

Quantum Information theory

Lectures 21-22

Qubit operations

Qubit operations: consider only trace-preserving operations

① Count parameters: $16 - 4 = 12$ parameters
 $\uparrow \quad \uparrow$
 D^* $+p$

② An operation is specified by its action on pure states. We only need to say what happens to the surface of the Bloch sphere.

$$Q(I) = aI + \vec{c} \cdot \vec{\sigma}$$

$$Q(\sigma_j) = b_j I + \sum_k \sigma_k M_{kj}$$

$$\Rightarrow Q(\rho) = Q\left(\frac{1}{2}(I + \vec{S} \cdot \vec{\sigma})\right)$$

$$= \frac{1}{2} \left(\underbrace{(a + \vec{S} \cdot \vec{b})}_{+p \Rightarrow \vec{b}=0 \text{ and } a=1} I + \vec{c} \cdot \vec{\sigma} + \sum_{j,k} \underbrace{\sigma_k M_{kj}}_{M \vec{S} \cdot \vec{\sigma} = \vec{S} \cdot M^T \vec{\sigma}} S_j \right)$$

$$+p \Rightarrow \vec{b}=0 \text{ and } a=1$$

$$M \vec{S} \cdot \vec{\sigma} = \vec{S} \cdot M^T \vec{\sigma}$$

Summary:

$$Q(I) = I + \vec{c} \cdot \vec{\sigma} \quad \begin{matrix} \nwarrow 3 \text{ parameters} \\ \nearrow \end{matrix}$$

$$Q(\sigma_j) = \sum_k \sigma_k M_{kj} \quad \begin{matrix} \nwarrow 9 \text{ parameters} \\ \nearrow \end{matrix}$$

$$Q(\rho) = \frac{1}{2} \left(1 + \vec{\sigma} \cdot \underbrace{(M \vec{S} + \vec{c})}_{\vec{S}'} \right)$$

affine transformation of the Bloch vector

$$\vec{S}' = M \vec{S} + \vec{c}$$

$\uparrow$
displacement

Look at M more closely.

Polar decomposition:

$$M = O \sqrt{M^T M}$$

real symmetric positive matrix

$$\sqrt{M^T M} = R_1^T D R_1$$

rotation

diagonal with positive entries

orthogonal matrix: $\det O = \pm 1$

$$O = R P$$

identity or a reflection or inversion, we can choose a standard reflection, say thru x-y plane, but prefer to leave it general

rotation: $\det R = +1$

$$M = R P R_1^T D R_1 = R R_2^T P D R_1$$

$$P R_1^T = P R_1^T P P = R_2^T P$$

I rotation

diagonal matrix, with entries of either sign

Summarize: $M = R_2 T R_1$

$$T = \begin{pmatrix} t_x & 0 & 0 \\ 0 & t_y & 0 \\ 0 & 0 & t_z \end{pmatrix}$$

3 parameters

3 parameters

$$T_{jk} = t_j \delta_{jk}$$

$$\begin{pmatrix} \text{contraction} \\ \text{along 3} \\ \text{coordinate} \\ \text{axes} \end{pmatrix} \times \begin{pmatrix} \text{identity} \\ \text{OR} \\ \text{a reflection} \\ \text{or inversion} \end{pmatrix}$$

3 parameters

$$\vec{S}' = R_2 T R_1 \vec{S} + \vec{c} = R_2 (T R_1 \vec{S} + \vec{d}), \quad \vec{d} = R_2^T \vec{c}$$

displacement of center of Bloch sphere

How does this look in terms of the quantum operation? Define a new operation that only has the contraction and displacement in it,

$$\mathcal{B}(\rho) = \frac{1}{2}(\mathbf{I} + \vec{\sigma} \cdot (\mathbf{T}\vec{S} + \vec{d}))$$

We claim that

$$Q(\rho) = U_{R_2} \mathcal{B}(U_{R_1} \rho U_{R_1}^\dagger) U_{R_2}^\dagger$$

$\uparrow$ initial unitary
 $\uparrow$ final unitary

$$\rho = \frac{1}{2}(\mathbf{I} + \vec{S} \cdot \vec{\sigma})$$

$$\mathcal{B}(\mathbf{I}) = \mathbf{I} + \vec{\sigma} \cdot \vec{d}$$

$$\mathcal{B}(X) = t_x X$$

$$\mathcal{B}(Y) = t_y Y$$

$$\mathcal{B}(Z) = t_z Z$$

$$\mathcal{B} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ d_x & t_x & 0 & 0 \\ d_y & 0 & t_y & 0 \\ d_z & 0 & 0 & t_z \end{pmatrix}$$

Check:

$$U_{R_1} \rho U_{R_1}^\dagger = \frac{1}{2} (1 + \underbrace{U_{R_1} \vec{\sigma} U_{R_1}^\dagger}_{R_1^T \vec{\sigma} = \vec{\sigma} \cdot R_1} \cdot \vec{S}) = \frac{1}{2} (1 + \vec{\sigma} \cdot R_1 \vec{S})$$

$$\mathcal{B}(U_{R_1} \rho U_{R_1}^\dagger) = \frac{1}{2} (1 + \vec{\sigma} \cdot (\mathbf{T} R_1 \vec{S} + \vec{d}))$$

$$\begin{aligned}
 U_{R_2} \mathcal{B}(U_{R_1} \rho U_{R_1}^\dagger) U_{R_2}^\dagger &= \frac{1}{2} \left(1 + \underbrace{U_{R_2} \vec{\sigma} U_{R_2}^\dagger}_{= R_2^T \vec{\sigma} = \vec{\sigma} \cdot R_2} \cdot (\mathbf{T} R_1 \vec{S} + \vec{d}) \right) \\
 &= \vec{\sigma} \cdot (R_2 \mathbf{T} R_1 \vec{S} + R_2 \vec{d}) \\
 &= \vec{\sigma} \cdot (M \vec{S} + \vec{c}) \\
 &= \frac{1}{2} (1 + \vec{\sigma} \cdot (M \vec{S} + \vec{c})) \\
 &= Q(\rho)
 \end{aligned}$$

Complete positivity:

①

$$\frac{1}{2} \text{tr}(\sigma_\alpha^\dagger \mathcal{B}(\sigma_\beta)) = \mathcal{B}_{\alpha\beta} \rightarrow$$

"

$$\frac{1}{2} (\sigma_\alpha | \mathcal{B}^\# | \sigma_\beta)$$

↑
Choi
matrix

$$\mathcal{B}_{00} = 1$$

$$\mathcal{B}_{j0} = d_j$$

$$\mathcal{B}_{0j} = 0$$

$$\mathcal{B}_{kj} = T_{kj} = t_j \delta_{jk}$$

This matrix is
our present
characterization
of $\mathcal{B}$.

$$\Rightarrow \mathcal{B}^\# = \sum_{\alpha, \beta} \frac{1}{2} |\sigma_\alpha\rangle \frac{1}{2} \langle \sigma_\alpha | \mathcal{B}^\# | \sigma_\beta \rangle \langle \sigma_\beta| = \frac{1}{2} \sum_{\alpha, \beta} \mathcal{B}_{\alpha\beta} \sigma_\alpha \otimes \sigma_\beta^\dagger$$

② For complete positivity, we require that

$$\frac{1}{2} (\sigma_\alpha | \mathcal{B} | \sigma_\beta) = \frac{1}{2} \text{tr}(\sigma_\alpha^\dagger \mathcal{B}^\#(\sigma_\beta)) = \mathcal{B}_{\alpha\beta}^\# = X_{\alpha\beta}$$

be a positive matrix.

↑
process
matrix

$$\mathcal{B}_{\alpha\beta}^\# = \frac{1}{2} \text{tr}(\sigma_\alpha^\dagger \mathcal{B}^\#(\sigma_\beta)) = \frac{1}{4} \sum_{\gamma, \delta} \mathcal{B}_{\gamma\delta} \text{tr}(\sigma_\alpha \sigma_\gamma \sigma_\beta \sigma_\delta)$$

$X_{\alpha\beta}$

Choi
transformation

$\mathcal{C}_P$

this matrix has
nonnegative eigenvalues

process matrix

↓

$$\mathcal{B} = \frac{1}{2} \sum_{\alpha, \beta} X_{\alpha\beta} \sigma_\alpha \otimes \sigma_\beta^\dagger = \frac{1}{2} \sum_{\alpha, \beta} X_{\alpha\beta} |\sigma_\alpha\rangle \langle \sigma_\beta|$$

↑

Right-left diagonalizing this superoperator gives
orthogonal Kraus operators.

Positivity: $|\vec{S}'| = |\vec{M}\vec{S} + \vec{c}| \leq 1$ for all $\vec{S}$

Kraus decompositions } What happened to these?
 Complete positivity } See insert (A)

↑
 Only need to worry about $\mathcal{B}$ -contractions and displacement

Examples: $\vec{d}=0$

① $T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is cp

↑
 projection of Bloch sphere onto z axis

Kraus operators: P_0, P_1 ; $Q = P_0 \otimes P_0 + P_1 \otimes P_1$

$Q(X) = P_0 X P_0 + P_1 X P_1 = 0$

$Q(Y) = P_0 Y P_0 + P_1 Y P_1 = 0 \Rightarrow T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$Q(Z) = P_0 Z P_0 + P_1 Z P_1 = Z$

This T corresponds to measurement in the Z basis.

② $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is not cp

↑
 projection of Bloch sphere onto equatorial plane

Try $Q = \frac{3}{4} I \otimes I + \frac{1}{4} X \otimes X + \frac{1}{4} Y \otimes Y - \frac{1}{4} Z \otimes Z$

$Q(I) = I, Q(X) = X, Q(Y) = Y, Q(Z) = 0$

$\Rightarrow T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

These are the left-right eigenvalues of Q. One is negative, so Q is not cp.

Interesting qubit operations:

① Stochastic flip operations:

Kraus operators: $\sqrt{p} I, \sqrt{1-p} \sigma_j$

$$Q = p \underbrace{I \otimes I}_Q + (1-p) \underbrace{\sigma_j \otimes \sigma_j}_{\text{flip}}$$

$$Q(p) = p p + (1-p) \sigma_j p \sigma_j$$

Probability p for nothing to happen; probability $1-p$ for flip

$\sigma_j = X \leftrightarrow$ bit flip

$\sigma_j = Z \leftrightarrow$ phase flip

$\sigma_j = Y = -iZX = iXZ \leftrightarrow$ bit-phase flip

Bloch-sphere description: Bit flip

$$Q = p I \otimes I + (1-p) X \otimes X$$

$$Q(I) = I \Rightarrow \vec{c} = 0$$

$$Q(X) = X$$

$$Q(Y) = (2p-1) Y$$

$$Q(Z) = (2p-1) Z$$

$$\longleftrightarrow T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2p-1 & 0 \\ 0 & 0 & 2p-1 \end{pmatrix}$$



contraction along y and z axes

$p=1$: identity

$p=1/2$: measurement in X basis

$p=0$: pure bit flip

② Depolarizing operation:

Kraus operators: $\frac{\sqrt{p}}{2} I, \frac{\sqrt{p}}{2} X, \frac{\sqrt{p}}{2} Y, \frac{\sqrt{p}}{2} Z, \sqrt{1-p} I$

$$Q = p \frac{1}{4} (I \otimes I + X \otimes X + Y \otimes Y + Z \otimes Z) + (1-p) I \otimes I$$

completely depolarizing operation: $p \rightarrow \frac{1}{2} I$

$$= I/2 \quad (\text{generally } I/D: \frac{1}{D} I(p) = 1/D)$$

↳ left-right identity superoperator

$$\rho(p) = p \frac{1}{2} \mathbb{1} + (1-p) \rho$$

Probability p for complete depolarization; probability $1-p$ for nothing

⑥

Bloch sphere description: $\rho(\mathbb{1}) = \mathbb{1} \Rightarrow \vec{c} = 0$

$$\rho(\sigma_j) = (1-p) \sigma_j \iff T = (1-p) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

uniform contraction

③ Amplitude damping (spontaneous decay from $|0\rangle$ to $|1\rangle$):

Measurement model:

$$|1\rangle | \text{environment ground state} \rangle \longrightarrow |1\rangle | \text{environment ground state} \rangle$$

U:

$$|0\rangle | \text{environment ground state} \rangle \longrightarrow \sqrt{1-\gamma} |0\rangle | \text{environment ground state} \rangle + \sqrt{\gamma} |1\rangle | \text{decay products} \rangle$$

↑
amplitude for decay

orthogonal

$$\begin{aligned} \rho \otimes | \text{egs} \rangle \langle \text{egs} | &= p_{11} |1\rangle | \text{egs} \rangle \langle 1| \langle \text{egs} | \\ &+ p_{10} |1\rangle | \text{egs} \rangle \langle 0| \langle \text{egs} | \\ &+ p_{01} |0\rangle | \text{egs} \rangle \langle 1| \langle \text{egs} | \\ &+ p_{00} |0\rangle | \text{egs} \rangle \langle 0| \langle \text{egs} | \end{aligned}$$

$$\begin{aligned} \xrightarrow{u} & p_{11} |1\rangle | \text{egs} \rangle \langle 1| \langle \text{egs} | \\ &+ p_{10} |1\rangle | \text{egs} \rangle (\sqrt{1-\gamma} \langle 0| \langle \text{egs} | + \sqrt{\gamma} \langle 1| \langle \text{dp} |) \\ &+ p_{01} (\sqrt{1-\gamma} |0\rangle | \text{egs} \rangle + \sqrt{\gamma} |1\rangle | \text{dp} \rangle) \langle 1| \langle \text{egs} | \\ &+ p_{00} (\sqrt{1-\gamma} |0\rangle | \text{egs} \rangle + \sqrt{\gamma} |1\rangle | \text{dp} \rangle) (\sqrt{1-\gamma} \langle 0| \langle \text{egs} | + \sqrt{\gamma} \langle 1| \langle \text{dp} |) \end{aligned}$$

$Q(\rho)$ is obtained by tracing out the environment: (7)

$$\begin{aligned} Q(\rho) &= \rho_{11} |1\rangle\langle 1| + \sqrt{1-x} \rho_{10} |1\rangle\langle 0| + \sqrt{1-x} \rho_{01} |0\rangle\langle 1| \\ &\quad + \rho_{00}(1-x) |0\rangle\langle 0| + x |1\rangle\langle 1| \\ &= (|1\rangle\langle 1| + \sqrt{1-x} |0\rangle\langle 0|) \rho (|1\rangle\langle 1| + \sqrt{1-x} |0\rangle\langle 0|) \\ &\quad + \sqrt{x} |1\rangle\langle 0| \rho |0\rangle\langle 1| \sqrt{x} \end{aligned}$$

Kraus operators: $A_1 = |1\rangle\langle 1| + \sqrt{1-x} |0\rangle\langle 0| \longleftrightarrow \begin{pmatrix} \sqrt{1-x} & 0 \\ 0 & 1 \end{pmatrix}$

$A_2 = \sqrt{x} |1\rangle\langle 0| \longleftrightarrow \begin{pmatrix} 0 & \sqrt{x} \\ \sqrt{x} & 0 \end{pmatrix}$

$$Q = A_1 \odot A_1^\dagger + A_2 \odot A_2^\dagger$$

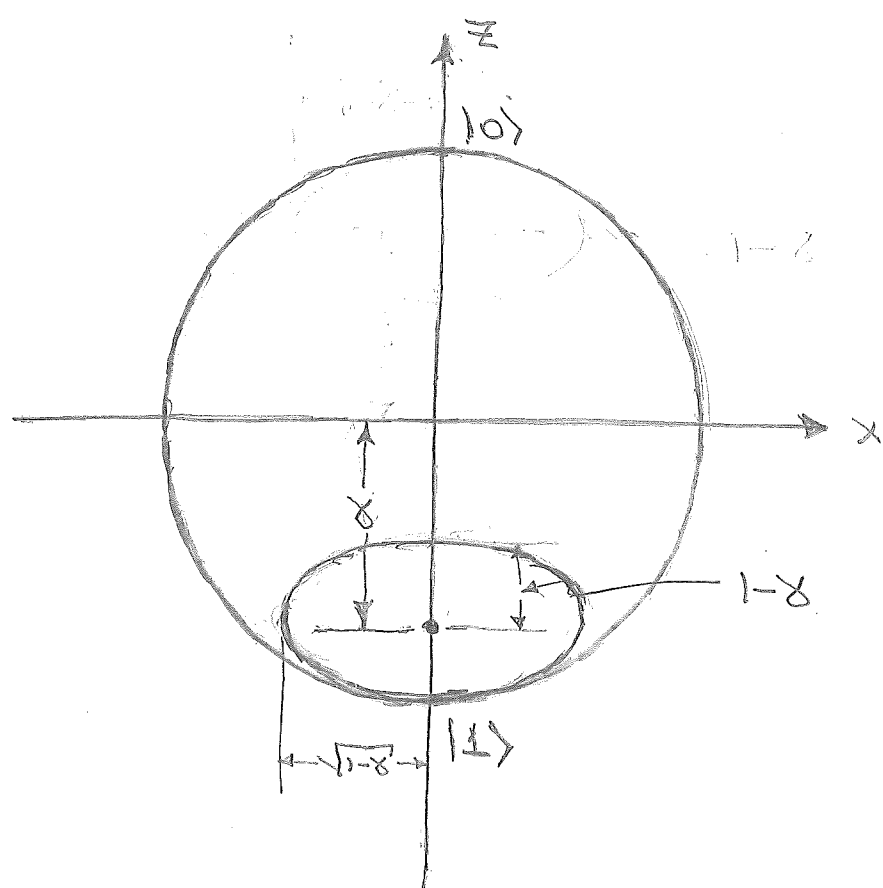
Bloch-sphere description:

$$Q(1) = \underbrace{A_1 A_1^\dagger}_{\begin{pmatrix} 1-x & 0 \\ 0 & 1 \end{pmatrix}} + \underbrace{A_2 A_2^\dagger}_{\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}} = \begin{pmatrix} 1-x & 0 \\ 0 & 1+x \end{pmatrix} = 1-xZ \Rightarrow \vec{c} = -x\vec{e}_z$$

$$Q(X) = \underbrace{A_1 X A_1^\dagger}_{\sqrt{1-x} X} + \underbrace{A_2 X A_2^\dagger}_0 = \sqrt{1-x} X \rightarrow T = \begin{pmatrix} \sqrt{1-x} & 0 & 0 \\ 0 & \sqrt{1-x} & 0 \\ 0 & 0 & 1-x \end{pmatrix}$$

$$Q(Y) = \sqrt{1-x} Y$$

$$Q(Z) = \underbrace{A_1 Z A_1^\dagger}_{\begin{pmatrix} 1-x & 0 \\ 0 & -1 \end{pmatrix}} + \underbrace{A_2 Z A_2^\dagger}_{\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}} = \begin{pmatrix} 1-x & 0 \\ 0 & -(1-x) \end{pmatrix} = (1-x)Z$$



Amplitude damping
Spontaneous decay to $|1\rangle$

→ This is called amplitude damping because it damps the amplitude to be in $|0\rangle$.

Convert to a differential equation that describes decay:

$$\gamma = R \Delta t$$

$$\vec{S}' = T\vec{S} + \vec{C} = \begin{pmatrix} 1 - \frac{1}{2}R\Delta t & 0 & 0 \\ 0 & 1 - \frac{1}{2}R\Delta t & 0 \\ 0 & 0 & 1 - R\Delta t \end{pmatrix} \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} + R\Delta t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\frac{dS_x}{dt} = -\frac{1}{2}R S_x, \quad \frac{dS_y}{dt} = -\frac{1}{2}R S_y, \quad \frac{dS_z}{dt} = -R S_z - R$$

$$S_x(t) = e^{-Rt/2} S_x(0), \quad S_y(t) = e^{-Rt/2} S_y(0), \quad S_z(t) = -1 + e^{-Rt} (1 + S_z(0))$$