deeply quantum mechanical.

This view of teleportation as a quantum-mechanical deconstruction of a trivial classical state-swapping circuit generalizes readily from qubits to d-state systems (qudits). If we are dealing with a d-valued classical register, we can generalize the controlled-NOT gate to the controlled bit rotation

$$\mathsf{cX}: \quad |x\rangle|y\rangle \to |x\rangle|y\oplus x\rangle, \quad 0\le x,y<d, \tag{9}$$

where $\oplus$ now denotes addition modulo d. This extends by linearity to a unitary operation on quantum d-state systems, which is only self-inverse when $d=2$. In the general case the inverse is

$$\mathsf{cX}^\dagger: \quad |x\rangle|y\rangle \to |x\rangle|y\ominus x\rangle, \quad 0\le x,y<d, \tag{10}$$

where $\ominus$ denotes subtraction modulo d. The classical circuits of Figs. 2 and 3 thus become

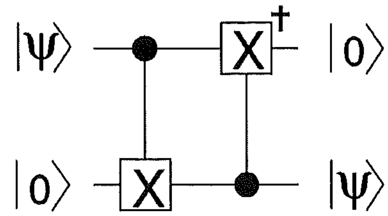

FIG. 11.

and

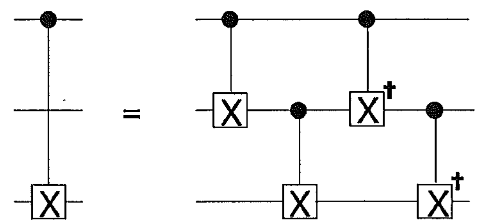

FIG. 12.

We generalize the Hadamard transformation H on a single qubit to the quantum Fourier transform F on a single d-state system

$$\mathsf{F}:|y\rangle \to \frac{1}{\sqrt{d}}\sum_z e^{2\pi i zy/d}|z\rangle, \tag{11}$$

and its inverse

$$\mathsf{F}^\dagger:|y\rangle \to \frac{1}{\sqrt{d}}\sum_z e^{-2\pi i zy/d}|z\rangle. \tag{12}$$

Note that $\mathsf{F}|0\rangle = \mathsf{F}^\dagger|0\rangle$ is invariant under an arbitrary bit rotation so that

$$(\mathsf{cX})(1\otimes \mathsf{F})|\psi\rangle|0\rangle = |\psi\rangle|0\rangle. \tag{13}$$

[This is the generalization of Eq. (6) from qubits to qudits.] A maximally entangled state is prepared by

$$(\mathsf{cX})(\mathsf{F}\otimes 1)|0\rangle|0\rangle = \frac{1}{\sqrt{d}}\sum_z |z\rangle|z\rangle. \tag{14}$$

An appropriate generalization to d-state systems of controlled-σ_z operation is

$$\mathsf{cZ}:|x\rangle|y\rangle \to e^{-2\pi i xy/d}|x\rangle|y\rangle, \tag{15}$$

which remains symmetric in control and target qubits and has the inverse

$$\mathrm{cZ}^{\dagger}: |x\rangle|y\rangle \rightarrow e^{2\pi i xy/d}|x\rangle|y\rangle. \tag{16}$$

In the above definitions of cX, $\mathrm{cX}^{\dagger}$, cZ, $\mathrm{cZ}^{\dagger}$, the state on the left is the control and the state on the right is the target. More generally, in the relations below, let $(\mathrm{cX})_{ij}$ denote a cX operation in which state i is the control and state j is the target, and let $(\mathrm{F})_i$ denote a Fourier transform acting on state i.

One easily verifies that

$$(\mathrm{cX})_{12}(\mathrm{F})_2 = (\mathrm{F})_2(\mathrm{cZ})_{12} \tag{17}$$

and, therefore,

$$\mathrm{cX}_{12} = (\mathrm{F})_2(\mathrm{cZ})_{12}(\mathrm{F}^{\dagger})_2, \tag{18}$$

so

$$(\mathrm{cX}^{\dagger})_{12} = (\mathrm{F})_2(\mathrm{cZ}^{\dagger})_{12}(\mathrm{F}^{\dagger})_2 = (\mathrm{F})_2(\mathrm{cZ}^{\dagger})_{21}(\mathrm{F}^{\dagger})_2, \tag{19}$$

which has the circuit representation (the generalization of Fig. 4) [14],

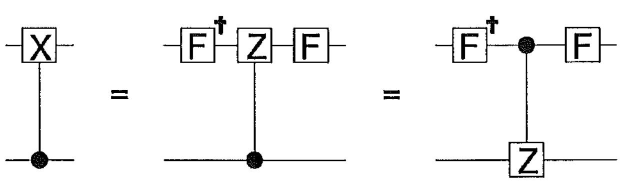

FIG. 13.

Therefore, following the same sequence of expansions as in the case of two-state systems, we arrive at the generalization of the BBC circuit of Fig. 7,

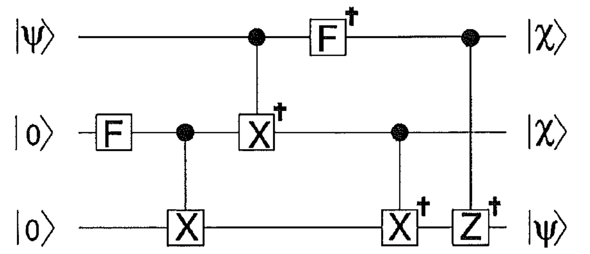

FIG. 14.

where

$$|\chi\rangle = \mathrm{F}|0\rangle = \mathrm{F}^{\dagger}|0\rangle. \tag{20}$$

One can go from this to the generalization of Fig. 10

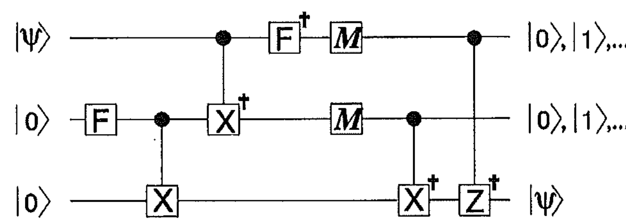

FIG. 15.

since the remark [10], that measurement of several control qubits commutes with multiqubit controlled operations, applies equally well to d-state systems even when d is not a power of 2.

The teleportation circuit of Fig. 15 for d-state systems neatly encapsulates the protocol for teleporting d-state systems spelled out in the original teleportation paper [1], along with its relation to the protocol of Fig. 10 for teleporting qubits.

I thank Gilles Brassard and Igor Devetak for useful comments on an earlier version of this paper, and Chris Fuchs for asking why I found it interesting. This work was supported by the National Science Foundation, Grant Nos. PHY9722065 and PHY0098429.

[1] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).

[2] The unitary controlled-NOT gate operates on the computational basis, i.e., the basis of classically meaningful states, $|0\rangle|0\rangle$, $|0\rangle|1\rangle$, $|1\rangle|0\rangle$, $|1\rangle|1\rangle$, as the identity of the state of the control qubit (indicated by a black dot in Fig. 1) is $|0\rangle$, and flips the state ($|0\rangle \leftrightarrow |1\rangle$) of the target qubit (indicated by the boxed X in Fig. 1) if the state of the control qubit is $|1\rangle$.

[3] I call a quantum circuit classical if it is classically meaningful when restricted to classically meaningful states, i.e., if every unitary gate takes computational-basis states into other computational basis states without introducing superpositions or phases. Because the circuit of Fig. 1 exchanges computational-basis states and acts linearly on superpositions of inputs, it also, of course, exchanges arbitrary quantum states.

[4] Alternatively one can note, in the computational basis, that if the state of the top wire is $|0\rangle$ then neither of the NOT operations act on the middle wire so the two lower self-inverse controlled-NOT operations act in direct succession, giving the identity. But if the state of the top wire is $|1\rangle$ then both NOT operations act on the middle wire, leaving its state unaltered, and ensuring that exactly one controlled-NOT operation acts on the lower wire regardless of that state.

[5] In either case controlled-Z gate acts as the identity on the computational basis states $|0\rangle|0\rangle$, $|0\rangle|1\rangle$, $|1\rangle|0\rangle$ and mul-

tiplies $|1\rangle|1\rangle$ by -1.

[6] Gilles Brassard, Samuel L. Braunstein, and Richard Cleve, Physica D **120**, 43 (1998); e-print quant-ph/9605035. BBC prefer to expand Z as HXH.

[7] It is also necessary to retrace this familiar ground to confirm that it supports the generalization to *d*-state systems described at the end of this paper.

[8] The Bell-basis states are $1/\sqrt{2}(|0\rangle|0\rangle \pm |1\rangle|1\rangle)$ and $1/\sqrt{2}(|0\rangle|1\rangle \pm |1\rangle|0\rangle)$. It is easiest to see that the controlled-NOT and Hadamard gates have this affect by looking at the inverse transformation.

[9] Conventional expositions of teleportation do indeed expand the state of Alice's two qubits in the Bell basis after the entangled pair is formed, having her then make a coherent two-qubit measurement in that basis. But it is simpler analytically when algebraically tracing the progress of a general $|\psi\rangle$ through the protocol, as well as more straightforward to implement physically, to take seriously the circuit of BBC, letting Alice explicitly apply the next controlled-NOT and Hadamard gates and follow this by independent qubit measurements in the ordinary computational basis. As BBC note, there is no need to mention the Bell basis at all.

[10] This is a straightforward extension to more than two qubits of the point made by R. B. Griffiths and C. S. Niu, Phys. Rev. Lett. **76**, 3228 (1996); e-print quant-ph/9511007, and invoked by BBC. The same situation holds for a unitary operation controlled by the 2^M different outcomes of a measurement on M control qubits. Such an operation has the form $\mathcal{U}=\Sigma_i \mathsf{P}_i \mathsf{U}_i$ where the $\mathsf{P}_i = |\Phi_i\rangle\langle\Phi_i|$ project onto a complete orthonormal set of states $|\Phi_i\rangle$ of the control bits, and U_i is the unitary transformation on the N target bits associated with the ith measurement outcome. (Since the U_i are unitary and the P_i commute with all the U_j and give a resolution of the identity into orthogonal projections, it follows that $\mathcal{U}$ is indeed unitary.) Clearly performing the von Neumann measurement associated with the P_i commutes with applying $\mathcal{U}$, in the sense that the same final states arise with the same probabilities.

[11] This extension of the Born's probability rule to cases in which only a subsystem is measured, which is crucial in quantum computation, receives surprisingly little explicit attention in most textbook introductions to quantum mechanics.

[12] The very first of the four controlled-NOT gates coming from the expansion in Fig. 3 of the middle coupling of Fig. 1 was crucially rendered unnecessary by the initial choice $\mathsf{H}|0\rangle$ for the state of the ancilla.

[13] The remaining controlled-NOT gate in Fig. 10 links Alice's qubit only to her ancilla. It can be viewed, if one wishes, as a part of the process of "measurement in the Bell basis."

[14] Note the unfortunate but firmly entrenched convention that in circuit diagrams operations on the left act first while in equations operations on the right act first.