Quantum information theory

Lecture 8

Quantum states: I. Mixed states

Pure states:

State vector $|\psi\rangle = \sum_j |e_j\rangle \underbrace{\langle e_j|\psi\rangle}_{c_j = \text{(probability amplitude)}}$

$$1 = \langle\psi|\psi\rangle = \sum_j |c_j|^2$$

Measurement in basis $|e_j\rangle$:

$$p_k = |\langle e_k|\psi\rangle|^2 = \langle e_k|\psi\rangle\langle\psi|e_k\rangle = \langle e_k|P_\psi|e_k\rangle$$

Mixed states: What if we know system is in state $|\psi_j\rangle$ with probability q_j (ensemble)

$$p_k = \sum_j \underbrace{p_{k|j}}_{|\langle e_k|\psi_j\rangle|^2 = \langle e_k|\psi_j\rangle\langle\psi_j|e_k\rangle} q_j = \langle e_k|\Big(\underbrace{\sum_j q_j|\psi_j\rangle\langle\psi_j|}_{\text{density operator } \rho}\Big)|e_k\rangle$$

$p_k = \mathrm{tr}(\rho|e_k\rangle\langle e_k|)$

Observable $A = \sum_k \lambda_k |e_k\rangle\langle e_k|$

$\langle A\rangle = \sum_k \lambda_k p_k = \mathrm{tr}(\rho A)$

density operator ρ

$\{q_j, |\psi_j\rangle\}$ or $\{\sqrt{q_j}\,|\psi_j\rangle\}$ is an ensemble decomposition of ρ.

[A density operator is a Hermitian operator with nonnegative eigenvalues that sum to 1 ($\mathrm{tr}\,\rho = 1$).]

(A)

Let $|\phi_j\rangle$, $j=1,\dots,D$ be a basis (i.e., D linearly independent, not necessarily orthogonal or normalized) for a D-dimensional Hilbert space. Any vector can be expanded uniquely as

$$|\psi\rangle = \sum_j c_j |\phi_j\rangle$$

If $|e_k\rangle$, $k=1,\dots,D$, is an orthonormal basis,

$$\langle e_k|\psi\rangle = \sum_j c_j \underbrace{\langle e_k|\phi_j\rangle}_{S_{kj}}$$

$$\begin{pmatrix} \langle e_1|\psi\rangle \\ \vdots \\ \langle e_D|\psi\rangle \end{pmatrix} = S \begin{pmatrix} c_1 \\ \vdots \\ c_D \end{pmatrix}$$

S is invertible because of the uniqueness of the c_j (i.e., the l.i. of $|\phi_j\rangle$).

$$\langle \phi_j|\psi\rangle = \sum_k \underbrace{\langle \phi_j|e_k\rangle}_{\langle e_k|\phi_j\rangle^* = S^*_{kj} = (S^\dagger)_{jk}} \langle e_k|\psi\rangle$$

$$(S^\dagger)^{-1} = (S^{-1})^\dagger$$

$$\begin{pmatrix} \langle \phi_1|\psi\rangle \\ \vdots \\ \langle \phi_D|\psi\rangle \end{pmatrix} = S^\dagger \begin{pmatrix} \langle e_1|\psi\rangle \\ \vdots \\ \langle e_D|\psi\rangle \end{pmatrix} = S^\dagger S \begin{pmatrix} c_1 \\ \vdots \\ c_D \end{pmatrix}$$

$$\begin{pmatrix} \langle e_1|\psi\rangle \\ \vdots \\ \langle e_D|\psi\rangle \end{pmatrix} = (S^{-1})^\dagger \begin{pmatrix} \langle \phi_1|\psi\rangle \\ \vdots \\ \langle \phi_D|\psi\rangle \end{pmatrix}$$

These inner products specify $|\psi\rangle$.

Hermitian operators are a real vector space; L_V is the complexification.

Back up 2 steps:

1. The operators on a D-dimensional Hilbert space V make up a D^2-dimensional complex vector space L_V, with inner product $(A, B) = \mathrm{tr}(A^\dagger B)$.

Orthonormal basis for L_V: $\tau_{jk} = |e_j\rangle\langle e_k|$

$$A = \sum_{j,k} A_{jk} |e_j\rangle\langle e_k|$$

$$\mathrm{tr}(\tau_{\ell m}^\dagger \tau_{jk}) = \mathrm{tr}(|e_m\rangle\langle e_\ell|e_j\rangle\langle e_k|)$$

$$= \langle e_k|e_m\rangle\langle e_\ell|e_j\rangle$$

$$= \delta_{j\ell}\,\delta_{km}$$

There is no such basis in a real vector space.

Basis of 1-d projectors: (pure states)

This is not an orthonormal basis. That's impossible: $\mathrm{tr}(|\psi\rangle\langle\psi|\phi\rangle\langle\phi|) = |\langle\phi|\psi\rangle|^2$

$|\phi_\alpha\rangle$, $\alpha = 1, \dots, D^2$

$j = 1, \dots, D$: $|e_j\rangle\langle e_j| = \tau_{jj}$

$j < k$: $|\chi_{jk}\rangle\langle\chi_{jk}| = \frac{1}{2}(\tau_{jj} + \tau_{kk} + \tau_{jk} + \tau_{kj})$

$|\xi_{jk}\rangle\langle\xi_{jk}| = \frac{1}{2}(\tau_{jj} + \tau_{kk} + i(-\tau_{jk} + \tau_{kj}))$

$$|\chi_{jk}\rangle \equiv \frac{1}{\sqrt{2}}(|e_j\rangle + |e_k\rangle)$$

$$|\xi_{jk}\rangle \equiv \frac{1}{\sqrt{2}}(|e_j\rangle + i|e_k\rangle)$$

An operator A is specified by the inner products

$$(|\phi_\alpha\rangle\langle\phi_\alpha|, A) = \mathrm{tr}(|\phi_\alpha\rangle\langle\phi_\alpha| A) = \langle\phi_\alpha|A|\phi_\alpha\rangle$$

An operator A is (over) specified by $\underbrace{\langle\psi|A|\psi\rangle}_{\text{"sandwiches"}}$ for all $|\psi\rangle$.

An operator A is Hermitian iff $\langle\psi|A|\psi\rangle$ is real for all $|\psi\rangle$.

② A positive operator G is one such that $\langle\psi|G|\psi\rangle$ is real and nonnegative for all $|\psi\rangle$. This is denoted $G\ge 0$. An operator G is positive iff it is Hermitian with nonnegative eigenvalues.

[Positive operators G_1 and G_2 are orthogonal, i.e. $\mathrm{tr}(G_1G_2)=0$, iff $G_1G_2=0$.]

Positive definite $G>0$ means $\langle\psi|G|\psi\rangle>0$ for all $|\psi\rangle$.
$\iff$ (i) Hermitian with positive eigenvalues
(ii) Positive and invertible.

Positive operators have a square root: $\sqrt{G}=\sum_j\sqrt{\lambda_j}\,|e_j\rangle\langle e_j|$.

Characterization of a density operator. ρ is a density operator iff (i) $\rho\ge 0$ and (ii) $\mathrm{tr}(\rho)=1$.

Pure states: $\rho=|\psi\rangle\langle\psi|$ is a 1-d (rank-1) projector.

① A unit-trace Hermitian operator ρ is a pure state iff $\rho^2=\rho$

[A Hermitian operator P is a projector iff $P^2=P$. $\begin{pmatrix}1&1\\0&0\end{pmatrix}$ squares to itself, but is not a projector.]

② A density operator ρ is a pure state iff $\mathrm{tr}(\rho^2)=1$

$\Leftarrow$: trivial

$\Rightarrow$: $0=\mathrm{tr}(G_1G_2)=\mathrm{tr}\left(G_2^{1/2}G_1^{1/2}G_1^{1/2}G_2^{1/2}\right)=\mathrm{tr}\left(\left(G_1^{1/2}G_2^{1/2}\right)^\dagger G_1^{1/2}G_2^{1/2}\right)$

$\Rightarrow\ 0=G_1^{1/2}G_2^{1/2}\ \Rightarrow\ 0=G_1G_2$

Convex Sets: A set S is convex if for any $v_1, v_2 \in S$,

$\lambda v_1 + (1-\lambda) v_2 \in S$, $0 \le \lambda \le 1$.

$\Downarrow$

$\underbrace{\sum_j \lambda_j v_j}_{\text{convex combination or mixture}} \in S$, $\lambda_j \ge 0$, $\sum_j \lambda_j = 1$

Examples:

An extreme point of S is a point that can't be written as a (proper) convex combination ($0<\lambda<1$) of any other points.

If S is closed and bounded, every point in S can be written as a convex combination of extreme points.

Simplex: Closed & bounded convex set where expansion in terms of extreme points is unique

[The density operators are a closed and bounded convex set whose extreme points are the pure states.]

Qubits:

$$\rho = A_0 \mathbb{1} + \vec{A}\cdot\vec{\sigma} = (A_0 + |\vec{A}|)\,|\vec{n}\rangle\langle\vec{n}| + (A_0 - |\vec{A}|)\,|-\vec{n}\rangle\langle-\vec{n}|$$

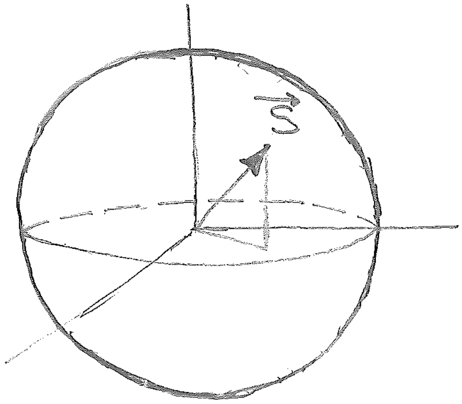
Hermitian $\Rightarrow$ $A_0, \vec{A}$ real

$1 = \mathrm{tr}(\rho) \Rightarrow A_0 = \frac{1}{2}$

positive $\Rightarrow$ $|\vec{A}| \le A_0 = 1/2$

$$\boxed{\rho = \tfrac{1}{2}(\mathbb{1} + \vec{S}\cdot\vec{\sigma}),\quad |\vec{S}| \le 1 \qquad \vec{S} = \mathrm{tr}(\rho\vec{\sigma}) = \binom{\text{Bloch}}{\text{vector}}}$$

Bloch sphere



Pure states on surface

Mixed states in interior

Contrast higher dimensions:

Mixed states: (d^2-1)-dimensional manifold

Pure states: $2(d-1)$ dimensional

Multiple boundaries for every number of zero eigenvalues from $1, \ldots, d-1$.

Freedom in ensemble decompositions for qubits:

$$\rho = \sum_j q_j \underbrace{|\psi_j\rangle\langle\psi_j|}_{\frac{1}{2}(1 + \vec{n}_j \cdot \vec{\sigma})} = \frac{1}{2}\Big(1 + \vec{\sigma} \cdot \underbrace{\sum_j q_j \vec{n}_j}_{\vec{S}}\Big)$$

<u>Freedom in ensemble decompositions</u> (HJW theorem; Schrödinger). Space of density operators is not a simplex.

Eigendecomposition $\rho = \sum_{j=1}^{D} \lambda_j |e_j\rangle\langle e_j| = \sum_{j=1}^{D} |\tilde{e}_j\rangle\langle\tilde{e}_j|$

$$|\tilde{e}_j\rangle = \sqrt{\lambda_j}\, |e_j\rangle$$

<u>Support</u> S of ρ is the subspace spanned by eigenvectors with nonzero eigenvalue. <u>Null subspace</u> (kernel) K of ρ is the subspace spanned by eigenvectors with zero eigenvalue. These subspaces are orthogonal. The subspace orthogonal to a subspace S is called the <u>orthocomplement</u> of S.

The <u>rank</u> of ρ is the dimension of the support

$$P_S = \sum_{\lambda_j \neq 0} |e_j\rangle\langle e_j| \qquad P_K = \sum_{\lambda_j = 0} |e_j\rangle\langle e_j|$$

Ensemble decomposition: $\{p_\alpha, |\psi_\alpha\rangle\}$ or $\{|\tilde{\psi}_\alpha\rangle = \sqrt{p_\alpha}\,|\psi_\alpha\rangle\}$

$$\rho = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| = \sum |\tilde{\psi}_\alpha\rangle\langle\tilde{\psi}_\alpha|$$

All states in the ensemble decomposition lie in the support of ρ [Proof: If $|e_j\rangle$ lies in the null subspace, then $0 = \langle e_j|\rho|e_j\rangle = \sum_\alpha |\langle e_j|\tilde{\psi}_\alpha\rangle|^2$ $\Rightarrow \langle e_j|\tilde{\psi}_\alpha\rangle = 0$ for all $\alpha \Rightarrow |\tilde{\psi}_\alpha\rangle$ is orthogonal to the null subspace and thus lies in the support.]

<u>HJW theorem.</u> Two ensemble decompositions, $\{|\tilde{\psi}_\alpha\rangle\}$ and $\{|\tilde{\phi}_\alpha\rangle\}$, correspond to the same density operator iff there exists a unitary matrix $U_{\alpha\beta}$ such that

$$|\tilde{\phi}_\alpha\rangle = \sum U_{\alpha\beta}|\tilde{\psi}_\beta\rangle .$$

Add zero vectors to ensemble with fewer elements.

Column-vector notation

<u>Proof:</u>

$\Leftarrow$

$$\sum_\alpha |\tilde{\phi}_\alpha\rangle\langle\tilde{\phi}_\alpha| = \sum_{\alpha,\beta,\gamma} U_{\alpha\beta}|\tilde{\psi}_\beta\rangle\langle\tilde{\psi}_\gamma| U^*_{\alpha\gamma}$$

$$= \sum_{\beta,\gamma} |\tilde{\psi}_\beta\rangle\langle\tilde{\psi}_\gamma| \underbrace{\sum_\alpha U_{\alpha\beta} U^*_{\alpha\gamma}}_{\delta_{\beta\gamma}}$$

$$= \sum_\beta |\tilde{\psi}_\beta\rangle\langle\tilde{\psi}_\beta|$$

$$\Longrightarrow \quad \rho = \sum_{\alpha} |\tilde{\psi}_\alpha\rangle\langle\tilde{\psi}_\alpha| = \underbrace{\sum_{|e_j\rangle\in S} |\tilde{e}_j\rangle\langle\tilde{e}_j|}_{\text{eigendecomposition}}$$

We know $|\tilde{\psi}_\alpha\rangle \in S$, so this sum can be restricted to the support

$$|\tilde{\psi}_\alpha\rangle = \sum_{|e_j\rangle\in S} |e_j\rangle\langle e_j|\tilde{\psi}_\alpha\rangle = \sum_{|e_j\rangle\in S} |\tilde{e}_j\rangle \underbrace{\frac{\langle e_j|\tilde{\psi}_\alpha\rangle}{\sqrt{\lambda_j}}}_{M_{\alpha j}}$$

$\lambda_j \neq 0$ because $|e_j\rangle \in S$

$$\sum_{\alpha} M_{\alpha j} M^*_{\alpha k} = \left\langle e_j \right| \frac{\overbrace{\sum_\alpha |\tilde{\psi}_\alpha\rangle\langle\tilde{\psi}_\alpha|}^{\rho}}{\sqrt{\lambda_j \lambda_k}} \left| e_k \right\rangle$$

$$= \frac{\langle e_j|\rho|e_k\rangle}{\sqrt{\lambda_j \lambda_k}}$$

$$= \frac{\lambda_j \delta_{jk}}{\sqrt{\lambda_j \lambda_k}}$$

$$= \delta_{jk}$$

$M_{\alpha j}$ is a $N \times R$ matrix, where N is the number of elements in the ensemble decomposition and R is the rank of ρ.

M can be extended to be unitary.