1

8.1.
$$\rho = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| = \sum_\alpha |\bar\psi_\alpha\rangle\langle\bar\psi_\alpha| , \qquad |\bar\psi_\alpha\rangle = \sqrt{p_\alpha}\,|\psi_\alpha\rangle$$

$$E_\alpha = \rho^{-1/2}|\bar\psi_\alpha\rangle\langle\bar\psi_\alpha|\rho^{-1/2} = |\bar\phi_\alpha\rangle\langle\bar\phi_\alpha| , \qquad |\bar\phi_\alpha\rangle = \rho^{-1/2}|\bar\psi_\alpha\rangle$$

$$\sum_\alpha E_\alpha = I$$

$$\begin{aligned} q_\alpha = \mathrm{tr}(\rho E_\alpha) &= \mathrm{tr}(\rho\,\rho^{-1/2}|\bar\psi_\alpha\rangle\langle\bar\psi_\alpha|\rho^{-1/2}) \\ &= \mathrm{tr}(\underbrace{\rho^{-1/2}\rho\,\rho^{-1/2}}_{I}|\bar\psi_\alpha\rangle\langle\bar\psi_\alpha|) \\ &= \langle\bar\psi_\alpha|\bar\psi_\alpha\rangle \\ &= p_\alpha \end{aligned}$$

(a) $$\sum_\alpha q_\alpha \log(\mathrm{tr}(E_\alpha)) = \sum_\alpha q_\alpha \log(\langle\bar\phi_\alpha|\bar\phi_\alpha\rangle)$$

$$E_\alpha = |\bar\phi_\alpha\rangle\langle\bar\phi_\alpha| = \langle\bar\phi_\alpha|\bar\phi_\alpha\rangle \underbrace{\frac{|\bar\phi_\alpha\rangle}{\langle\bar\phi_\alpha|\bar\phi_\alpha\rangle^{1/2}}\frac{\langle\bar\phi_\alpha|}{\langle\bar\phi_\alpha|\bar\phi_\alpha\rangle^{1/2}}}_{\text{1d projector}}$$ has a

single nonzero eigenvalue, $\langle\bar\phi_\alpha|\bar\phi_\alpha\rangle$, which has to be $\le 1$, since $E_\alpha$ is a POVM element. So

$$\langle\bar\phi_\alpha|\bar\phi_\alpha\rangle \le 1 \Longrightarrow \log(\langle\bar\phi_\alpha|\bar\phi_\alpha\rangle) \le 0,$$

which implies that

$$\sum_\alpha q_\alpha \log(\langle\bar\phi_\alpha|\bar\phi_\alpha\rangle) \le 0.$$

Thus the POVM inequality is a tighter constraint on $H(\vec p)$, i.e., forces $H(\vec p)$ to be bigger.

(b) POVM inequality:

$$\underbrace{H(\vec{p}) + \sum_\alpha p_\alpha \log\left(\operatorname{tr}(E_\alpha)\right)}_{} \geq S(\rho)$$

$$= \log\left(\langle \bar{\phi}_\alpha | \bar{\phi}_\alpha \rangle\right) = \log\left(p_\alpha \langle \psi^{y}_\alpha | \rho^{-1} | \psi^{y}_\alpha \rangle\right)$$

$$= \log p_\alpha + \log\left(\langle \psi^{y}_\alpha | \rho^{-1} | \psi^{y}_\alpha \rangle\right)$$

$$\underbrace{H(\vec{p}) + \sum_\alpha p_\alpha \log p_\alpha}_{=0} + \sum_\alpha p_\alpha \log\left(\langle \psi^{y}_\alpha | \rho^{-1} | \psi^{y}_\alpha \rangle\right) \geq S(\rho)$$

$$\Leftrightarrow \boxed{\sum_\alpha p_\alpha \log\left(\langle \psi^{y}_\alpha | \rho^{-1} | \psi^{y}_\alpha \rangle\right) \geq S(\rho)}$$

(c) This transformation relies on the fact that

$$p_\alpha \langle \psi^{y}_\alpha | \rho^{-1} | \psi^{y}_\alpha \rangle = \langle \bar{\phi}_\alpha | \bar{\phi}_\alpha \rangle \leq 1,$$

so we have, since log is an increasing function,

$$\log\left(\langle \psi^{y}_\alpha | \rho^{-1} | \psi^{y}_\alpha \rangle\right) \leq \log p_\alpha^{-1} = -\log p_\alpha,$$

which implies

$$\sum_\alpha p_\alpha \log\left(\langle \psi^{y}_\alpha | \rho^{-1} | \psi^{y}_\alpha \rangle\right) \leq -\sum_\alpha p_\alpha \log p_\alpha \leq H(\vec{p}),$$

thus giving a different perspective on why the POVM inequality is tighter than the preparation inequality.