7.4.
$$\rho = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| = \sum_\alpha |\tilde{\psi}_\alpha\rangle\langle\tilde{\psi}_\alpha| \,, \qquad |\tilde{\psi}_\alpha\rangle = \sqrt{p_\alpha}\,|\psi_\alpha\rangle$$

$$E_\alpha = \rho^{-1/2}|\tilde{\psi}_\alpha\rangle\langle\tilde{\psi}_\alpha|\rho^{-1/2} = |\tilde{\phi}_\alpha\rangle\langle\tilde{\phi}_\alpha| \,, \qquad |\tilde{\phi}_\alpha\rangle = \rho^{-1/2}|\tilde{\psi}_\alpha\rangle$$

$$\sum_\alpha E_\alpha = I$$

(a)
$$\begin{aligned} q_\alpha = \mathrm{tr}(\rho E_\alpha) &= \mathrm{tr}(\rho\,\rho^{-1/2}|\tilde{\psi}_\alpha\rangle\langle\tilde{\psi}_\alpha|\rho^{-1/2}) \\ &= \mathrm{tr}(\underbrace{\rho^{-1/2}\rho\,\rho^{-1/2}}_{I}|\tilde{\psi}_\alpha\rangle\langle\tilde{\psi}_\alpha|) \\ &= \langle\tilde{\psi}_\alpha|\tilde{\psi}_\alpha\rangle \\ &= p_\alpha \end{aligned}$$

(b) The linear independence of $\{|\tilde{\psi}_\alpha\rangle\}$ immediately implies the linear independence of $\{|\tilde{\phi}_\alpha\rangle\}$. If a set of linearly independent vectors resolves the identity, as does the set $\{|\tilde{\phi}_\alpha\rangle\}$, the vectors are orthonormal. We see this by expanding the vectors $|\tilde{\phi}_\alpha\rangle$ in an orthonormal basis, i.e.,

$$|\tilde{\phi}_\alpha\rangle = \sum_j M_{\alpha j}|e_j\rangle \,, \qquad M \text{ is a } D\times D \text{ matrix,}$$

and then writing

$$\sum_j |e_j\rangle\langle e_j| = I = \sum_\alpha |\tilde{\phi}_\alpha\rangle\langle\tilde{\phi}_\alpha| = \sum_{j,k}\Big(\sum_\alpha M_{\alpha j}M^*_{\alpha k}\Big)|e_j\rangle\langle e_k| \,,$$

which implies that

$$\sum_\alpha M_{\alpha j}M^*_{\alpha k} = \delta_{jk} .$$

Thus M is a $D\times D$ unitary matrix, and hence the vectors $|\tilde{\phi}_\alpha\rangle$ are orthonormal.

(b) Two states, $|\vec{n}_1\rangle$ and $|\vec{n}_2\rangle$, with equal prior probabilities.

$$\rho = \sum_\alpha p_\alpha |\vec{n}_\alpha\rangle\langle\vec{n}_\alpha| = \frac{1}{2}(I + S\vec{n}\cdot\vec{\sigma})$$

$$S\vec{n} = \frac{1}{2}(\vec{n}_1 + \vec{n}_2) \Longrightarrow \vec{n} = \frac{\vec{n}_1+\vec{n}_2}{|\vec{n}_1+\vec{n}_2|}$$

$$S = \frac{1}{2}|\vec{n}_1+\vec{n}_2| = \frac{1}{\sqrt{2}}(1+\vec{n}_1\cdot\vec{n}_2)^{1/2}$$

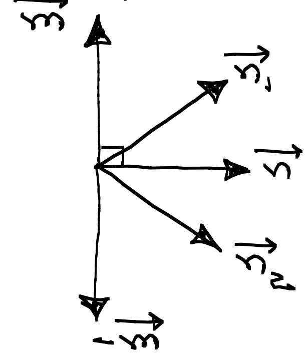
$$|\bar{\phi}_\alpha\rangle = \frac{1}{\sqrt{2}}\rho^{-1/2}|\vec{n}_\alpha\rangle$$

The measurement that minimizes error probability is in the basis $|\pm\vec{m}\rangle$, where

$$\vec{m} = \frac{\vec{n}_1-\vec{n}_2}{|\vec{n}_1-\vec{n}_2|}, \quad \vec{m}\cdot\vec{n} = 0$$

$$|\vec{n}_1-\vec{n}_2| = \sqrt{2}(1-\vec{n}_1\cdot\vec{n}_2)^{1/2}$$

Geometry:



We need to show that

$$\begin{cases} |\vec{m}\rangle\langle\vec{m}| = |\bar{\phi}_1\rangle\langle\bar{\phi}_1| = \frac{1}{2}\rho^{-1/2}|\vec{n}_1\rangle\langle\vec{n}_1|\rho^{-1/2} \\ |-\vec{m}\rangle\langle-\vec{m}| = |\bar{\phi}_2\rangle\langle\bar{\phi}_2| = \frac{1}{2}\rho^{-1/2}|\vec{n}_2\rangle\langle\vec{n}_2|\rho^{-1/2} \end{cases}$$

It's really obvious that this works, because of the symmetry of the situation, but let's work it out anyway.

$$\Longleftrightarrow \begin{cases} \text{Sum: } I = \frac{1}{2}\rho^{-1/2}\underbrace{\left(|\vec{n}_1\rangle\langle\vec{n}_1| + |\vec{n}_2\rangle\langle\vec{n}_2|\right)}_{2\rho}\rho^{-1/2} = I \quad \text{This is satisfied} \\ \text{Difference: } \vec{\sigma}\cdot\vec{m} = \frac{1}{2}\rho^{-1/2}\underbrace{\left(|\vec{n}_1\rangle\langle\vec{n}_1| - |\vec{n}_2\rangle\langle\vec{n}_2|\right)}\rho^{-1/2} \end{cases}$$

$$= \frac{1}{2}(\vec{n}_1-\vec{n}_2)\cdot\vec{\sigma} = \frac{1}{\sqrt{2}}(1-\vec{n}_1\cdot\vec{n}_2)^{1/2}\vec{m}\cdot\vec{\sigma}$$

$$\vec{\sigma}\cdot\vec{m} = \frac{1}{2\sqrt{2}}(1-\vec{n}_1\cdot\vec{n}_2)^{1/2}\rho^{-1/2}\vec{\sigma}\cdot\vec{m}\,\rho^{-1/2}$$

$$\rho^{1/2}\,\vec{\sigma}\cdot\vec{m}\,\rho^{1/2} = \frac{1}{2\sqrt{2}}(1-\vec{n}_1\cdot\vec{n}_2)^{1/2}\,\vec{\sigma}\cdot\vec{m}$$

This is what we have to show.

$$\rho = \frac{1}{2}(I + S\vec{n}\cdot\vec{\sigma}) = \frac{1}{2}(1+S)\,|\vec{n}\rangle\langle\vec{n}| + \frac{1}{2}(1-S)\,|-\vec{n}\rangle\langle-\vec{n}|$$

$$\rho^{1/2} = \underbrace{\sqrt{\frac{1+S}{2}}}_{A}\,|\vec{n}\rangle\langle\vec{n}| + \underbrace{\sqrt{\frac{1-S}{2}}}_{B}\,|-\vec{n}\rangle\langle-\vec{n}|$$

$$= \frac{1}{2}(A+B)\,I + \frac{1}{2}(A-B)\,\vec{\sigma}\cdot\vec{n}$$

$$\rho^{1/2}\,\vec{\sigma}\cdot\vec{m}\,\rho^{1/2} = \frac{1}{4}(A+B)^2\,\vec{\sigma}\cdot\vec{m} + \frac{1}{4}(A^2-B^2)\Big[\overbrace{(\vec{\sigma}\cdot\vec{n})(\vec{\sigma}\cdot\vec{m}) + (\vec{\sigma}\cdot\vec{m})(\vec{\sigma}\cdot\vec{n})}^{2\vec{n}\cdot\vec{m}=0}\Big]$$
$$+ \frac{1}{4}(A-B)^2\underbrace{(\vec{\sigma}\cdot\vec{n})(\vec{\sigma}\cdot\vec{m})(\vec{\sigma}\cdot\vec{n})}_{-\vec{\sigma}\cdot\vec{m}}$$

$$= AB\,\vec{\sigma}\cdot\vec{m}$$

$$= \frac{1}{2}\underbrace{\sqrt{1-S^2}}_{\frac{1}{\sqrt{2}}(1-\vec{n}_1\cdot\vec{n}_2)^{1/2}}\,\vec{\sigma}\cdot\vec{m}$$

$$= \frac{1}{2\sqrt{2}}(1-\vec{n}_1\cdot\vec{n}_2)^{1/2}\,\vec{\sigma}\cdot\vec{m}\quad\checkmark$$