6.1

(a) <u>Method 1</u>: The cp of $\mathcal{B}$ is equivalent to the positivity of $\mathcal{I}\otimes\mathcal{B}(|\beta_{00}\rangle\langle\beta_{00}|)$.

$$\mathcal{I}\otimes\mathcal{B}(|\beta_{00}\rangle\langle\beta_{00}|)$$
$$= \mathcal{I}\otimes\mathcal{B}\left(\tfrac{1}{4}(1\otimes 1 + X\otimes X - Y\otimes Y + Z\otimes Z)\right)$$
$$= \tfrac{1}{4}\Big(1\otimes \underbrace{\mathcal{B}(1)}_{1} + X\otimes \underbrace{\mathcal{B}(X)}_{t_x X} - Y\otimes \underbrace{\mathcal{B}(Y)}_{t_y Y} + Z\otimes \underbrace{\mathcal{B}(Z)}_{t_z Z}\Big)$$
$$= \tfrac{1}{4}\left(1\otimes 1 + t_x X\otimes X - t_y Y\otimes Y + t_z Z\otimes Z\right)$$

The operators $\sigma_j\otimes\sigma_j$ can be written in terms of the Bell states. Inverting the Pauli representations for the Bell states (Lectures 6-7), we get

	$\lvert\beta_{00}\rangle\langle\beta_{00}\rvert$	$\lvert\beta_{10}\rangle\langle\beta_{10}\rvert$	$\lvert\beta_{01}\rangle\langle\beta_{01}\rvert$	$\lvert\beta_{11}\rangle\langle\beta_{11}\rvert$
$1\otimes 1 =$	+	+	+	+
$X\otimes X =$	+	−	+	−
$Y\otimes Y =$	−	+	+	−
$Z\otimes Z =$	+	+	−	−

Inserting this above, we get

<u>Method 2.</u> The cp of $\mathcal{B}$ is equivalent to its being left-right positive.

$$\mathcal{B}(1) = \mathcal{B}^{\#}|1) = 1$$
$$\mathcal{B}(\sigma_j) = \mathcal{B}^{\#}|\sigma_j) = t_j\sigma_j$$

$\Rightarrow$ The Pauli operators are the left-right eigenoperators of $\mathcal{B}^{\#}$

$$\Rightarrow \mathcal{B}^{\#} = \tfrac{1}{2}|1)(1| + \tfrac{1}{2}\sum_j t_j|\sigma_j)(\sigma_j|$$
$$= \tfrac{1}{2}\, 1\odot 1 + \tfrac{1}{2}\sum_j t_j\,\sigma_j\odot\sigma_j$$
$$\mathcal{B} = \tfrac{1}{2}(1\odot 1)^{\#} + \tfrac{1}{2}\sum_j t_j(\sigma_j\odot\sigma_j)^{\#}$$

$$(1\odot 1)^{\#} = \mathcal{I}^{\#} = \mathbb{I} = \tfrac{1}{2}\sum_\alpha |\sigma_\alpha)(\sigma_\alpha| = \tfrac{1}{2}\sum_\alpha \sigma_\alpha\odot\sigma_\alpha$$

$$\sigma_z\odot\sigma_z = (|0\rangle\langle 0| - |1\rangle\langle 1|)\odot(|0\rangle\langle 0| - |1\rangle\langle 1|)$$
$$= |0\rangle\langle 0|\odot|0\rangle\langle 0| + |1\rangle\langle 1|\odot|1\rangle\langle 1|$$
$$- |0\rangle\langle 0|\odot|1\rangle\langle 1| - |1\rangle\langle 1|\odot|0\rangle\langle 0|$$

exchanges 2nd and 4th slots.

$$(\sigma_z\odot\sigma_z)^{\#} = \underbrace{|0\rangle\langle 0|}_{\frac{1}{2}(1+Z)}\odot\underbrace{|0\rangle\langle 0|}_{\frac{1}{2}(1+Z)} + \underbrace{|1\rangle\langle 1|}_{\frac{1}{2}(1-Z)}\odot\underbrace{|1\rangle\langle 1|}_{\frac{1}{2}(1-Z)}$$
$$- \underbrace{|0\rangle\langle 1|}_{\frac{1}{2}(X+iY)}\odot\underbrace{|1\rangle\langle 0|}_{\frac{1}{2}(X-iY)} - \underbrace{|1\rangle\langle 0|}_{\frac{1}{2}(X-iY)}\odot\underbrace{|0\rangle\langle 1|}_{\frac{1}{2}(X+iY)}$$
$$= \tfrac{1}{2}\left(1\odot 1 - X\odot X - Y\odot Y + Z\odot Z\right)$$

$$\mathcal{D}\otimes\mathcal{E}(|\beta_{00}\rangle\langle\beta_{00}|)$$

$$= \frac{1}{4}\Big(|\beta_{00}\rangle\langle\beta_{00}| (1+t_x+t_y+t_z)$$
$$+ |\beta_{10}\rangle\langle\beta_{10}| (1-t_x-t_y+t_z)$$
$$+ |\beta_{01}\rangle\langle\beta_{01}| (1+t_x-t_y-t_z)$$
$$+ |\beta_{11}\rangle\langle\beta_{11}| (1-t_x+t_y-t_z)\Big)$$

$\uparrow$ These are the eigenvalues. They have to be ≥ 0.

Similarly,

$$(\sigma_x\otimes\sigma_x)^{\#} = \frac{1}{2}(1\otimes 1 + X\otimes X - Y\otimes Y - Z\otimes Z)$$
$$(\sigma_y\otimes\sigma_y)^{\#} = \frac{1}{2}(1\otimes 1 - X\otimes X + Y\otimes Y - Z\otimes Z)$$

$$\Rightarrow \mathcal{E} = \frac{1}{4}\Big(1\otimes 1\, (1+t_x+t_y+t_z)$$
$$+ X\otimes X\, (1+t_x-t_y-t_z)$$
$$+ Y\otimes Y\, (1-t_x+t_y-t_z)$$
$$+ Z\otimes Z\, (1-t_x-t_y+t_z)\Big)$$

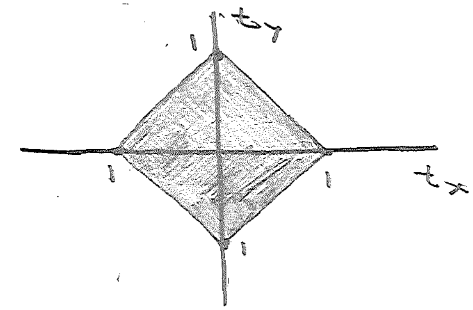
$\uparrow$ These are the eigenvalues. They have to be ≥ 0.

Constraints are

$$\left.\begin{array}{l} -(t_x+t_y) \leq 1+t_z \\ t_x+t_y \leq 1+t_z \end{array}\right\} \Longleftrightarrow |t_x+t_y| \leq 1+t_z$$

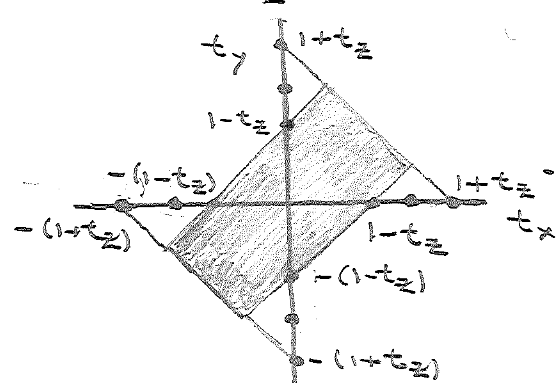
$$\left.\begin{array}{l} t_y-t_x \leq 1-t_z \\ t_x-t_y \leq 1-t_z \end{array}\right\} \Longleftrightarrow |t_x-t_y| \leq 1-t_z$$

$t_z=0$: $|t_x+t_y|\leq 1$, $|t_x-t_y|\leq 1$

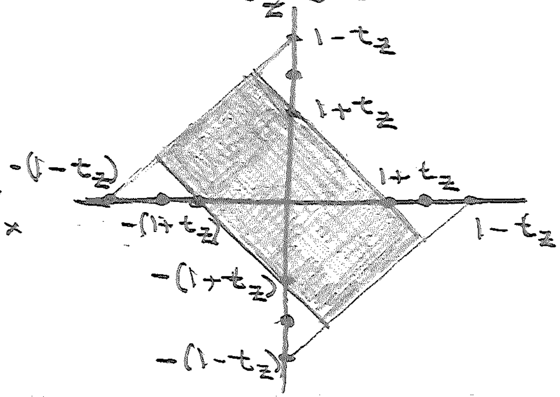


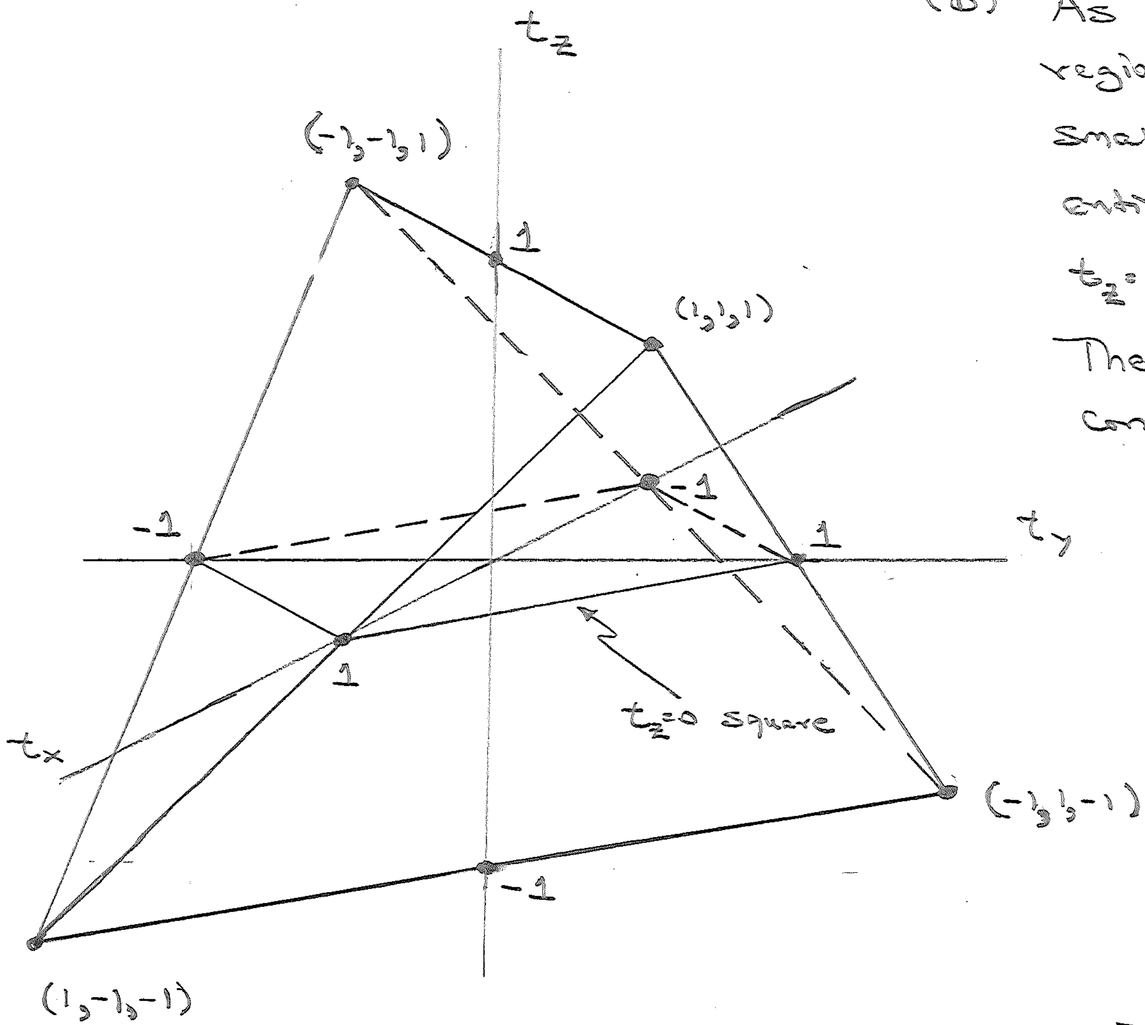
The 3-d region is obtained by stacking these squares on top of each other. It is shown on the next page.

$t_z>0$



$t_z<0$





The allowed region is a tetrahedron with vertices at $(1,1,1)$, $(1,-1,-1)$, $(-1,1,-1)$, and $(-1,-1,1)$. Positive superoperators are those within the cube that contains this tetrahedron.

(b) As t_z moves away from zero, the regions containing reflections ($\det T < 0$) get smaller and smaller, disappearing entirely when $t_z = \pm 1$. Indeed, when $t_z = 1$, we must have $-1 \le t_x = t_y \le 1$. The case where $t_x = t_y < 0$ can be converted to $t_x = t_y > 0$ by a $180°$ rotation about z. So, when $t_z = +1$, we are dealing with a uniform contraction in the x and y directions (or that followed by a $180°$ rotation about z). This is the stochastic bit flip operation.

(c) The extreme points of the unital qubit operations are the vertices, i.e., nothing happens or two reflections, which are equivalent to a $180°$ rotations. The edges, as noted in (b), are equivalent to the stochastic flip operations.

Notice that despite our formulation's seeming to single out the z direction, the three Cartesian directions are actually equivalent.

No matter how you slice it, our result says that any unital, trace-preserving qubit quantum operation can be written as

$$p_0\, I \odot I + p_x\, X \odot X + p_y\, Y \odot Y + p_z\, Z \odot Z, \qquad p_0 + p_x + p_y + p_z = 1$$

extreme points

This is the nonisotropic version of the depolarizing chanel.

$$p_0 = \frac{1}{4}(1 + t_x + t_y + t_z) \qquad t_x = p_0 + p_x - p_y - p_z$$

$$p_x = \frac{1}{4}(1 + t_x - t_y - t_z) \qquad t_y = p_0 - p_x + p_y - p_z$$

$$p_y = \frac{1}{4}(1 - t_x + t_y - t_z) \qquad t_z = p_0 - p_x - p_y + p_z$$

$$p_z = \frac{1}{4}(1 - t_x - t_y + t_z)$$