5.3.

(a) The Neumark extension is constructed by expanding the POVM states in the standard basis:

$$|\bar{\psi}_{\vec{y}}\rangle \otimes |0\rangle^{\otimes(n-1)} = \sum_{x_1} |x_1 0\cdots 0\rangle\langle x_1 0\cdots 0|\bar{\psi}_{\vec{y}} 0\cdots 0\rangle$$

$$= \sum_{x_1} |x_1 0\cdots 0\rangle \underbrace{\langle x_1|\bar{\psi}_{\vec{y}}\rangle}_{\equiv\, U_{\vec{y},x_1}}$$

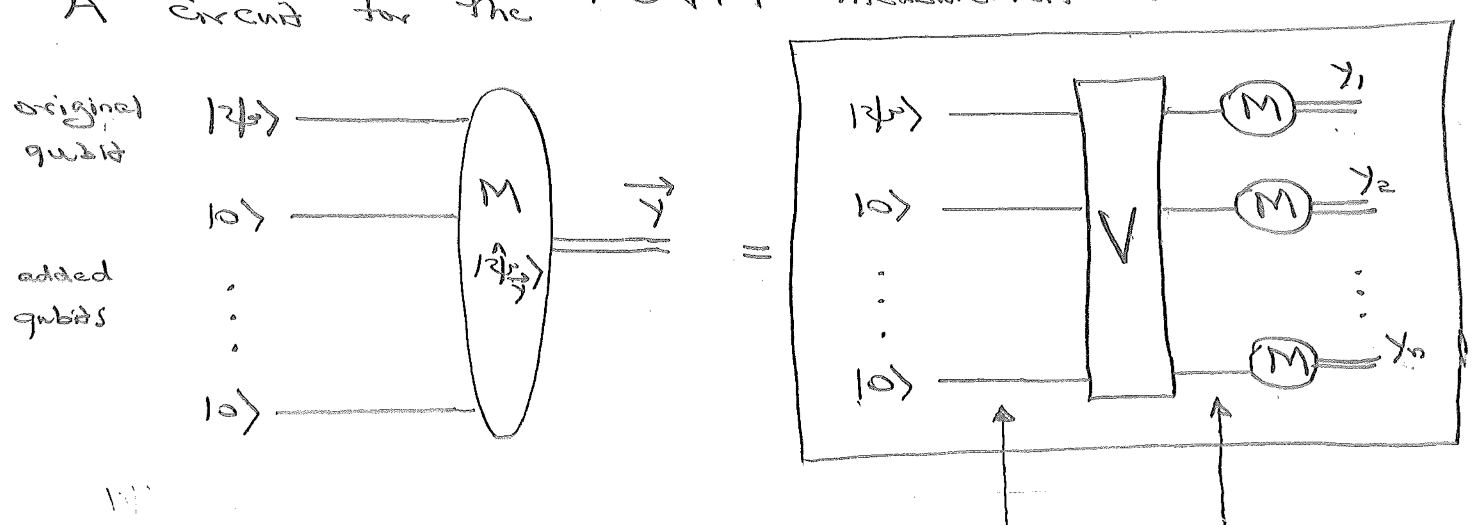
What we showed in a previous homework is that these are the 1st two columns, $U_{\vec{y},x_1 0\cdots 0}$, of a unitary matrix $U_{\vec{y}\vec{x}}$, which is used to construct a Neumark extension

$$|\hat{\psi}_{\vec{y}}\rangle = \sum_{\vec{x}} \underbrace{U_{\vec{y}\vec{x}}}_{\equiv \langle\vec{x}|V^\dagger|\vec{y}\rangle} |\vec{x}\rangle = V^\dagger|\vec{y}\rangle$$

unitary operator with $V_{\vec{x}\vec{y}} = (V^\dagger)^*_{\vec{y}\vec{x}} = U^*_{\vec{x}\vec{y}}$.

$$\boxed{P|\hat{\psi}_{\vec{y}}\rangle = |\bar{\psi}_{\vec{y}}\rangle \otimes |0\rangle^{\otimes(n-1)}}$$

A circuit for the POVM measurement is



$$|\psi\rangle \otimes |0\rangle^{\otimes(n-1)} = \sum_{\vec{y}} |\hat{\psi}_{\vec{y}}\rangle \underbrace{\langle \hat{\psi}_{\vec{y}} | \left(|\psi\rangle \otimes |0\rangle^{\otimes(n-1)} \right)}$$

$$= \langle \hat{\psi}_{\vec{y}} | P | \left(|\psi\rangle \otimes |0\rangle^{\otimes(n-1)} \right)$$

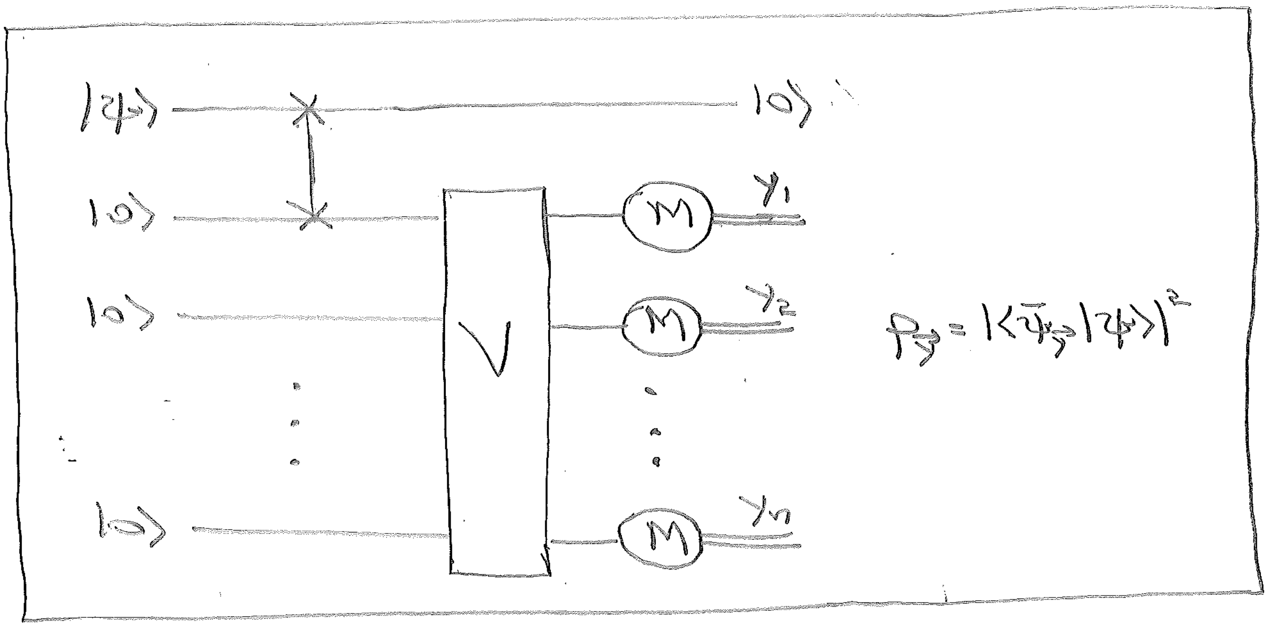
$$= \langle \bar{\psi}_{\vec{y}} | \psi \rangle$$

$$= \sum_{\vec{y}} |\hat{\psi}_{\vec{y}}\rangle \langle \bar{\psi}_{\vec{y}} | \psi \rangle$$

$$V |\psi\rangle \otimes |0\rangle^{\otimes(n-1)} = \sum_{\vec{y}} |\vec{y}\rangle \langle \bar{\psi}_{\vec{y}} | \psi \rangle$$

The meters read out the successive bits of $\vec{y}$, with overall probabilities, $p_{\vec{y}} = |\langle \bar{\psi}_{\vec{y}} | \psi \rangle|^2$, given by the POVM.

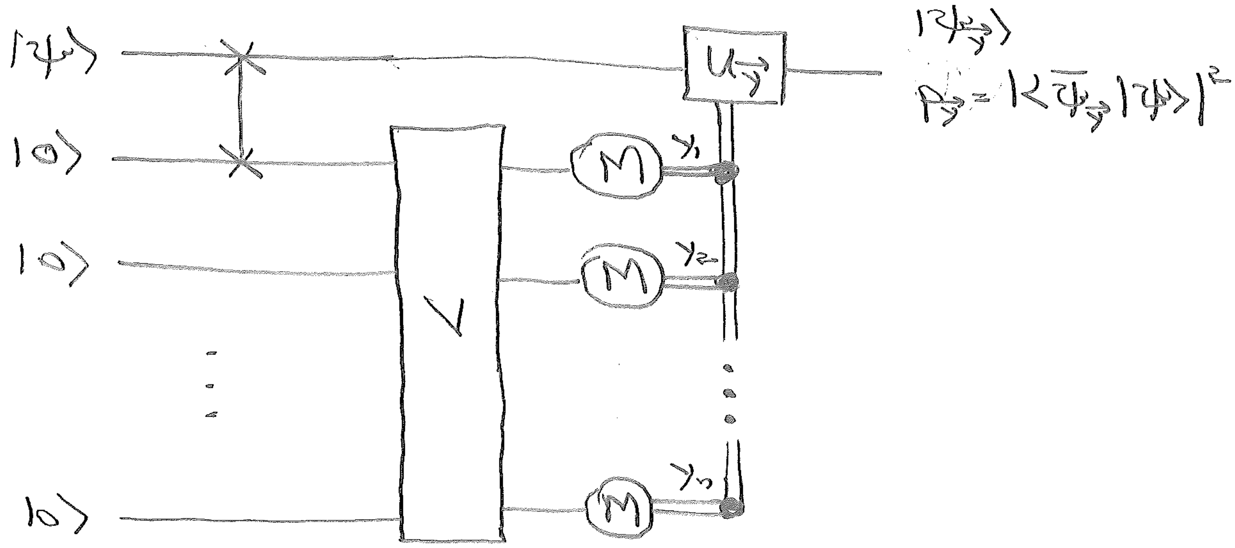
(b) In order not to make a direct measurement on the ancilla qubits, we swap its state into an additional ancilla qubit and then make the measurement on the n ancilla qubits.



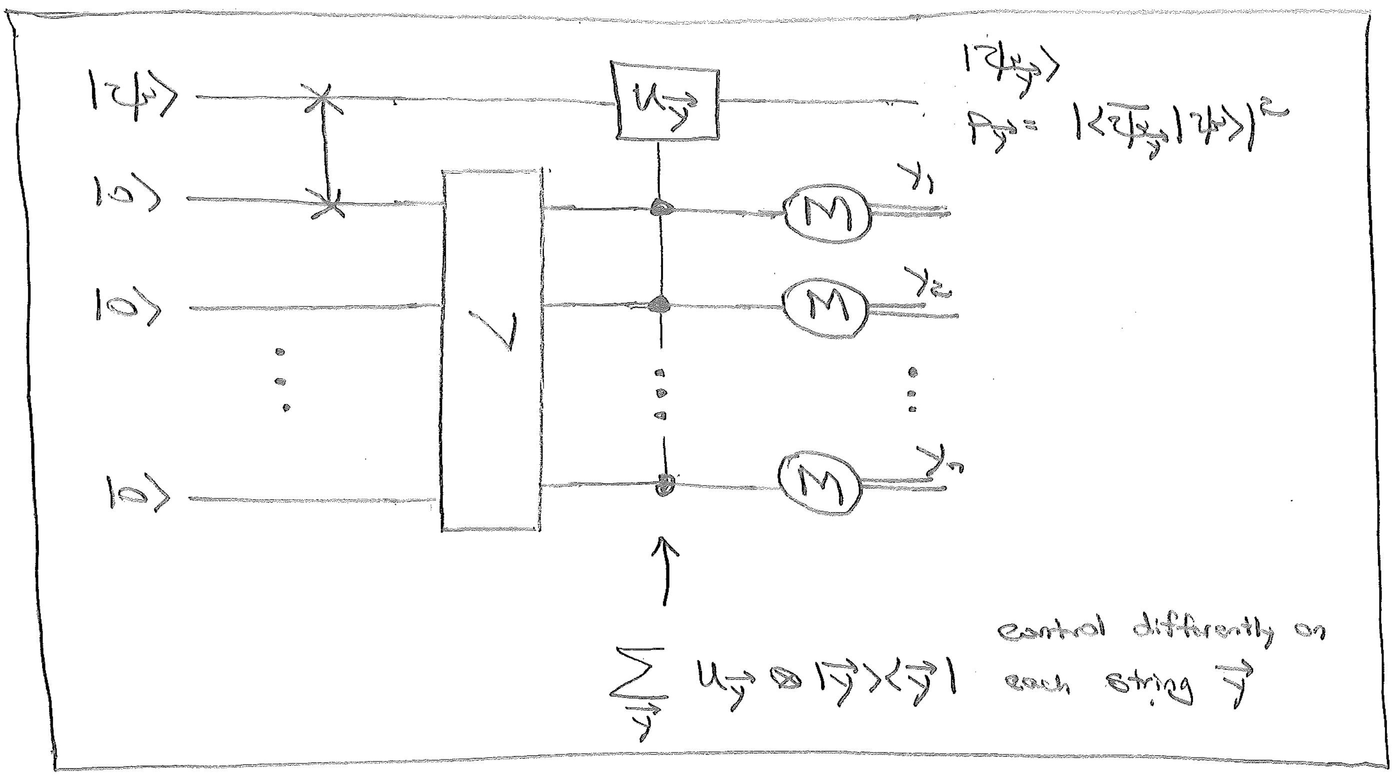
In this model, the statistics are those of the POVM $\{E_{\vec{y}}\}$, but the original qubit always ends up in the state $|0\rangle$, so the Kraus operators are

$$A_{\vec{y}} = |0\rangle\langle\tilde{\Phi}_{\vec{y}}| \Rightarrow E_{\vec{y}} = A_{\vec{y}}^{\dagger} A_{\vec{y}} = |\tilde{\Phi}_{\vec{y}}\rangle\langle\tilde{\Phi}_{\vec{y}}|$$

(c) Let $U_{\vec{y}}$ be a unitary that maps $|0\rangle$ to $|\psi_{\vec{y}}\rangle = |\tilde{\Phi}_{\vec{y}}\rangle/\sqrt{\mu_{\vec{y}}}$, i.e., $U_{\vec{y}}|0\rangle = |\psi_{\vec{y}}\rangle$. Now we control off the measurement results to put the original qubit in the state $|\psi_{\vec{y}}\rangle$ after the measurement.



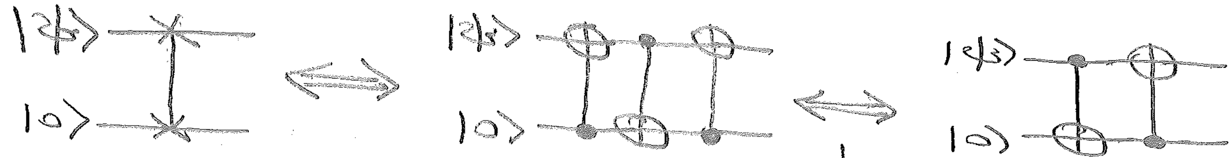
We can move the controlled unitary to the other side of the measurement, provided we understand what the notation means



Kraus operators: $A_{\vec{y}} = |\psi_{\vec{y}}\rangle\langle\bar{\psi}_{\vec{y}}| = U_{\vec{y}}|0\rangle\langle\bar{\psi}_{\vec{y}}|$ ← can always do a conditional post-measurement unitary based on the measurement result

$$E_{\vec{y}} = A^{\dagger}_{\vec{y}} A_{\vec{y}} = |\bar{\psi}_{\vec{y}}\rangle\langle\bar{\psi}_{\vec{y}}|$$

We can replace the SWAP by 3 C-NOTs:

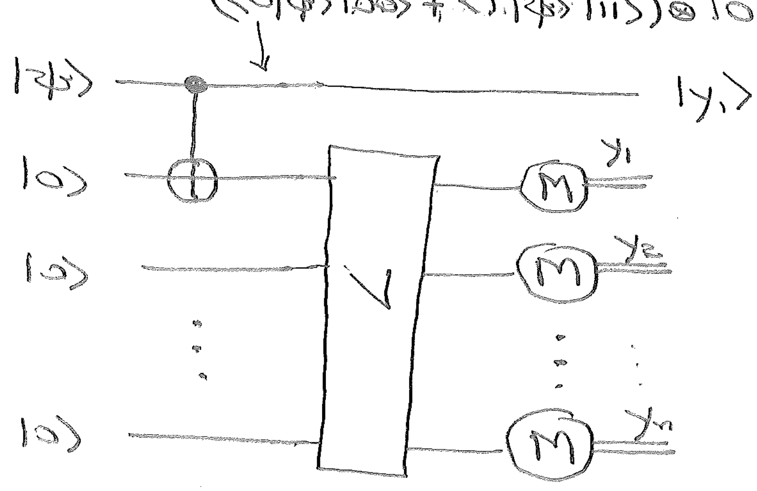


leave out 1st C-NOT since it does nothing

There is a tendency to think that in part (b), we could use a single C-NOT in place of the SWAP, since the C-NOT is the canonical measurement gate. The problem with this is that we don't want to <u>measure</u> the primary qubit in the $|0\rangle - |1\rangle$ basis; instead, we want to <u>transfer</u> its state to the second qubit.

So the following circuit,

$(\langle 0|\psi\rangle |00\rangle + \langle 1|\psi\rangle |11\rangle) \otimes |0\rangle^{\otimes(n-1)}$



has the statistics of

$$p(\vec{y}) = \mathrm{tr}\Big(\big((\langle 0|\psi\rangle|00\rangle + \langle 1|\psi\rangle|11\rangle) \otimes |0\rangle^{\otimes(n-1)}\big)|\hat{\psi}_{\vec{y}}\rangle\langle\hat{\psi}_{\vec{y}}|$$
$$\times\ \big(\langle 00|\langle 0|\psi\rangle^* + \langle 11|\langle 1|\psi\rangle^*\big)\Big)$$

do the trace on the 1st qubit

$$= \langle\hat{\psi}_{\vec{y}}|\underbrace{\big(|\langle 0|\psi\rangle|^2 |0\rangle\langle 0| + |\langle 1|\psi\rangle|^2 |1\rangle\langle 1|\big)\otimes P_0^{\otimes(n-1)}}_{\rho}|\hat{\psi}_{\vec{y}}\rangle$$

$\rho \leftarrow$ density operator with coherence in the $|0\rangle - |1\rangle$ basis removed

$$= \langle\bar{\psi}_{\vec{y}}|\rho|\bar{\psi}_{\vec{y}}\rangle \qquad \text{not} \quad |\langle\bar{\psi}_{\vec{y}}|\psi\rangle|^2$$