7.2.

Suppose ρ_1 and ρ_2 have prior probabilities q_1 and q_2

(a) $P_e = P_{1|\rho_2} q_2 + P_{2|\rho_1} q_1 = q_2 \operatorname{tr}(E_1 \rho_2) + q_1 \operatorname{tr}(E_2 \rho_1)$

$$E_1 + E_2 = 1$$

$$P_e = q_1 - \operatorname{tr}(E_1(q_1\rho_1 - q_2\rho_2))$$

(b) Lemma: $\max_{0\le E\le 1} \operatorname{tr}(E(\underbrace{q_1\rho_1 - q_2\rho_2}_{A=A^\dagger})) = \frac{1}{2}\operatorname{tr}(|q_1\rho_1 - q_2\rho_2|) + \frac{1}{2}(q_1 - q_2)$

$$A = \sum_j \lambda_j |e_j\rangle\langle e_j| \,, \quad |A| = \sqrt{A^\dagger A} = \sum_j |\lambda_j| |e_j\rangle\langle e_j|$$

$$q_1 - q_2 = \operatorname{tr}(A) = \sum_j \lambda_j = \sum_{\lambda_j>0} |\lambda_j| - \sum_{\lambda_j<0} |\lambda_j|$$

$$\operatorname{tr}(|A|) = \sum_j |\lambda_j| = \sum_{\lambda_j\ge 0} |\lambda_j| + \sum_{\lambda_j<0} |\lambda_j|$$

$$\sum_{\lambda_j>0} |\lambda_j| = \frac{1}{2}(\operatorname{tr}(|A|) + q_1 - q_2) \qquad \operatorname{tr}(|A|) = 2\sum_{\lambda_j>0}|\lambda_j| - (q_1 - q_2)$$

$$\sum_{\lambda_j<0} |\lambda_j| = \frac{1}{2}(\operatorname{tr}(|A|) - (q_1 - q_2)) \qquad = 2\sum_{\lambda_j<0}|\lambda_j| + q_1 - q_2$$

Proof: $\operatorname{tr}(EA) = \sum_j \lambda_j \langle e_j|E|e_j\rangle$

$$= \underbrace{\sum_{\lambda_j>0} \lambda_j \langle e_j|E|e_j\rangle}_{\ge 0} + \underbrace{\sum_{\lambda_j<0} \lambda_j \langle e_j|E|e_j\rangle}_{\le 0}$$

$$\le \sum_{\lambda_j>0} \lambda_j \underbrace{\langle e_j|E|e_j\rangle}_{\le 1}$$

$$\le \sum_{\lambda_j>0} \lambda_j$$

equality iff

$E = P_+ + E'$, $E' = P_0 E' P_0$

$P_+ = \sum_{\lambda_j>0} |e_j\rangle\langle e_j|$

$P_- = \sum_{\lambda_j<0} |e_j\rangle\langle e_j|$

$P_0 = I - P_+ - P_-$

$$= \sum_{\lambda_j > 0} |\lambda_j|$$

$$= \frac{1}{2}\left(\mathrm{tr}(|A|) + q_1 - q_2\right)$$

(c) $(P_e)_{min} = \frac{1}{2} - \frac{1}{2}\mathrm{tr}\left(|q_1\rho_1 - q_2\rho_2|\right)$

$E_1 = P_+ + E_1'$

$E_2 = P_- + E_2'$

$E_1' + E_2' = P_0 = I - P_+ - P_-$

We could use, for example, $E_1' = P_0$ and $E_2' = 0$

(d) As discussed in the lecture, we can specialize to a two-outcome POVM. Outcome E_1 results in a decision for $|\psi_1\rangle$, and outcome E_2 results in a decision for $|\psi_2\rangle$. The error probability is

$$P_e = \underbrace{P_{1|\psi_2}}_{=\langle\psi_2|E_1|\psi_2\rangle = tr(E_1 P_2)} g_2 + \underbrace{P_{2|\psi_1}}_{=\langle\psi_1|E_2|\psi_1\rangle = tr(E_2 P_1)} g_1$$

$$= tr(E_1 P_2 g_2 + \underbrace{E_2}_{1 - E_1} P_1 g_1)$$

$$= g_1 - tr\left(E_1(g_1 P_1 - g_2 P_2)\right)$$

Minimizing P_e means maximizing $tr(E_1(g_1P_1 - g_2P_2))$ over all POVM elements E_1. We only have to worry about the 2-d subspace spanned by $|\psi_1\rangle$ and $|\psi_2\rangle$. Go to the Bloch-sphere description in that subspace.

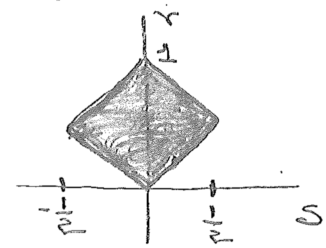
$$P_j = \frac{1}{2}(1 + \vec{\sigma}\cdot\vec{n}_j)$$

$$g_1 P_1 - g_2 P_2 = \frac{1}{2}(g_1 - g_2) + \frac{1}{2}\vec{\sigma}\cdot\underbrace{(g_1\vec{n}_1 - g_2\vec{n}_2)}_{\vec{m}\,|g_1\vec{n}_1 - g_2\vec{n}_2|}$$

Notice that $|g_1 - g_2| \le |g_1\vec{n}_1 - g_2\vec{n}_2| \le g_1 + g_2$.

E_1 is a POVM element, so it can be written as

$$E_1 = r\mathbb{1} + s\,\vec{n}\cdot\vec{\sigma}, \qquad 0 \le r+s \le 1, \quad 0 \le r-s \le 1$$



$$\mathrm{tr}\big(E_1(g_1P_1 - g_2P_2)\big) = r(g_1 - g_2) + s\,\vec{n}\cdot\vec{m}\,|g_1\vec{n}_1 - g_2\vec{n}_2|$$

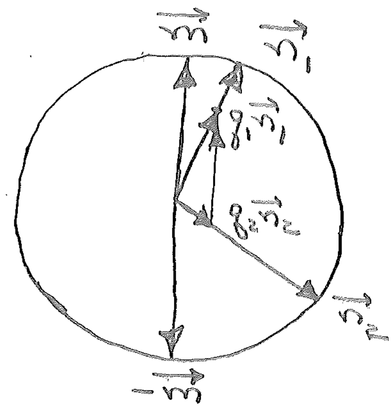
Since $|g_1\vec{n}_1 - g_2\vec{n}_2| \ge g_1 - g_2$, it is clear we want to choose $\vec{n} = \vec{m}$ and $r = s = 1/2$, giving $E_1 = \frac{1}{2}(\mathbb{1} + \vec{m}\cdot\vec{\sigma})$
$= |\vec{m}\rangle\langle\vec{m}|$.

↑

ODOP measurement

$E_1 = |\vec{m}\rangle\langle\vec{m}|$

$E_2 = |-\vec{m}\rangle\langle-\vec{m}|$



$$\mathrm{tr}\big(E_1(g_1P_1 - g_2P_2)\big) = \frac{1}{2}(g_1 - g_2) + \frac{1}{2}|g_1\vec{n}_1 - g_2\vec{n}_2|$$

$$\boxed{\begin{aligned}(P_e)_{min} &= \frac{1}{2} - \frac{1}{2}|g_1\vec{n}_1 - g_2\vec{n}_2| \\ &= \frac{1}{2} - \frac{1}{2}\big(g_1^2 + g_2^2 - 2g_1g_2\underbrace{\vec{n}_1\cdot\vec{n}_2}_{2|\langle\psi_1|\psi_2\rangle|^2 - 1}\big)^{1/2} \\ &= \frac{1}{2} - \frac{1}{2}\big(1 - 4g_1g_2|\langle\psi_1|\psi_2\rangle|^2\big)^{1/2}\end{aligned}}$$

This result also follows from the general result (part (c)) for the minimum error probability in distinguishing two mixed states, ρ_1 and ρ_2:

$$(P_e)_{min} = \frac{1}{2} - \frac{1}{2}\mathrm{tr}\left(|q_1\rho_1 - q_2\rho_2|\right).$$

Here

$$q_1\rho_1 - q_2\rho_2 = q_1 P_1 - q_2 P_2$$
$$= \frac{1}{2}(q_1 - q_2) + \frac{1}{2}|q_1\vec{n}_1 - q_2\vec{n}_2|\,\vec{\sigma}\cdot\vec{m}$$

$$\downarrow$$

Eigenvalues $\lambda_{\pm} = \frac{1}{2}(q_1 - q_2) \pm \frac{1}{2}|q_1\vec{n}_1 - q_2\vec{n}_2| \begin{matrix} \geq 0 \\ \leq 0 \end{matrix}$

$$\mathrm{tr}\left(|q_1P_1 - q_2P_2|\right) = \lambda_+ - \lambda_- = |q_1\vec{n}_1 - q_2\vec{n}_2|$$

So

$$(P_e)_{min} = \frac{1}{2} - \frac{1}{2}|q_1\vec{n}_1 - q_2\vec{n}_2|$$