7.1.

(a) $I = \alpha \int d\Gamma_{|\phi\rangle} |\phi\rangle\langle\phi|$

Take the trace of both sides.

$$D = tr(I) = \alpha \int d\Gamma_{|\phi\rangle} = \alpha \underbrace{\int_0^{\pi/2} d\theta \cos\theta \sin^{2D-3}\theta}_{} \underbrace{\int dS_{2D-3}}_{S_{2D-3}}$$

$x = \sin\theta$
$dx = \cos\theta\, d\theta$

$$\int_0^1 dx\, x^{2D-3} = \frac{1}{2D-2}$$

$$D = \frac{\alpha}{2(D-1)} S_{2D-3} \Longrightarrow \boxed{\alpha = \frac{2D(D-1)}{S_{2D-3}}}$$

(b)

① Direct integration:

$$\int d\Gamma_{|\phi\rangle} \underbrace{|\langle\phi|\psi\rangle|^2}_{\cos^2\theta} = \underbrace{\int_0^{\pi/2} d\theta \cos^3\theta \sin^{2D-3}\theta}_{} \underbrace{\int dS_{2D-3}}_{S_{2D-3}}$$

$$\int_0^1 dx\,(1-x^2)x^{2D-3} = \frac{1}{2D-2} - \frac{1}{2D} = \frac{1}{2D(D-1)}$$

$$\boxed{\int d\Gamma_{|\phi\rangle} |\langle\phi|\psi\rangle|^2 = \frac{S_{2D-3}}{2D(D-1)}}$$

② Had to be: $1 = \int dp(|\phi\rangle \,|\, |\psi\rangle) = \alpha \int d\Gamma_{|\phi\rangle} |\langle\phi|\psi\rangle|^2$

$$\Longrightarrow \boxed{\int d\Gamma_{|\phi\rangle} |\langle\phi|\psi\rangle|^2 = \frac{1}{\alpha} = \frac{S_{2D-3}}{2D(D-1)}}$$

③ Being clever: The integral is clearly independent of what $|\psi\rangle$ is, so

$$\int d\Gamma_{|\phi\rangle} |\langle\phi|\psi\rangle|^2 = \frac{1}{D}\sum_j \int d\Gamma_{|\phi\rangle} |\langle\phi|e_j\rangle|^2$$

$$= \frac{1}{D}\underbrace{\int d\Gamma_{|\phi\rangle}}_{\frac{S_{2D-3}}{2(D-1)}}$$

$$\boxed{\int d\Gamma_{|\phi\rangle} |\langle\phi|\psi\rangle|^2 = \frac{S_{2D-3}}{2D(D-1)}}$$

(c)

$$\frac{1}{\int d\Gamma_{|\phi\rangle}} = \frac{\alpha}{D}$$

$$\overline{F^2} = \left(\begin{array}{c}\text{average squared}\\ \text{fidelity of copies}\end{array}\right) = \underbrace{\int d\Gamma_{|\psi\rangle}\, p(|\psi\rangle)}_{\text{average over input}} \underbrace{\int dp(|\phi\rangle\,|\,|\psi\rangle)\, F^2(|\phi\rangle, |\psi\rangle)}_{\text{average over output}}$$

$$= \alpha \int d\Gamma_{|\phi\rangle} |\langle\phi|\psi\rangle|^2\, |\langle\phi|\psi\rangle|^2$$

$$= \alpha \underbrace{\int_0^{\pi/2} d\theta \cos^5\theta \sin^{2D-3}\theta}\ \underbrace{\int dS_{2D-3}}_{S_{2D-3}}$$

$$\int_0^1 dx\, (1-x^2)^2 x^{2D-3} = \int_0^1 dx\, (1-2x^2+x^4)\, x^{2D-3}$$

$$= \frac{1}{2D-2} - \frac{2}{2D} + \frac{1}{2D+2}$$

$$= \frac{1}{D(D-1)(D+1)}$$

$$\overline{F^2} = \frac{\alpha\, S_{2D-3}}{D(D-1)(D+1)} = \boxed{\frac{2}{D+1} = \overline{F^2}}$$

Thus this becomes a very bad cloning method for D large, but bad or not, it is the best one can do in cloning <u>all</u> states.

In stating this problem, we <u>assumed</u> that an unweighted integral over all projectors is a multiple of I. We should actually <u>show</u> this. We take the matrix elements of the integral in some basis:

$$\langle e_j|\left(\int d\Gamma_{|\phi\rangle}\,|\phi\rangle\langle\phi|\right)|e_k\rangle = \int d\Gamma_{|\phi\rangle}\,\langle e_j|\phi\rangle\langle\phi|e_k\rangle$$

$$= \delta_{jk}\int d\Gamma_{|\phi\rangle}\,\underbrace{|\langle e_j|\phi\rangle|^2}$$

independent of j by unitary invariance

Why? If $j\neq k$, the contribution from $|\phi\rangle = \sum_l c_l|e_l\rangle = c_j|e_j\rangle + c_k|e_k\rangle + \sum_{l\neq j,k} c_l|e_l\rangle$ is cancelled by the contribution from $|\phi\rangle = c_j|e_j\rangle - c_k|e_k\rangle + \sum_{l\neq j,k} c_l|e_l\rangle$.

So

$$\int d\Gamma_{|\phi\rangle}\, |\phi\rangle\langle\phi| = I \underbrace{\int d\Gamma_{|\phi\rangle}\, |\langle\phi|\psi\rangle|^2}_{\frac{S_{2D-3}}{2D(D-1)}}$$

$$= \frac{S_{2D-3}}{2D(D-1)}\, I$$

Notice also that the average fidelity of two randomly selected states is

$$\int d\Gamma_{|\phi\rangle}\, \underbrace{p(|\phi\rangle)}\int d\Gamma_{|\psi\rangle}\, \underbrace{p(|\psi\rangle)}\, |\langle\phi|\psi\rangle|^2 = \frac{\alpha}{D}\,\frac{S_{2D-3}}{2D(D-1)} = \frac{1}{D} < \frac{2}{D+1}$$

$$= \frac{1}{\int d\Gamma_{|\phi\rangle}} = \frac{\alpha}{D}$$

Asymptotically in D, the approximate cloning is better than flipping a coin by the factor

$$\frac{\frac{2}{D+1}}{\frac{1}{D}} = \frac{2D}{D+1} = \frac{2}{1+1/D}$$