Quantum information theory

Lectures 13-14

Quantum dynamics. I-II. Generalized measurements

Dynamics and Measurement

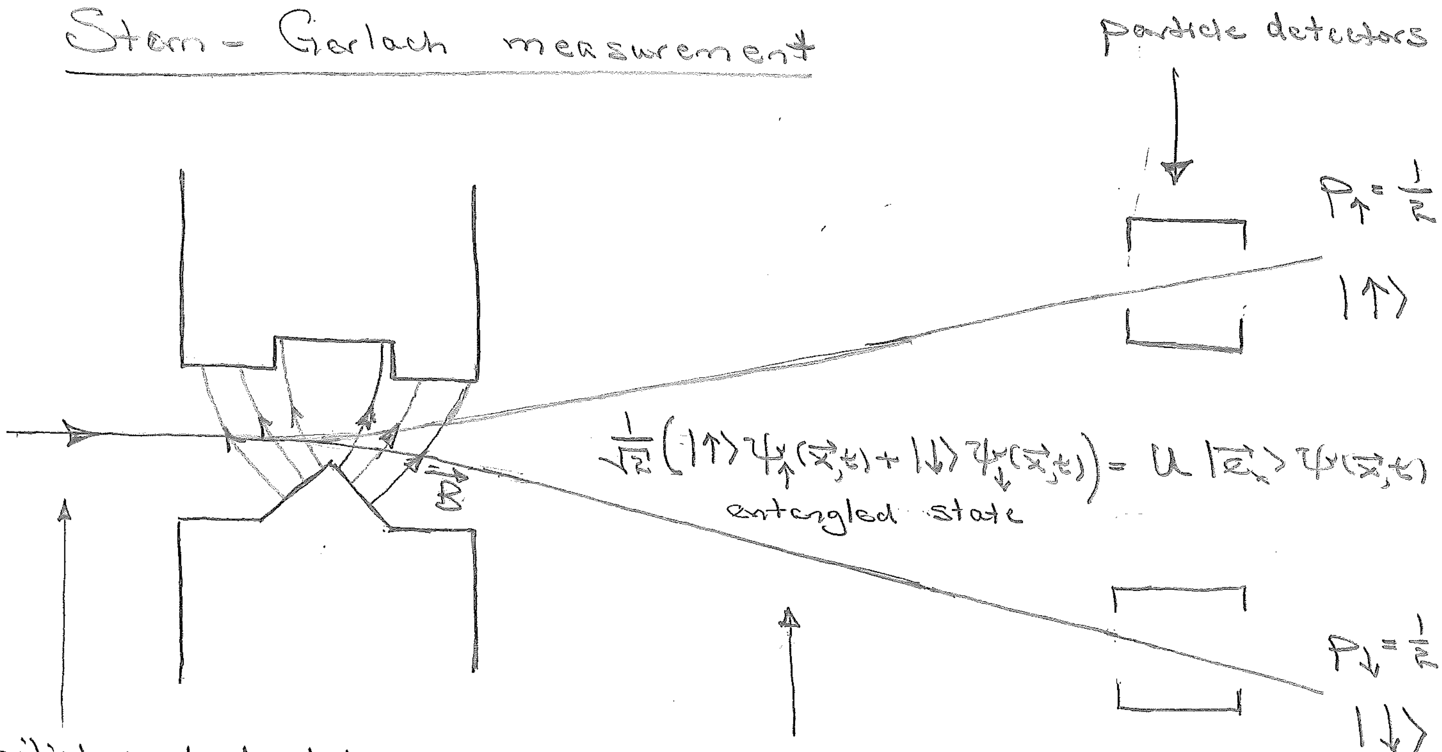
Open systems
Couple system to environment (ancilla)

Trace-preserving quantum operations

Measurement models
Couple system to "measuring apparatus" (ancilla) and then observe apparatus

Trace-decreasing quantum operations

Stern-Gerlach measurement



$$\frac{1}{\sqrt{2}}\left(|\uparrow\rangle \psi_\uparrow(\vec{x},t) + |\downarrow\rangle \psi_\downarrow(\vec{x},t)\right) = U\,|\vec{e}_x\rangle \psi(\vec{x},t)$$

entangled state

not a measurement

$$= \frac{1}{\sqrt{2}}\left[|\vec{e}_x\rangle \frac{1}{\sqrt{2}}\left(\psi_\uparrow(\vec{x},t) + \psi_\downarrow(\vec{x},t)\right) + |-\vec{e}_x\rangle \frac{1}{\sqrt{2}}\left(\psi_\uparrow(\vec{x},t) - \psi_\downarrow(\vec{x},t)\right)\right]$$

initial product state

$$|\vec{e}_x\rangle \psi(\vec{x},t) = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right)\psi(\vec{x},t)$$

System Q   apparatus

Relation to polarizing beamsplitter

Measurement statistics:

$$p_\uparrow = \langle\uparrow|\rho|\uparrow\rangle = \mathrm{tr}(\rho P_\uparrow) \qquad p_\downarrow = \langle\downarrow|\rho|\downarrow\rangle = \mathrm{tr}(\rho P_\downarrow)$$

Post-measurement state:

$$\rho_\uparrow = \frac{P_\uparrow \rho P_\uparrow}{p_\uparrow} = |\uparrow\rangle\langle\uparrow| \qquad \rho_\downarrow = \frac{P_\downarrow \rho P_\downarrow}{p_\downarrow} = |\downarrow\rangle\langle\downarrow|$$

Ignorance of result: $\rho' = p_\uparrow \rho_\uparrow + p_\downarrow \rho_\downarrow = P_\uparrow \rho P_\uparrow + P_\downarrow \rho P_\downarrow$

Measurement statistics — POVM (Positive-operator-valued measure)

Post-measurement state — Quantum operation

We can get at POVMs from abstract considerations.

Orthogonal projectors (OP)	POVM $\{E_\alpha\}$
Observable $A = \sum_\lambda \lambda P_\lambda$ ⟶	?
① $p(\lambda) = \mathrm{tr}(\rho P_\lambda) \ge 0 \iff P_\lambda \ge 0$ ⟶	$E_\alpha \ge 0$ $\quad p_\alpha = \mathrm{tr}(\rho E_\alpha) \ge 0$ ↖ POVM element
② Normalization: $\sum_\lambda p(\lambda) = 1 \iff \sum_\lambda P_\lambda = 1$ ⟶	$\sum_\alpha E_\alpha = 1$ $\quad \sum_\alpha p_\alpha = 1$ ↖ resolution of 1 completeness relation
③ For each λ there are states with support restricted to S_λ give certainty. $\iff P_\lambda P_{\lambda'} = \delta_{\lambda\lambda'} P_\lambda$ (orthogonal projectors) ⟶	Generally there is no state that gives certainty for result α
④ von Neumann measurement: $\rho_\lambda = P_\lambda \rho P_\lambda / p(\lambda)$ ⟶	?

$0 \le E_\alpha \le 1$

$\Updownarrow$

$E_\alpha = \sum_j \lambda_j |e_j\rangle\langle e_j|, \quad 0 \le \lambda_j \le 1$

Qubit examples:

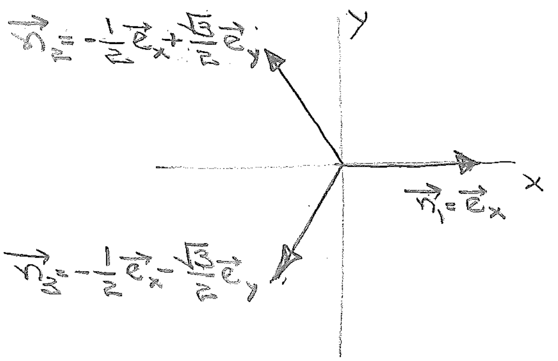
1. $E_\uparrow = P_\uparrow, \quad E_\downarrow = P_\downarrow$ One-dimensional orthogonal projectors (ODOP)

2. $$E_1 = \tfrac{1}{2}|\vec{e}_z\rangle\langle\vec{e}_z| = \tfrac{1}{4}(1+Z)$$
$$E_2 = \tfrac{1}{2}|-\vec{e}_z\rangle\langle-\vec{e}_z| = \tfrac{1}{4}(1-Z)$$
$$E_3 = \tfrac{1}{2}|\vec{e}_x\rangle\langle\vec{e}_x| = \tfrac{1}{4}(1+X)$$
$$E_4 = \tfrac{1}{2}|-\vec{e}_x\rangle\langle-\vec{e}_x| = \tfrac{1}{4}(1-X)$$

Coin flip to decide on measurement of z-spin or x-spin.

$\begin{pmatrix}\text{max probability}\\ \text{for any result}\end{pmatrix} = \frac{1}{2}$

3. Trine measurement:



$$E_1 = \tfrac{2}{3}|\vec{n}_1\rangle\langle\vec{n}_1| = \tfrac{1}{3}(1+X)$$
$$E_2 = \tfrac{2}{3}|\vec{n}_2\rangle\langle\vec{n}_2| = \tfrac{1}{3}\left(1-\tfrac{1}{2}X+\tfrac{\sqrt{3}}{2}Y\right)$$
$$E_3 = \tfrac{2}{3}|\vec{n}_3\rangle\langle\vec{n}_3| = \tfrac{1}{3}\left(1-\tfrac{1}{2}X-\tfrac{\sqrt{3}}{2}Y\right)$$

$\begin{pmatrix}\text{max probability}\\ \text{for any result}\end{pmatrix} = \frac{2}{3}$

4. $$E_1 = \tfrac{1}{3}(1+X) = \tfrac{2}{3}|\vec{e}_x\rangle\langle\vec{e}_x|$$

coarse graining

$$E_{23} = E_2 + E_3 = \tfrac{2}{3}\left(1-\tfrac{1}{2}X\right) = \tfrac{1}{3}|\vec{e}_x\rangle\langle\vec{e}_x| + |-\vec{e}_x\rangle\langle-\vec{e}_x|$$

5. Tetrahedron measurement

6. General: $E_\alpha = g_\alpha(1+\vec{r}_\alpha\cdot\vec{\sigma}) \longrightarrow$

$$\boxed{\begin{aligned} E_\alpha \ge 0 &\Rightarrow g_\alpha \ge 0,\ |\vec{r}_\alpha| \le 1\\ E_\alpha \le 1 &\Rightarrow g_\alpha(1+|\vec{r}_\alpha|) \le 1\\ \sum_\alpha E_\alpha = 1 &\Rightarrow \sum_\alpha g_\alpha = 1\\ &\ \sum_\alpha g_\alpha \vec{r}_\alpha = 0\end{aligned}}$$

7. HO example: Simultaneous measurement of x and p

$$E_\alpha = \frac{1}{\pi}|\alpha\rangle\langle\alpha|$$

$$\int d^2\alpha\, E_\alpha = \int \frac{d^2\alpha}{\pi}|\alpha\rangle\langle\alpha| = 1$$

$$Q(\alpha) = \mathrm{tr}(\rho E_\alpha) = \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle$$

Q function
Husimi distribution

POVMs:

1. Coarse graining
2. Unsharp $\longleftrightarrow$ no states that give certainty $\longleftrightarrow$ simultaneous measurement of incompatible observables $\longleftrightarrow$ qubit examples

Quantum operations:

1. Open systems: Ancilla
   Environment
   Measuring apparatus
   Measurement model

Measurement model

System Q Initial state ρ

Ancilla A Initial state σ $\longrightarrow$ $\rho\otimes\sigma$

$\downarrow$

Unitary evolution U : $U\rho\otimes\sigma U^\dagger$

OP measurement on ancilla: P_α

$p_\alpha = \mathrm{tr}(P_\alpha U\rho\otimes\sigma U^\dagger)$

Measurement statistics

$$\rho_\alpha = \frac{\mathrm{tr}_A(P_\alpha U\rho\otimes\sigma U^\dagger P_\alpha)}{p_\alpha} = \frac{\mathrm{tr}_A(P_\alpha U\rho\otimes\sigma U^\dagger)}{p_\alpha}$$

Post-measurement state

$$\boxed{\mathcal{Q}_\alpha(\rho) = \mathrm{tr}_A(P_\alpha U\rho\otimes\sigma U^\dagger) = p_\alpha\rho_\alpha}$$

Quantum operations

trace-decreasing

trace-preserving

Ignorance of result: $\rho' = \sum_\alpha p_\alpha\rho_\alpha$

$= \sum_\alpha \mathrm{tr}_A(P_\alpha U\rho\otimes\sigma U^\dagger)$

$= \mathrm{tr}_A(U\rho\otimes\sigma U^\dagger)$

$$\mathcal{Q}(\rho) = \sum_\alpha \mathcal{Q}_\alpha(\rho) = \mathrm{tr}_A(U\rho\otimes\sigma U^\dagger)$$

Use capital script letters for quantum operations.

2 Kraus decomposition (operator-sum representation). Description of quantum operations in terms of <u>system</u> operators.

$$\sigma = \sum_k \lambda_k |e_k\rangle\langle e_k| \longleftarrow$$ initial ancilla density operator

$$P_\alpha = \sum_j |f_{\alpha j}\rangle\langle f_{\alpha j}| \quad , \quad 1 = \sum_\alpha P_\alpha = \sum_{\alpha j} |f_{\alpha j}\rangle\langle f_{\alpha j}|$$

orthonormal basis for A

$$\mathcal{A}_\alpha(\rho) = \mathrm{tr}_A(P_\alpha U \rho\otimes\sigma U^\dagger) = \sum_{j,k} \lambda_k \mathrm{tr}_A(|f_{\alpha j}\rangle\langle f_{\alpha j}| U \rho\otimes|e_k\rangle\langle e_k| U^\dagger)$$

$$= \sum_{j,k} \underbrace{\sqrt{\lambda_k}\langle f_{\alpha j}|U|e_k\rangle}_{\equiv A_{\alpha jk}} \rho \underbrace{\langle e_k|U^\dagger|f_{\alpha j}\rangle\sqrt{\lambda_k}}_{= A^\dagger_{\alpha jk}}$$

system operator

$$U|\psi\rangle\otimes|e_k\rangle = \sum_{\alpha,j} \frac{1}{\sqrt{\lambda_k}} A_{\alpha jk}|\psi\rangle\otimes|f_{\alpha j}\rangle$$

relative-state decomposition

$$\mathcal{A}_\alpha(\rho) = \sum_{j,k} A_{\alpha jk}\,\rho\, A^\dagger_{\alpha jk} \longleftarrow$$ [Kraus decomposition / Operator-sum representation]

trace-decreasing quantum operation

Kraus operator / Operation element $\longrightarrow$ any operator A with $A^\dagger A \le 1$

$$p_\alpha = \mathrm{tr}(\mathcal{A}_\alpha(\rho)) = \mathrm{tr}\Big(\rho \underbrace{\sum_{j,k} A^\dagger_{\alpha jk} A_{\alpha jk}}_{0\le E_\alpha\le 1}\Big)$$

Measurement statistics are always described by a POVM

$$\sum_\alpha E_\alpha = \sum_{\alpha,j,k} A^\dagger_{\alpha jk} A_{\alpha jk} = \sum_{\alpha,j,k} \lambda_k \langle e_k|U^\dagger|f_{\alpha j}\rangle\langle f_{\alpha j}|U|e_k\rangle$$

$$= 1 \underbrace{\sum_k \lambda_k \langle e_k|e_k\rangle}_{\mathrm{tr}(\sigma)}$$

$$= 1$$

Summarize:

Measurement statistics

$$p_\alpha = \mathrm{tr}(\mathcal{Q}_\alpha(\rho)) = \mathrm{tr}\Big(\rho \overbrace{\sum_{jk} A^\dagger_{\alpha jk} A_{\alpha jk}}^{E_\alpha}\Big)$$

Post-measurement state

$$\rho_\alpha = \frac{\mathcal{Q}_\alpha(\rho)}{p_\alpha} = \frac{1}{p_\alpha}\sum_{j,k} A_{\alpha jk}\,\rho\, A^\dagger_{\alpha jk}$$

$$0 \le E_\alpha \le 1\,, \qquad 1 = \sum_\alpha E_\alpha = \sum_{\alpha jk} A^\dagger_{\alpha jk} A_{\alpha jk}$$

Ignorance of result

$$\rho' = \sum_\alpha p_\alpha \rho_\alpha = \sum_\alpha \mathcal{Q}_\alpha(\rho) = \sum_{\alpha jk} A_{\alpha jk}\,\rho\, A^\dagger_{\alpha jk} \equiv \mathcal{Q}(\rho)$$

① Any interaction with an environment can be regarded as a measurement with results ignored.

② The jk in $A_{\alpha jk}$ can be regarded as a coarse graining over unavailable results, j from coarse-grained measurement on A, k from not knowing initial state of A.

<u>Kraus representation theorem.</u> Given a set of superoperators, $\mathcal{Q}_\alpha(\rho) = \sum_j A_{\alpha j}\,\rho\, A^\dagger_{\alpha j}$, with $\sum_{\alpha j} A^\dagger_{\alpha j} A_{\alpha j} = 1$, there exists an ancilla with initial state $|e_0\rangle\langle e_0|$, unitary $\mathcal{U}$ on QA, and orthogonal projectors P_α such that

$$\mathcal{Q}_\alpha(\rho) = \mathrm{tr}_A\big(P_\alpha\, \mathcal{U}\, \rho \otimes |e_0\rangle\langle e_0|\, \mathcal{U}^\dagger\big).$$

Quantum operations $\Longleftrightarrow$ measurement model

Why pure-state ancilla? Can always purify σ.

What about a single quantum operation? Complete w/ $A_2 = \sqrt{1-E_1}$

Do we need to consider quantum operations on the ancilla?

No, the theorem shows these could always be gotten from QPs on a yet bigger ancilla.

Proof: Take any ancilla pure state $|e_0\rangle\langle e_0|$ and any ancilla orthonormal basis $|f_{\alpha j}\rangle$. Partially define a joint QA operator U by

$$\underbrace{\langle f_{\alpha j}|U|e_0\rangle = A_{\alpha j}}_{\text{partial definition}} \Longleftrightarrow \underbrace{U|\psi\rangle\otimes|e_0\rangle = \sum_{\alpha j} A_{\alpha j}|\psi\rangle\otimes|f_{\alpha j}\rangle}_{\text{relative-state decomposition}}$$

Is this partial definition consistent with U being unitary? It is if it preserves inner products:

$$\big(\langle\phi|\otimes\langle e_0|U^\dagger\big)\big(U|\psi\rangle\otimes|e_0\rangle\big) = \sum_{\substack{\alpha,j\\ \beta,k}} \langle\phi|A^\dagger_{\beta k}A_{\alpha j}|\psi\rangle \underbrace{\langle f_{\beta k}|f_{\alpha j}\rangle}_{\delta_{\alpha\beta}\delta_{jk}}$$

$$= \langle\phi|\underbrace{\sum_{\alpha,j} A^\dagger_{\alpha j}A_{\alpha j}}_{1}|\psi\rangle$$

$$= \langle\phi|\psi\rangle$$

This means that U maps the D-dimensional subspace $\mathcal{H}_Q\otimes R_0$, where R_0 is the 1-d subspace spanned by $|e_0\rangle$, unitarily to a D-dimensional subspace S_0 of $\mathcal{H}_Q\otimes\mathcal{H}_A$. U can be extended to be a unitary operator on all of $\mathcal{H}_Q\otimes\mathcal{H}_A$ by defining it to map the subspace $\mathcal{H}_Q\otimes R_\perp$, where $R_\perp$ is orthogonal to R_0, unitarily to the subspace orthogonal to S_0.

Now

$$
\begin{aligned}
Q_\alpha(\rho) &= \sum_j A_{\alpha j}\,\rho\, A_{\alpha j}^\dagger \\
&= \sum_j \langle f_{\alpha j}|\, U\, \rho\otimes|e_0\rangle\langle e_0|\, U^\dagger\, |f_{\alpha j}\rangle \\
&= \mathrm{tr}_A\Big(\underbrace{\sum_j |f_{\alpha j}\rangle\langle f_{\alpha j}|}_{P_\alpha}\, U\rho\otimes|e_0\rangle\langle e_0|\,U^\dagger\Big)
\end{aligned}
$$

$\blacksquare$

<u>Quantum state tomography</u>

ρ recoverable from $p_\alpha = \mathrm{tr}(\rho E_\alpha)$

operator basis
rank-one?

$$\mathcal{G} = \sum_\alpha |E_\alpha)(E_\alpha| \longleftarrow$$ positive superoperator, so invertible

$$\rho = \sum_\alpha \mathcal{G}^{-1}|E_\alpha)(E_\alpha|\rho) = \sum_\alpha p_\alpha \mathcal{G}^{-1}|E_\alpha)$$

Qubit:

① Measure X, Y, and Z — 6 probabilities, 3 independent $\Longrightarrow$ $\langle X\rangle$, $\langle Y\rangle$, $\langle Z\rangle$ $\Longrightarrow \rho$

② Tetrahedron

Quantum process tomography

$\mathcal{R}(\rho)$

State tomography for a basis ρ_α of input states