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Bell-Kochen-Specker theorem: A proof with 18 vectors

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Abstract

We present a "state-independent" proof of the Bell-Kochen-Specker theorem using only 18 four-dimensional vectors, which is a record for this kind of proof. This set of vectors contains subsets which allow us to develop a "state-specific" proof with ten vectors (also a record) and a "probabilistic" proof with seven vectors which reflects the algebraic structure of Hardy's nonlocality theorem.

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The Bell-Kochen-Specker (BKS) theorem [1,2] asserts that there is no consistent way of ascribing non-contextual definite answers to some sets of yes-no questions regarding an individual physical system. There are different versions of the theorem: "state-independent", "state-specific", and "probabilistic" [3,4].

Since the original state-independent proof by Kochen and Specker involving projectors over 117 three-dimensional real vectors was formulated, successive demonstrations have reduced the size of the set to only 20 four-dimensional vectors [5]; see for instance the references in Refs. [3,4].

In this paper we present a state-independent proof with only 18 real vectors in dimension 4. We also find subsets with ten and seven vectors making possible state-specific and probabilistic proofs, respectively. Finally, we show the relation between our probabilistic proof and a theorem by Hardy [6] on the incompatibility between quantum mechanics and local realistic theories.

Given an individual physical system, let $v(\boldsymbol{u})$ denote the answer (1 = yes, 0 = no) in said system to the proposition $P_{\boldsymbol{u}}$ (mathematically represented by the projector $|\boldsymbol{u}\rangle\langle\boldsymbol{u}|$) in a non-contextual hidden-variables (NCHV) theory. In order to simplify the notation we will write $\boldsymbol{u}$ as a row vector, omit its normalization constant, and speak indistinctly of propositions and projectors.

The premises behind the BKS theorem can be formulated as follows:

(a) In an individual system each proposition $P_{\boldsymbol{u}_i}$ has a unique answer, 0 or 1, which is independent of which other observables are being considered jointly (non-contextuality).

(b) For each set of one-dimensional projectors whose sum is the unit matrix in the n-dimensional

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Hilbert space of the states of the system, the answer to one and only one of the projectors is 1, and the answers to the other $n-1$ projectors are 0.

Following these rules, if the answer to the projector over a given vector is 1, the answers to the projectors over all orthogonal vectors must be zero.

Let us consider the answers to the projectors over the following nine sets of orthogonal four-dimensional vectors,

$$v(0,0,0,1)+v(0,0,1,0)+v(1,1,0,0)+v(1,-1,0,0)=1, \tag{1}$$

$$v(0,0,0,1)+v(0,1,0,0)+v(1,0,1,0)+v(1,0,-1,0)=1, \tag{2}$$

$$v(1,-1,1,-1)+v(1,-1,-1,1)+v(1,1,0,0)+v(0,0,1,1)=1, \tag{3}$$

$$v(1,-1,1,-1)+v(1,1,1,1)+v(1,0,-1,0)+v(0,1,0,-1)=1, \tag{4}$$

$$v(0,0,1,0)+v(0,1,0,0)+v(1,0,0,1)+v(1,0,0,-1)=1, \tag{5}$$

$$v(1,-1,-1,1)+v(1,1,1,1)+v(1,0,0,-1)+v(0,1,-1,0)=1, \tag{6}$$

$$v(1,1,-1,1)+v(1,1,1,-1)+v(1,-1,0,0)+v(0,0,1,1)=1, \tag{7}$$

$$v(1,1,-1,1)+v(-1,1,1,1)+v(1,0,1,0)+v(0,1,0,-1)=1, \tag{8}$$

$$v(1,1,1,-1)+v(-1,1,1,1)+v(1,0,0,1)+v(0,1,-1,0)=1. \tag{9}$$

There are 18 different vectors in (1)–(9). S will denote this set of vectors, and P the set of the corresponding propositions. Our state-independent version of the BKS theorem can be enunciated as follows:

There is no set of answers satisfying (a) and (b) to the set of propositions P.

The proof is straightforward: the sum of the right-hand sides of (1)–(9) is *odd*, whereas the sum of the left-hand sides is necessarily *even*, because each answer appears twice. The previous record [5] involved 11 equations with 20 vectors, appearing either twice or four times each.

The vectors in S can be interpreted as pure spin states of a system of two spin-1/2 particles. The 12 vectors in (1)–(4) are factorizable (i.e., of the form $(a,b)^{(1)}\otimes(c,d)^{(2)}$), and the corresponding projectors are products of local observables. For instance, in the usual Pauli representation, the (unnormalized) vector $(1,-1,0,0)$ represents the state $|\sigma_z=+1\rangle^{(1)}\otimes|\sigma_x=-1\rangle^{(2)}$, and its corresponding projector represents the proposition: does the observable $\widehat{\sigma}_z^{(1)}$ have a well-defined (hidden) value $+1$ *and* the observable $\widehat{\sigma}_x^{(2)}$ a well-defined (hidden) value -1, with $v(1,-1,0,0)=1$ if the answer is "yes", and $v(1,-1,0,0)=0$ otherwise?

The remaining six vectors in (5)–(9) are entangled, and the corresponding propositions cannot be factorized in terms of local observables. Each one can be expressed in terms of a pair of the observables $\widehat{\sigma}_z^{(1)}\otimes\widehat{\sigma}_z^{(2)}$, $\widehat{\sigma}_z^{(1)}\otimes\widehat{\sigma}_x^{(2)}$, $\widehat{\sigma}_x^{(1)}\otimes\widehat{\sigma}_z^{(2)}$ and $\widehat{\sigma}_x^{(1)}\otimes\widehat{\sigma}_x^{(2)}$ [7,8]. For instance, $(1,-1,1,1)$ is an eigenvector of $\widehat{\sigma}_z^{(1)}\otimes\widehat{\sigma}_x^{(2)}$ and $\widehat{\sigma}_x^{(1)}\otimes\widehat{\sigma}_z^{(2)}$ with eigenvalues -1 and $+1$, respectively, and therefore can be associated with the proposition: do the observables $\widehat{\sigma}_z^{(1)}\otimes\widehat{\sigma}_x^{(2)}$ *and* $\widehat{\sigma}_x^{(1)}\otimes\widehat{\sigma}_z^{(2)}$ have well-defined (hidden) values -1 *and* $+1$, respectively? Each of Eqs. (5)–(8) involves a pair of these entangled vectors, whereas (9) involves four.

A state-independent BKS proof is said to be "critical" [9] if it is based on a set of propositions not having any subset also making possible a state-independent proof. Peres' set of 24 vectors[2] is not critical; it contains Kernaghan's 20-vector critical set [5] and 95 other critical sets of 20, plus our previous set S and 15 other critical sets of 18. From the

[2] The vectors in Peres' set [7,8] can be geometrically interpreted as vectors along the 24 directions that join the center of a four-dimensional hypercube (tesseract) with the (pairwise opposite) centers of its eight three-dimensional faces (cubes), the centers of the 24 two-dimensional intersections of them (squares), and the 16 vertices. The sets of vectors in several other BKS "state-independent" proofs have been nicknamed according to their aspect (Kochen–Specker's 117-vector set [2] is also known as the "cat's cradle" [10], Peres' 33-vector set [7,8] as the "quantum polyhedron" [11], and Penrose's 40-vector set [9,12] as the "magic dodecahedron" [13]); therefore we suggest naming Peres' 24-vector set the "quantum tesseract".

definition of critical set there follows that none of these 18-vector sets are contained in any of the 96 *critical* 20-vector sets, which probably explains why they were not obtained previously. Peres' set does not contain any subset with fewer than 18 vectors allowing for a state-independent BKS proof. The assertions in this paragraph can be checked by means of a computer program generalizing to dimension 4 the one in Ref. [8].

In Ref. [4] we proved how, by increasing the number of vectors, we can go from probabilistic demonstrations to state-specific and then to state-independent ones. Here we will illustrate the reverse procedure, showing how our 18-vector set S contains subsets allowing for state-specific and probabilistic BKS proofs.

Each vector in our set S is orthogonal to seven other vectors in the set; therefore, we can prepare the system in a state that assigns the answer 1 to the projector over one of these vectors and the answer 0 to the other seven projectors over orthogonal vectors. For instance, if we prepare the system in the singlet state,

$$|\psi\rangle = \left(|+-\rangle - |-+\rangle\right)/\sqrt{2}, \tag{10}$$

then, by definition,

$$v(0,1,-1,0) = 1, \tag{11}$$

and we can discard from Eqs. (1)–(9) the vector $(0,1,-1,0)$ and those orthogonal to it, whose associated values are zero,

$$\begin{aligned} v(0,0,0,1) &= v(1,-1,-1,1) = v(1,1,1,1) \\ &= v(1,0,0,-1) = v(1,1,1,-1) \\ &= v(-1,1,1,1) = v(1,0,0,1) = 0. \end{aligned} \tag{12}$$

Therefore, only seven equations with ten different vectors remain,

$$v(0,0,1,0) + v(1,1,0,0) + v(1,-1,0,0) = 1, \tag{13}$$

$$v(0,1,0,0) + v(1,0,1,0) + v(1,0,-1,0) = 1, \tag{14}$$

$$v(1,-1,1,-1) + v(1,1,0,0) + v(0,0,1,1) = 1, \tag{15}$$

$$v(1,-1,1,-1) + v(1,0,-1,0) + v(0,1,0,-1) = 1, \tag{16}$$

$$v(0,0,1,0) + v(0,1,0,0) = 1, \tag{17}$$

$$v(1,1,-1,1) + v(1,-1,0,0) + v(0,0,1,1) = 1, \tag{18}$$

$$v(1,1,-1,1) + v(1,0,1,0) + v(0,1,0,-1) = 1. \tag{19}$$

There is no way of assigning definite answers to the ten propositions appearing in these equations. The proof is the same as before: the sum of the right-hand sides of (13)–(19) is *odd*, whereas the sum of the left-hand sides is necessarily *even*, because each answer appears twice.

Apparently this conclusion rests on the impossibility of unique answers to $10 + 8$ propositions: the ten different ones in (13)–(19), plus the one for the initial state (11) and the seven for orthogonal vectors (12). But in fact we can justify Eqs. (13)–(19) without the assistance of (12), using the following argument [3]: each subset of two or three vectors in the left-hand sides of (13)–(19) spans a subspace that contains the vector $(0,1,-1,0)$ (we can check that this vector can be expressed as a linear combination of the ones in each subset); therefore, even if the sums of the corresponding projectors are not the 4×4 unit matrix, the system is in an eigenstate, with eigenvalue 1, of each sum of projectors, and the sums of the corresponding answers must be 1. In consequence, our state-specific proof uses only ten vectors (or $10(+1)$, if we also count the initial state). The previous record [3] involved seven equations with 13 (or $13(+1)$, if we include the initial state) different eight-dimensional vectors, appearing either twice or four times each. Note nevertheless that the state-specific proof in Ref. [3] has the desirable property of using only factorizable vectors (i.e., of the form $(a,b)^{(1)} \otimes (c,d)^{(2)} \otimes (e,f)^{(3)}$), as opposed to our state-specific and state-independent proofs, or the state-independent one in Ref. [5].

Our state-specific BKS theorem cannot be interpreted as a contradiction between quantum mechanics (QM in the following) and NCHV in terms of *local* measurements, because, although it is possible to prepare the system in an entangled state (the singlet, in our previous choice), we cannot eliminate the remaining five non-factorizable propositions, and still reach a contradiction. For instance, our previous choice (13)–(19) contains the entangled state $(1,1,-1,1)$, and the answer to the corresponding propositions cannot be determined by means of a local measurement on particle 1 and a local measurement on particle 2.

We will now obtain a probabilistic version of the BKS theorem using only factorizable projectors, interpretable in terms of local measurements, and showing the incompatibility between QM and local realistic theories. Other correspondences between the BKS and Bell's theorems have been discussed in the literature [14].

Suppose we prepare ("preselect") two spin-1/2 particles in the entangled (but no "maximally entangled") Hardy state [6],

$$|\eta\rangle = \left(|{+}{+}\rangle - |{+}{-}\rangle - |{-}{+}\rangle\right)/\sqrt{3}. \tag{20}$$

Then, by definition,

$$v(1,-1,-1,0) = 1. \tag{21}$$

The answers to the projectors over any vector orthogonal to[3] $(1,-1,-1,0)$, must be zero; in particular,

$$v(0,0,0,1) = v(1,1,0,0) = v(1,0,1,0) = 0. \tag{22}$$

Let us assume that a subsequent measurement ("postselection") finds the system in the state

$$|\varphi\rangle = |\sigma_x = +1\rangle^{(1)} \otimes |\sigma_x = +1\rangle^{(2)} \tag{23}$$

(this is possible because $\langle\varphi|\eta\rangle \neq 0$); then,

$$v(1,1,1,1) = 1. \tag{24}$$

In the individual systems postselected in state (23), the answer to all propositions over vectors orthogonal to $(1,1,1,1)$ is 0; in particular

$$v(1,-1,0,0) = v(1,0,-1,0) = 0. \tag{25}$$

Replacing (22) and (25) in (1) and (2) leads to

$$v(0,1,0,0) = v(0,0,1,0) = 1. \tag{26}$$

[3] This vector does not belong to S nor to Peres' 24-vector set. It can be geometrically interpreted as the direction that joins the centers of a pair of opposite edges of a tesseract (see footnote 2); the other 15 directions joining the centers of the remaining opposite edges also represent Hardy states [6].

But $(0,1,0,0)$ and $(0,0,1,0)$ are orthogonal, and therefore the answers to the corresponding propositions cannot both be 1: we have reached a contradiction.

This probabilistic demonstration of the BKS theorem uses 7 (+2) vectors (seven in (22), (25), (26) plus the states η and φ). The term "probabilistic" follows from the fact that preparing the system in the initial state η (i.e., preselecting η) gives only a nonzero probability of finding the system in the final state φ (i.e., of postselecting φ), not a certainty.

Now we are going to show how this result relates to Hardy's nonlocality theorem [6]. Note that all the vectors involved in the previous proof are factorizable, with the exception of the initial state η. The answer to a factorizable proposition can be expressed in terms of the answers to the corresponding factors,

$$v\left[(a,b)^{(1)} \otimes (c,d)^{(2)}\right] = 1 \quad \Leftrightarrow \quad v(a,b)^{(1)} = v(c,d)^{(2)} = 1, \tag{27}$$

$$v\left[(a,b)^{(1)} \otimes (c,d)^{(2)}\right] = 0 \quad \Leftrightarrow \quad v(a,b)^{(1)} \times v(c,d)^{(2)} = 0. \tag{28}$$

In particular, if we preselect the state η,

$$v(1,-1,-1,0) = 1 \quad \Rightarrow \quad v(1,1,0,0) = 0 \quad \Leftrightarrow \quad v(1,0)^{(1)} \times v(1,1)^{(2)} = 0. \tag{29}$$

Similarly, postselecting φ (i.e., $v[(1,1)^{(1)} \otimes (1,[1]1)^{(2)}] = 1$) implies, using (27),

$$v(1,1)^{(1)} = 1, \tag{30}$$

$$v(1,1)^{(2)} = 1. \tag{31}$$

Then, (29) and (31) imply

$$v(1,0)^{(1)} = 0. \tag{32}$$

If we use premises (a) and (b) in the two-dimensional spin space of the first particle, from (32) we conclude that

$$v(0,1)^{(1)} = 1. \tag{33}$$

The answers (21), (31), (33) correspond in QM terms to the following value for the conditional probability of finding $\sigma_z^{(1)} = -1$ in a system prepared in the state η, *if* $\sigma_x^{(2)} = +1$,

$$P_\eta\left(\sigma_z^{(1)} = -1 \,\middle|\, \sigma_x^{(2)} = +1\right) = 1. \tag{34}$$

If we interchange the roles of particles 1 and 2, a similar reasoning leads us to

$$P_\eta\left(\sigma_z^{(2)}=-1\,\middle|\,\sigma_x^{(1)}=+1\right)=1. \tag{35}$$

Eq. (21) and the first part of (22) ($v(0,0,0,1)=0$) translate into

$$P_\eta\left(\sigma_z^{(1)}=-1,\sigma_z^{(2)}=-1\right)=0. \tag{36}$$

Finally, the fact that the system can be postselected in the state φ, used to obtain (24), means that

$$P_\eta\left(\sigma_x^{(1)}=+1,\sigma_x^{(2)}=+1\right)>0. \tag{37}$$

Eqs. (34)–(37) translate into QM terms the set of answers used in our probabilistic BKS theorem. If we assume that particles 1 and 2 are localized in two space-like separated regions (this assumption was not necessary for the BKS theorem), Eqs. (34)–(37) are just those in Hardy's nonlocality theorem, which we can summarize as follows:

Let us consider a system of two space-like separated particles prepared in the spin state η, and suppose that we accept EPR's sufficient condition for existence of elements of reality [15]. In those individual systems in which $\sigma_x^{(2)}=+1$, and $\sigma_x^{(1)}=+1$ (a condition that can be fulfilled because of (37)), Eqs. (34), (35) imply that we can jointly infer two elements of reality, $\sigma_z^{(1)}=-1$ and $\sigma_z^{(2)}=-1$. But these results can never be obtained in a joint measurement in the state η, because of (36): QM and elements of reality are not compatible, q.e.d.

We could also establish the inverse correspondence: Eqs. (34)–(37) in Hardy's theorem can be translated into a set of definite answers to propositions that proves our probabilistic BKS theorem; we omit the details for brevity.

In summary: we have found sets of four-dimensional real vectors that allow us to develop state-independent, state-specific and probabilistic BKS proofs, illustrating the relations between these three versions of the theorem. In the first two cases, our sets are the most economical yet, in terms of vectors used, in any dimension. On the other hand, the probabilistic proof shows the same kind of contradiction as Hardy's theorem, and suggest an algebraic reading of it.

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