Quantum information theory

Lectures 23-24

Clonability and distinguishability

Introduction.

	Clonability	Distinguishability
Classical	✓	✓
Quantum	×	× ⟵ much richer set of possibilities

Clonability.

Set of pure states, $\{|\psi_j\rangle,\ j=1,\dots,N\}$

Cloning: $|\psi_j\rangle \otimes |S\rangle \longrightarrow |\psi_j\rangle \otimes |\psi_j\rangle$

↑ standard state of system receiving copy

(1) Is there a unitary U such that

$$U|\psi_j\rangle \otimes |S\rangle = |\psi_j\rangle \otimes |\psi_j\rangle \ ?$$

$$\langle\psi_j|\psi_k\rangle = \langle\psi_j|\otimes\langle S|\,U^\dagger U\,|\psi_k\rangle\otimes|S\rangle = \langle\psi_j|\psi_k\rangle^2 \Longrightarrow \langle\psi_j|\psi_k\rangle = 0 \text{ or } 1$$

$\Longrightarrow$ States in list that are not identical must be orthogonal

↑ Classical info

<u>Nonorthogonal states cannot be cloned.</u>

(2) Is there a trace-preserving operation $\mathcal{A}$ such that

$$\mathcal{A}\big(|\psi_j\rangle\langle\psi_j|\otimes|S\rangle\langle S|\big) = |\psi_j\rangle\langle\psi_j|\otimes|\psi_j\rangle\langle\psi_j|$$

$\mathcal{A}$ has a measurement model: $\mathcal{A}(\rho) = \mathrm{tr}_A\big(U\,\rho\otimes|\phi\rangle\langle\phi|\,U^\dagger\big)$

$$\mathcal{A}\big(|\psi_j\rangle\langle\psi_j|\otimes|S\rangle\langle S|\big) = \mathrm{tr}_A\big(U\,|\psi_j\rangle\langle\psi_j|\otimes|S\rangle\langle S|\otimes|\phi\rangle\langle\phi|\,U^\dagger\big)$$

$$= |\psi_j\rangle\langle\psi_j|\otimes|\psi_j\rangle\langle\psi_j|$$

$$U|\psi_j\rangle \otimes |S\rangle \otimes |\phi\rangle = |\psi_j\rangle \otimes |\psi_j\rangle \otimes |\phi_j\rangle$$

Why? State inside tr_A is a pure state; the only way the partial trace can be pure is for it to be a product state.

$$\langle\psi_j|\psi_k\rangle = \langle\psi_j| \otimes \langle S| \otimes \langle\phi|\, U^\dagger U\, |\psi_k\rangle \otimes |S\rangle \otimes |\phi\rangle$$

$$= \langle\psi_j|\psi_k\rangle^2 \langle\phi_j|\phi_k\rangle$$

$$\Longrightarrow \quad \langle\psi_j|\psi_k\rangle = 0 \quad \text{or} \quad \langle\psi_j|\psi_k\rangle = 1 = \langle\phi_j|\phi_k\rangle$$

$\Longrightarrow$ States in the list that are not identical must be orthogonal.

Generalizations: Multiple copies

Probabilistic exact cloning $\longrightarrow$ probability of success

Approximate deterministic cloning $\longrightarrow$ average fidelity of copies

Different probabilities of inputs

One comment: Qubits

Measure the <u>isotropic</u> <u>POVM</u> $\quad E_{\vec{n}} = \frac{1}{2\pi}|\vec{n}\rangle\langle\vec{n}| = \frac{1}{4\pi}(1 + \vec{\sigma}\cdot\vec{n})$

$$\int d\Omega\, E_{\vec{n}} = 1$$

Input state $|\vec{m}\rangle$.
Get result $\vec{n}$; make as many copies as you want of $|\vec{n}\rangle$.

$$\begin{pmatrix}\text{squared fidelity of output} \\ |\vec{n}\rangle \text{ with input } |\vec{m}\rangle\end{pmatrix} = |\langle\vec{n}|\vec{m}\rangle|^2$$

$$\begin{pmatrix}\text{average (fidelity)}^2\\ \text{of copies}\end{pmatrix} = \underbrace{\int \frac{d\Omega_{\vec m}}{4\pi}}_{\substack{\text{average over}\\ \text{input state } |\vec m\rangle}} \underbrace{\int d\Omega_{\vec n}}_{\substack{\text{average over}\\ \text{measurent results}}} \underbrace{p(\vec n|\vec m)}_{\substack{\text{probability for result } \vec n\\ \text{given input state } |\vec m\rangle\\ = \frac{1}{4\pi}|\langle\vec n|\vec m\rangle|^2}} \underbrace{|\langle\vec n|\vec m\rangle|^2}_{\substack{\text{(fidelity)}^2\text{ of}\\ \text{copy } |\vec n\rangle \text{ given}\\ \text{input } \vec m}}$$

$$|\langle\vec n|\vec m\rangle|^2 = \frac{1}{2}(1+\vec n\cdot\vec m)$$

$$p(\vec n|\vec m) = \langle\vec m|E_{\vec n}|\vec m\rangle = \frac{1}{4\pi}(1+\vec m\cdot\vec n)$$

$$\int d\Omega_{\vec n}\, p(\vec n|\vec m)\, |\langle\vec n|\vec m\rangle|^2 = \frac{1}{8\pi}\int d\Omega_{\vec n}\underbrace{(1+\vec n\cdot\vec m)^2}_{1+2\vec n\cdot\vec m+(\vec n\cdot\vec m)^2}$$

$$= \frac{1}{8\pi}\Big(4\pi + \underbrace{\int d\Omega_{\vec n}(\vec n\cdot\vec m)^2}_{\frac{4\pi}{3}}\Big)$$

$$= \frac{2}{3}$$

Average over $\vec m$ is irrelevant

$$\begin{pmatrix}\text{average (fidelity)}^2\\ \text{of copies}\end{pmatrix} = \frac{2}{3}$$

Generalization to D dimensions?

Distinguishability.

Relation to clonability.

Error probability.

Distinguish two pure states, $|\psi_1\rangle$ and $|\psi_2\rangle$, by measuring a POVM $\{F_\alpha, \alpha=1,\dots,N\}$

$$\underbrace{F_1,\dots,F_n}_{\text{decide on } |\psi_1\rangle} \qquad \underbrace{F_{n+1},\dots,F_N}_{\text{decide on } |\psi_2\rangle}$$

$$E_1 = \sum_{\alpha=1}^{n} F_\alpha \qquad E_2 = \sum_{\alpha=n+1}^{N} F_\alpha$$

always use a 2-outcome measurement for the decision

$$E_1 + E_2 = 1$$

$$\begin{pmatrix}\text{error}\\ \text{probability}\end{pmatrix} = P_e = \underbrace{P_{1|\psi_2}}\, \underset{1/2}{\underset{\parallel}{P_{\psi_2}}} + P_{2|\psi_1}\, \underset{1/2}{\underset{\parallel}{P_{\psi_1}}}$$

$$= \langle\psi_2|E_1|\psi_2\rangle \qquad = \langle\psi_1|E_2|\psi_1\rangle$$

$$= \mathrm{tr}(E_1 P_2) \qquad = \mathrm{tr}(E_2 P_1)$$

$$= \frac{1}{2}\mathrm{tr}\big(E_1 P_2 + \underset{1-E_1}{\underset{\uparrow}{E_2}} P_1\big)$$

$$P_e = \frac{1}{2} - \frac{1}{2}\mathrm{tr}\big(E_1(P_1 - P_2)\big)$$

Minimizing P_e means maximizing $\mathrm{tr}(E_1(P_1-P_2))$ over positive $E_1 \le 1$.

Go to a Bloch-sphere description in the 2-d subspace spanned by $|\psi_1\rangle$ and $|\psi_2\rangle$:

$$P_j = |\psi_j\rangle\langle\psi_j| = |\vec{n}_j\rangle\langle\vec{n}_j| = \frac{1}{2}(1 + \vec{\sigma}\cdot\vec{n}_j)$$

$$P_1 - P_2 = \frac{1}{2}\vec{\sigma}\cdot(\vec{n}_1 - \vec{n}_2) = \frac{1}{2}|\vec{n}_1 - \vec{n}_2|\,\vec{\sigma}\cdot\underbrace{\vec{m}}_{\frac{\vec{n}_1-\vec{n}_2}{|\vec{n}_1-\vec{n}_2|}}$$

$$= \frac{1}{2}|\vec{n}_1 - \vec{n}_2|\left(|\vec{m}\rangle\langle\vec{m}| - |-\vec{m}\rangle\langle-\vec{m}|\right)$$

$$\mathrm{tr}\left(E_1(P_1 - P_2)\right) = \frac{1}{2}|\vec{n}_1 - \vec{n}_2|\left(\langle\vec{m}|E_1|\vec{m}\rangle - \langle-\vec{m}|E_1|-\vec{m}\rangle\right)$$

$$\le \frac{1}{2}|\vec{n}_1 - \vec{n}_2|$$

Equality iff $E_1 = |\vec{m}\rangle\langle\vec{m}| = \frac{1}{2}(1 + \vec{\sigma}\cdot\vec{m})$

$$\cos^2\theta = |\langle\psi_1|\psi_2\rangle|^2 = \mathrm{tr}(P_1P_2) = \frac{1}{2}(1 + \vec{n}_1\cdot\vec{n}_2)$$

θ: Hilbert-space angle

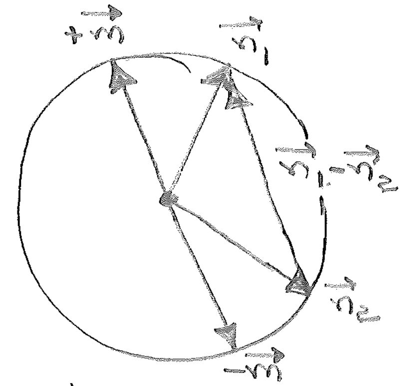
$$(P_e)_{min} = \frac{1}{2} - \frac{1}{4}\underbrace{|\vec{n}_1 - \vec{n}_2|}_{\sqrt{2 - 2\vec{n}_1\cdot\vec{n}_2} = \sqrt{2}\sqrt{1 - \vec{n}_1\cdot\vec{n}_2}}$$

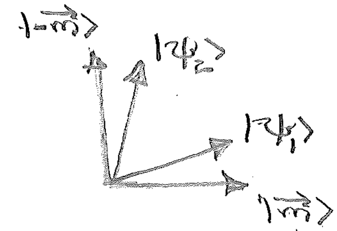
$$= \frac{1}{2} - \frac{\sqrt{2}}{4}\sqrt{1 - \vec{n}_1\cdot\vec{n}_2}$$

$$= \frac{1}{2} - \frac{1}{2}\sqrt{1 - |\langle\psi_1|\psi_2\rangle|^2}$$

$$= \frac{1}{2}(1 - \sin\theta)$$

minimum error determined by inner product

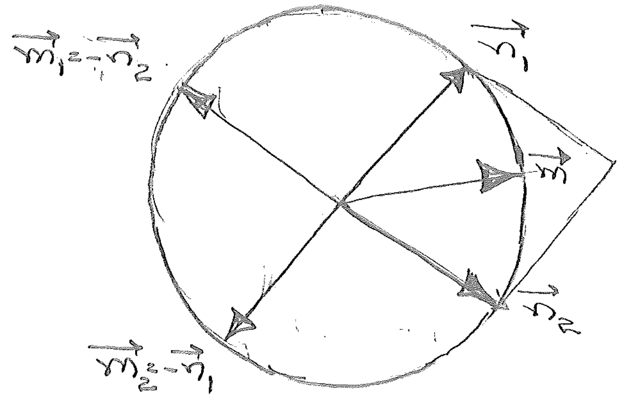




Nonorthogonal states cannot be reliably distinguished

Unambiguous state discrimination.

What if you can't afford a mistake, but can afford not to decide?



$$E_1 = \frac{\alpha}{2}(1 - \vec{\sigma}\cdot\vec{n}_2)$$ falsifies $\vec{n}_2$; confirms $\vec{n}_1$

$$E_2 = \frac{\alpha}{2}(1 - \vec{\sigma}\cdot\vec{n}_1)$$ falsifies $\vec{n}_1$; confirms $\vec{n}_2$

$$E_3 = 1 - E_1 - E_2$$

How big can α be?

$$1 \geq E_1 + E_2 = \alpha\left(1 - \frac{1}{2}(\vec{n}_1 + \vec{n}_2)\cdot\vec{\sigma}\right)$$

$$= \alpha\left(1 - \frac{1}{2}|\vec{n}_1 + \vec{n}_2|\, \vec{m}\cdot\vec{\sigma}\right)$$

$$\vec{m} = \frac{\vec{n}_1 + \vec{n}_2}{|\vec{n}_1 + \vec{n}_2|}$$

$$= \alpha\left(1 - \tfrac{1}{2}|\vec{n}_1 + \vec{n}_2|\right)|\vec{m}\rangle\langle\vec{m}| + \underbrace{\alpha\left(1 + \tfrac{1}{2}|\vec{n}_1 + \vec{n}_2|\right)}_{\leq 1}|-\vec{m}\rangle\langle-\vec{m}|$$

$$\therefore \alpha \leq \frac{1}{1 + \frac{1}{2}|\vec{n}_1 + \vec{n}_2|}$$

$$\frac{1}{2}|\vec{n}_1 + \vec{n}_2| = \frac{1}{\sqrt{2}}(1 + \vec{n}_1\cdot\vec{n}_2)^{1/2}$$
$$= |\langle\psi_1|\psi_2\rangle|$$

$$\alpha \leq \frac{1}{1 + |\langle\psi_1|\psi_2\rangle|}$$

$$\begin{pmatrix}\text{probability of} \\ \text{no decision}\end{pmatrix} = \frac{1}{2}\mathrm{tr}(E_3 P_1) + \frac{1}{2}\mathrm{tr}(E_3 P_2)$$

$$= 1 - \mathrm{tr}\left((E_1 + E_2)\underbrace{\tfrac{1}{2}(P_1 + P_2)}\right)$$

$$\tfrac{1}{2}(P_1+P_2) = \frac{1}{2}\left(1 + \frac{1}{2}(\vec{n}_1 + \vec{n}_2)\cdot\vec{\sigma}\right) = \frac{1}{2}\left(1 + \frac{1}{2}|\vec{n}_1 + \vec{n}_2|\,\vec{m}\cdot\vec{\sigma}\right)$$

$$= 1 - \alpha\left(1 - \frac{1}{4}|\vec{n}_1 + \vec{n}_2|^2\right)$$

$$\geq 1 - \left(1 - \frac{1}{2}|\vec{n}_1 + \vec{n}_2|\right)$$

$$\boxed{\begin{pmatrix}\text{probability of} \\ \text{no decision}\end{pmatrix} \geq \frac{1}{2}|\vec{n}_1 + \vec{n}_2| = \underbrace{|\langle\psi_1|\psi_2\rangle|}_{\cos\theta}}$$

Nonorthogonal states cannot be reliably distinguished.

Mixed states and distance measures

The only distinguishability measure for pure states is the inner product.
What about mixed states?

1. Trace distance generalizes $(P_e)_{min}$
2. Fidelity generalizes $|\langle\psi_1|\psi_2\rangle|$ (to square or not to square)

Trace distance

$$P_e = \frac{1}{2}\,tr\left(E_1\rho_2 + E_2\rho_1\right) = \frac{1}{2} - \frac{1}{2}\,tr\left(E_1(\rho_1-\rho_2)\right)$$

Why not the more obvious Euclidean distance $\sqrt{tr((\rho_1-\rho_2)^2)}$

Lemma: $$\max_{0\le E\le 1} tr\left(E(\rho_1-\rho_2)\right) = \frac{1}{2}\,tr\left(|\rho_1-\rho_2|\right) \equiv D(\rho_1,\rho_2)$$

$D(\rho_1,\rho_2)$: trace distance

$$|A| \equiv \sqrt{A^\dagger A}$$

$$tr(|A|) = tr\left(\sqrt{A^\dagger A}\right) = \left(\text{sum of the singular values of } A\right)$$

$$\rho_1-\rho_2 = \sum_j \lambda_j |e_j\rangle\langle e_j| = \underbrace{\sum_{\lambda_j>0} |\lambda_j||e_j\rangle\langle e_j|}_{P_+(\rho_1-\rho_2)P_+ \,=\, Q} - \underbrace{\sum_{\lambda_j<0} |\lambda_j||e_j\rangle\langle e_j|}_{P_-|\rho_1-\rho_2|P_- \,=\, S}$$

N&C

$$P_+ = \sum_{\lambda_j>0} |e_j\rangle\langle e_j|$$

$$P_- = \sum_{\lambda_j<0} |e_j\rangle\langle e_j|$$

$$P_0 = I - P_+ - P_- = \sum_{\lambda_j=0} |e_j\rangle\langle e_j|$$

$$|\rho_1-\rho_2| = \sum_j |\lambda_j||e_j\rangle\langle e_j| = \sum_{\lambda_j>0} |\lambda_j||e_j\rangle\langle e_j| + \sum_{\lambda_j<0} |\lambda_j||e_j\rangle\langle e_j|$$

$$0 = tr(\rho_1-\rho_2) = \sum_{\lambda_j>0}\lambda_j + \sum_{\lambda_j<0}\lambda_j = \sum_{\lambda_j>0}|\lambda_j| - \sum_{\lambda_j<0}|\lambda_j|$$

$$\Longrightarrow\quad tr\left(P_+(\rho_1-\rho_2)\right) = \sum_{\lambda_j>0}|\lambda_j| = \sum_{\lambda_j<0}|\lambda_j| = tr\left(P_-|\rho_1-\rho_2|\right)$$

$$D(\rho_1,\rho_2) = \frac{1}{2}\,tr\left(|\rho_1-\rho_2|\right) = \frac{1}{2}\sum_{\lambda_j>0}|\lambda_j| + \frac{1}{2}\sum_{\lambda_j<0}|\lambda_j| = \underbrace{\sum_{\lambda_j>0}|\lambda_j|}_{tr(P_+(\rho_1-\rho_2))} = \underbrace{\sum_{\lambda_j<0}|\lambda_j|}_{tr(P_-|\rho_1-\rho_2|)}$$

Proof: $\mathrm{tr}(E(\rho_1-\rho_2)) = \sum_j \lambda_j \langle e_j|E|e_j\rangle$

$$= \underbrace{\sum_{\lambda_j>0} \lambda_j \langle e_j|E|e_j\rangle}_{\geq 0} + \underbrace{\sum_{\lambda_j<0} \lambda_j \langle e_j|E|e_j\rangle}_{\leq 0}$$

$$\leq \sum_{\lambda_j\geq 0} \lambda_j \underbrace{\langle e_j|E|e_j\rangle}_{\leq 1}$$

$$\leq \sum_{\lambda_j\geq 0} |\lambda_j|$$

$$= D(\rho_1,\rho_2)$$

iff: $E = P_+ + E'$, $P_0 E' P_0 = E'$

■ QED.

$$(P_E)_{\min} = \frac{1}{2} - \frac{1}{4}\mathrm{tr}(|\rho_1-\rho_2|)$$
$$= \frac{1}{2} - \frac{1}{2}D(\rho_1,\rho_2)$$

$E_1 = P_+ + E_1'$

$E_2 = P_- + E_2'$

$E_1' + E_2' = P_0$

$\rho_1 = \rho_2$ in the P_0 subspace

Properties of trace distance:

① Pure states: $|\psi_1\rangle$, $|\psi_2\rangle$

$$\mathrm{tr}(|\underbrace{\rho_1-\rho_2}|) = |\vec{n}_1-\vec{n}_2|$$

$$\frac{1}{2}\vec{\sigma}\cdot(\vec{n}_1-\vec{n}_2) = \frac{1}{2}|\vec{n}_1-\vec{n}_2|\left(|\vec{m}\rangle\langle\vec{m}| - |-\vec{m}\rangle\langle-\vec{m}|\right)$$

$$D(\rho_1,\rho_2) = \frac{1}{2}\mathrm{tr}(|\rho_1-\rho_2|) = \frac{1}{2}|\vec{n}_1-\vec{n}_2| = (1-\underbrace{|\langle\psi_1|\psi_2\rangle|^2}_{\cos^2\theta})^{1/2} = \sin\theta$$

② Qubits: ρ_1, ρ_2

$$\mathrm{tr}(|\underbrace{\rho_1-\rho_2}|) = |\vec{S}_1-\vec{S}_2|$$

$$\frac{1}{2}(\vec{S}_1-\vec{S}_2)\cdot\vec{\sigma} = \frac{1}{2}|\vec{S}_1-\vec{S}_2|\left(|\vec{m}\rangle\langle\vec{m}| - |-\vec{m}\rangle\langle-\vec{m}|\right)$$

$$D(\rho_1, \rho_2) = \frac{1}{2} \mathrm{tr}(|\rho_1 - \rho_2|) = \frac{1}{2} |\vec{S}_1 - \vec{S}_2|$$ ← half the Euclidean distance on Bloch ball

③ Classical distance: $[\rho_1, \rho_2] = 0 \Rightarrow$ $\rho_1 = \sum_j p_j |e_j\rangle\langle e_j|$, $\rho_2 = \sum_j q_j |e_j\rangle\langle e_j|$

$$\mathrm{tr}(|\rho_1 - \rho_2|) = \sum_j |p_j - q_j|$$

$$D(\rho_1, \rho_2) = \frac{1}{2} \mathrm{tr}(|\rho_1 - \rho_2|) = \frac{1}{2} \sum_j |p_j - q_j| = D(\vec{p}, \vec{q})$$

$$= \sum_{p_j - q_j > 0} |p_j - q_j| = \sum_{p_j - q_j < 0} |p_j - q_j|$$

$$= \max_S [p(S) - q(S)]$$

④ Invariance under unitaries:

$$U(\rho_1 - \rho_2)U^\dagger = \sum_j \lambda_j U|e_j\rangle\langle e_j|U^\dagger$$

$$|U(\rho_1 - \rho_2)U^\dagger| = \sum_j |\lambda_j| U|e_j\rangle\langle e_j|U^\dagger$$

$$= U\Big(\sum_j |\lambda_j| |e_j\rangle\langle e_j|\Big)U^\dagger$$

$$= U|\rho_1 - \rho_2|U^\dagger$$

$$D(U\rho_1 U^\dagger, U\rho_2 U^\dagger) = \frac{1}{2}\mathrm{tr}|U(\rho_1 - \rho_2)U^\dagger| = \frac{1}{2}\mathrm{tr}(|\rho_1 - \rho_2|) = D(\rho_1, \rho_2)$$

⑤ Trace distance is a distance:

① $D(\rho_1, \rho_2) \geq 0$, equality iff $\rho_1 = \rho_2$.

② $D(\rho_1, \rho_2) = D(\rho_2, \rho_1)$

③ Triangle inequality: $D(\rho, \sigma) \leq D(\rho, \tau) + D(\tau, \sigma)$

$$D(\rho, \sigma) = \mathrm{tr}(P_+(\rho - \sigma)) = \underbrace{\mathrm{tr}(P_+(\rho - \tau))}_{\leq D(\rho, \tau)} + \underbrace{\mathrm{tr}(P_+(\tau - \sigma))}_{\leq D(\tau, \sigma)} \leq D(\rho, \tau) + D(\tau, \sigma)$$

⑥ $D(\rho_1,\rho_2) \underset{=}{\leq} 1$ — iff $\rho_1 \perp \rho_2$, i.e., $\mathrm{tr}(\rho_1\rho_2)=0 \Leftrightarrow \rho_1\rho_2=0$

$$D(\rho_1,\rho_2) = \mathrm{tr}(P_+(\rho_1-\rho_2)) \underset{=}{\leq} \mathrm{tr}(P_+\rho_1) \underset{=}{\leq} 1$$

iff support of ρ_1 lies in subspace of P_+.

iff $\mathrm{tr}(\rho_2 P_+)=0$, i.e., $\rho_2 \perp P_+$

⑦ Relation to measurements:

$$D(\rho,\sigma) = \max_{\{E_\alpha\}} D(\vec{p},\vec{q})$$

$q_\alpha = \mathrm{tr}(\sigma E_\alpha)$

$p_\alpha = \mathrm{tr}(\rho E_\alpha)$

maximum achieved by POVMs such that P_+ is the sum of some subset of POVM elements and P_- is the sum of the remaining POVM elements.

$$D(\vec{p},\vec{q}) = \frac{1}{2}\sum_\alpha |\mathrm{tr}(\rho E_\alpha) - \mathrm{tr}(\sigma E_\alpha)|$$

$$= \sum_{p_\alpha - q_\alpha \geq 0} \mathrm{tr}((\rho-\sigma)E_\alpha)$$

$$= \mathrm{tr}\Big((\rho-\sigma)\sum_{p_\alpha - q_\alpha \geq 0} E_\alpha\Big)$$

$$\leq \max_E \mathrm{tr}((\rho-\sigma)E) = \mathrm{tr}((\rho-\sigma)P_+)$$

$$= D(\rho,\sigma)$$

⑧ Marginalization: $D(\rho_A,\sigma_A) \leq D(\rho_{AB},\sigma_{AB})$,

$$D(\rho_A,\sigma_A) = \mathrm{tr}(P_+(\rho_A-\sigma_A)) = \mathrm{tr}\big(P_+\otimes I_B(\rho_{AB}-\sigma_{AB})\big) \leq D(\rho_{AB},\sigma_{AB})$$

(9) Contractivity: $D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \le D(\rho,\sigma)$, $\mathcal{E}$ trace-preserving quantum operation

Measurement model

$\mathcal{E}(\rho) = \mathrm{tr}_A(U\rho\otimes|0\rangle\langle 0|U^\dagger)$

$\mathcal{E}(\sigma) = \mathrm{tr}_A(U\sigma\otimes|0\rangle\langle 0|U^\dagger)$

$$D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \le D(U\rho\otimes|0\rangle\langle 0|U^\dagger, U\sigma\otimes|0\rangle\langle 0|U^\dagger)$$
$$= D(\rho\otimes|0\rangle\langle 0|, \sigma\otimes|0\rangle\langle 0|)$$
$$= D(\rho,\sigma)$$

$\mathcal{E}(\rho) - \mathcal{E}(\sigma) = \mathcal{E}(\rho-\sigma)$

Let Π_+ project onto the positive eigensubspace of $\mathcal{E}(\rho-\sigma)$

$$D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) = \mathrm{tr}(\Pi_+ \mathcal{E}(\rho-\sigma))$$

$$\mathcal{E}(\rho-\sigma) = \mathcal{E}(P_+(\rho-\sigma)P_+) - \mathcal{E}(P_-(\rho-\sigma)P_-)$$

These are positive operators because $\mathcal{E}$ maps positive operators to positive operators

$$\begin{aligned} D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) &= \mathrm{tr}(\Pi_+ \mathcal{E}(\rho-\sigma)) \\ &\le \mathrm{tr}(\Pi_+ \mathcal{E}(P_+(\rho-\sigma)P_+)) \\ &\le \mathrm{tr}(\mathcal{E}(P_+(\rho-\sigma)P_+)) \\ &\overset{\mathcal{E}\text{ t-p}}{=} \mathrm{tr}(P_+(\rho-\sigma)) \\ &= D(\rho,\sigma) \end{aligned}$$

(10) Convexity properties

Strong convexity: $D\left(\sum_j p_j\rho_j, \sum_j q_j\sigma_j\right) \le D(\vec{p},\vec{q}) + \sum_j p_j D(\rho_j,\sigma_j)$

$$\begin{aligned} D\left(\sum_j p_j\rho_j, \sum_j q_j\sigma_j\right) &= \mathrm{tr}\left(P_+\left(\sum_j p_j\rho_j - \sum_j q_j\sigma_j\right)\right) \\ &= \sum_j p_j \underbrace{\mathrm{tr}(P_+(\rho_j-\sigma_j))}_{\le D(\rho_j,\sigma_j)} + \underbrace{\sum_j (p_j-q_j)\mathrm{tr}(P_+\sigma_j)}_{\le \sum_{p_j-q_j\ge 0}(p_j-q_j) = D(\vec{p},\vec{q})} \\ &\le D(\vec{p},\vec{q}) + \sum_j p_j D(\rho_j,\sigma_j) \end{aligned}$$

Double convexity: $D\left(\sum_j p_j \rho_j, \sum_j p_j \sigma_j\right) \le \sum_j p_j D(\rho_j, \sigma_j)$

Convexity: $D\left(\sum_j p_j \rho_j, \sigma\right) \le \sum_j p_j D(\rho_j, \sigma)$

Fidelity

Properties we want:

Hilbert-space angle

① $F(|\psi_1\rangle, |\psi_2\rangle) = |\langle\psi_1|\psi_2\rangle| = \cos\theta$ To square or not to square?

② $F(\rho, |\psi\rangle) = \sqrt{\langle\psi|\rho|\psi\rangle} = \left(\begin{matrix}\text{probability to find } |\psi\rangle \text{ in} \\ \text{an ODOP measurement on } \rho\end{matrix}\right)^{1/2}$

③ $F(\rho, \rho) = 1$

Guesses:

① $F(\rho_1, \rho_2) = \sqrt{\mathrm{tr}(\rho_1\rho_2)}$, but then $F(\rho, \rho) = \sqrt{\mathrm{tr}(\rho^2)} = (\text{purity})^{1/2} < 1$ except for pure states.

② We might think $\sqrt{\rho}$ is analogous to an amplitude, and so guess $F(\rho_1, \rho_2) = \mathrm{tr}(\sqrt{\rho_1}\sqrt{\rho_2})$, but this gives $F(\rho, |\psi\rangle) = \langle\psi|\sqrt{\rho}|\psi\rangle$.

Definition: Let $|\rho\rangle$ denote a purification of ρ, i.e., $\mathrm{tr}_R(|\rho\rangle\langle\rho|) = \rho$. Define the fidelity as

$$F(\rho_1, \rho_2) \equiv \max_{|\rho_1\rangle, |\rho_2\rangle} |\langle\rho_1|\rho_2\rangle|$$

We put this in a different form using the following:

① $|\mathrm{tr}(GU)| \le \mathrm{tr}\, G$ iff U is ± identity on the support of G

$G \ge 0$

U unitary

$$|\mathrm{tr}(GU)| = \left|\sum_j G_j \langle e_j|U|e_j\rangle\right| \le \sum_j G_j |\langle e_j|U|e_j\rangle| \le \sum_j G_j = \mathrm{tr}(G)$$

iff $\langle e_j|U|e_j\rangle \ge 0$ or ≤ 0 $\forall\, G_j > 0$ iff $|\langle e_j|U|e_j\rangle| = 1$ $\forall\, G_j > 0$

② $\left|\mathrm{tr}(AU)\right| \le \mathrm{tr}\left(\sqrt{A^\dagger A}\right) = \mathrm{tr}\left(\sqrt{AA^\dagger}\right)$

$A = V\sqrt{A^\dagger A} = \sqrt{AA^\dagger}\,V$ $\left(\begin{array}{l}\text{polar decomposition} \\ \text{if } A \text{ is invertible, } V = A(A^\dagger A)^{-1/2} = (AA^\dagger)^{-1/2}A\end{array}\right)$

$\left|\mathrm{tr}(AU)\right| = \left|\mathrm{tr}\left(V\sqrt{A^\dagger A}\,U\right)\right| = \left|\mathrm{tr}\left(\sqrt{A^\dagger A}\,UV\right)\right| \underset{=}{\le} \mathrm{tr}\left(\sqrt{A^\dagger A}\right)$

if A is invertible
$\Leftrightarrow U = \pm V^\dagger = \pm(A^\dagger A)^{-1/2}A^\dagger = \pm A^\dagger(AA^\dagger)^{-1/2}$

Purifications: $\rho = \sum_j \lambda_j |e_j\rangle\langle e_j|$; any purification has the form $|\rho\rangle = \sum_j \sqrt{\lambda_j}\,|f_j\rangle \otimes |e_j\rangle = \sum_j |f_j\rangle \otimes \sqrt{\rho}\,|e_j\rangle$. The freedom in

$\uparrow$ orthonormal basis in reference system R

$|\rho\rangle$ is that we can use any reference-system basis $|f_j\rangle$; i.e., we can multiply $|\rho\rangle$ by any reference-system unitary U.

There is another, more useful way to write an arbitrary purification. Let $|f_j\rangle$ and $|e_j\rangle$ be <u>arbitrary</u> orthonormal bases. Then

$$|\Phi_{\sqrt{\rho}}\rangle = I \otimes \sqrt{\rho}\,|\Psi\rangle = \sum_j |f_j\rangle \otimes \sqrt{\rho}\,|e_j\rangle$$

$\uparrow$ VEC'ing $\qquad\qquad \uparrow$ $\sum_j |f_j\rangle \otimes |e_j\rangle$

is a purification, i.e., $\mathrm{tr}_R\left(|\Phi_{\sqrt{\rho}}\rangle\langle\Phi_{\sqrt{\rho}}|\right) = \sum_j \sqrt{\rho}\,|e_j\rangle\langle e_j|\sqrt{\rho} = \rho$.

The apparent freedom in $|\Phi_{\sqrt{\rho}}\rangle$ is to choose both bases arbitrarily, i.e., to multiply both by unitaries. So an arbitrary purification can be written as

$$|\rho\rangle = U \otimes \sqrt{\rho}\,V\,|\Psi\rangle = (I \otimes \sqrt{\rho})\underbrace{(U \otimes V)|\Psi\rangle}_{= I \otimes VU^T|\Psi\rangle = UV^T \otimes I|\Psi\rangle},$$

Special property of $|\Psi\rangle$ $\longrightarrow$ $= I \otimes VU^T|\Psi\rangle = UV^T \otimes I|\Psi\rangle$

which means we can get arbitrary purifications using either of the two unitaries. So an arbitrary purification can be written as

$$|\rho\rangle = U\otimes I|\Phi_{\sqrt{\rho}}\rangle = U\otimes\sqrt{\rho}|\Phi\rangle = I\otimes\sqrt{\rho}U^T|\Phi\rangle.$$

Now we're set.

$$\max_{|\rho\rangle,|\sigma\rangle} |\langle\rho|\sigma\rangle| = \max_{U,V} |\langle\Phi|(I\otimes U^\dagger\sqrt{\rho})(I\otimes\sqrt{\sigma}V)|\Phi\rangle|$$

$$= \max_{U,V} \Big|\sum_j \langle e_j|U^\dagger\sqrt{\rho}\sqrt{\sigma}V|e_j\rangle\Big|$$

$$= \max_{U,V} |tr(U^\dagger\sqrt{\rho}\sqrt{\sigma}V)|$$

$$= \max_{U,V} |tr(\sqrt{\rho}\sqrt{\sigma}VU^\dagger)|$$

We only need to maximize over one unitary, which means we only need to maximize over one purification, either $|\rho\rangle$ or $|\sigma\rangle$.

$$= \max_{U} |tr(\sqrt{\rho}\sqrt{\sigma}U)|$$

$$= tr\left(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)$$

If ρ and σ are invertible, the U that achieves the maximum is

$$U = \pm(\sqrt{\sigma}\rho\sqrt{\sigma})^{-1/2}\sqrt{\sigma}\sqrt{\rho}$$
$$= \pm\sqrt{\sigma}\sqrt{\rho}(\sqrt{\rho}\sigma\sqrt{\rho})^{-1/2}$$

So we have

$$\boxed{\begin{aligned} F(\rho_1,\rho_2) &= \max_U |tr(\sqrt{\rho_1}\sqrt{\rho_2}U)| \\ &= tr\left(\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}\right) \\ &= tr\left(\sqrt{\sqrt{\rho_2}\rho_1\sqrt{\rho_2}}\right) = F(\rho_2,\rho_1) \end{aligned}}$$

Properties of fidelity

$$\sqrt{\rho_1} = |\psi\rangle\langle\psi|$$

$$\sqrt{\rho_1}\,\rho_2\sqrt{\rho_1} = |\psi\rangle\langle\psi|\rho_2|\psi\rangle\langle\psi|$$

$$\sqrt{\sqrt{\rho_1}\,\rho_2\sqrt{\rho_1}} = |\psi\rangle\langle\psi|\sqrt{\langle\psi|\rho_2|\psi\rangle}$$

① $\rho_1 = |\psi\rangle\langle\psi|, \quad \rho_2 = \rho:$

$$F(|\psi\rangle\langle\psi|, \rho) = \sqrt{\langle\psi|\rho_2|\psi\rangle}$$

② Qubits: $\rho_j = \frac{1}{2}(I + \vec{\sigma}\cdot\vec{S}_j)$

$$F(\rho_1, \rho_2) = \frac{1}{\sqrt{2}}\Big(1 + \underbrace{\vec{S}_1\cdot\vec{S}_2 + \sqrt{(1-S_1^2)(1-S_2^2)}}_{\vec{N}_1\cdot\vec{N}_2}\Big)^{1/2}$$

$$\vec{N}_j = \vec{S}_j + \sqrt{1-S_j^2}\,\vec{e}_0$$

Unit vector on northern hemisphere of a sphere in 4 dimensions

$$= \cos\Big(\underbrace{\tfrac{1}{2}\cos^{-1}(\vec{N}_1\cdot\vec{N}_2)}_{\theta}\Big)$$

③ Classical fidelity: $[\rho_1, \rho_2] = 0$

$$\rho_1 = \sum_j p_j |e_j\rangle\langle e_j|$$

$$\rho_2 = \sum_j q_j |e_j\rangle\langle e_j|$$

$$F(\rho_1, \rho_2) = \mathrm{tr}\left(\sqrt{\rho_1\rho_2}\right) = \sum_j \sqrt{p_j q_j} = F(\vec{p}, \vec{q}) = \cos\left(\begin{array}{c}\text{Bhattacharyya-}\\ \text{Wootters}\\ \text{distance}\end{array}\right)$$

④ Invariance under unitaries:

$$F(U\rho_1 U^\dagger, U\rho_2 U^\dagger) = F(\rho_1, \rho_2)$$

⑤ $0 \leq F(\rho_1, \rho_2) \leq 1$

iff $\rho_1 \perp \rho_2$ — iff $\rho_1 = \rho_2$

6. Fidelity is not a distance, but $\Theta(\rho_1,\rho_2) \equiv \cos^{-1} F(\rho_1,\rho_2)$ is a Riemannian distance that generalizes Hilbert-space angle to mixed states

① $\Theta(\rho_1,\rho_2) \geq 0$, with equality iff $\rho_1 = \rho_2$

② $\Theta(\rho_1,\rho_2) = \Theta(\rho_2,\rho_1)$

③ Triangle inequality: $\Theta(\rho,\sigma) \leq \Theta(\rho,\tau) + \Theta(\tau,\sigma)$.

Let $|\rho\rangle$, $|\tau\rangle$, and $|\sigma\rangle$ be such that $F(\rho,\tau) = |\langle\rho|\tau\rangle|$ and $F(\sigma,\tau) = |\langle\sigma|\tau\rangle|$. Adjust the overall phases of $|\rho\rangle$, $|\tau\rangle$, and $|\sigma\rangle$ so that $\langle\rho|\sigma\rangle$, $\langle\rho|\tau\rangle$, and $\langle\sigma|\tau\rangle$ are real and positive. Now we're effectively in a real vector space, where it is obvious that

$$\cos^{-1}\langle\rho|\sigma\rangle \leq \cos^{-1}\langle\rho|\tau\rangle + \cos^{-1}\langle\tau|\sigma\rangle = \Theta(\rho,\tau) + \Theta(\tau,\sigma).$$

But we have $F(\rho,\sigma) \geq \langle\rho|\sigma\rangle$, which implies

$$\Theta(\rho,\sigma) \leq \cos^{-1}\langle\rho|\sigma\rangle \leq \Theta(\rho,\tau) + \Theta(\tau,\sigma).$$

7. Relation to measurements: $F(\rho,\sigma) = \min_{\{E_\alpha\}} F(\vec{p},\vec{q})$

$p_\alpha = tr(\rho E_\alpha)$ $\qquad$ $q_\alpha = tr(\sigma E_\alpha)$

$$F(\rho,\sigma) = tr(\sqrt{\rho}\sqrt{\sigma}\,U)$$

if ρ,σ invertible, $U = \sqrt{\sigma}\sqrt{\rho}(\sqrt{\rho}\,\sigma\sqrt{\rho})^{-1/2} = (\sqrt{\sigma}\,\rho\sqrt{\sigma})^{-1/2}\sqrt{\sigma}\sqrt{\rho}$

$$= \sum_\alpha tr(\sqrt{\rho}\,E_\alpha\sqrt{\sigma}\,U) \longleftarrow \text{completeness of POVM}$$

$$= \sum_\alpha tr\left((\sqrt{E_\alpha}\sqrt{\rho})^\dagger\sqrt{E_\alpha}\sqrt{\sigma}\,U\right)$$

$$\leq \sum_\alpha \mathrm{tr}\left((\sqrt{E_\alpha}\sqrt{\rho})^\dagger \sqrt{E_\alpha}\sqrt{\rho}\right)^{1/2} \mathrm{tr}\left((\sqrt{E_\alpha}\sqrt{\sigma}U)^\dagger(\sqrt{E_\alpha}\sqrt{\sigma}U)\right)^{1/2}$$

iff $\sqrt{E_\alpha}\sqrt{\rho} = \lambda_\alpha \sqrt{E_\alpha}\sqrt{\sigma}U$

Schwarz inequality for operator inner product

$$= \sum_\alpha \sqrt{\mathrm{tr}(\rho E_\alpha)\,\mathrm{tr}(\sigma E_\alpha)}$$

$$= F(\vec{p}, \vec{q})$$

We get equality by choosing a POVM that satisfies

$$\sqrt{E_\alpha}\sqrt{\rho} = \lambda_\alpha \sqrt{E_\alpha}\sqrt{\sigma}\,U$$

Assuming ρ and σ are invertible (noninvertible ρ and σ follow by continuous limits to the boundary), we have

$$\sqrt{E_\alpha} = \lambda_\alpha \sqrt{E_\alpha}\sqrt{\sigma}\,U\rho^{-1/2} = \lambda_\alpha \sqrt{E_\alpha}\,\underbrace{\sqrt{\sigma}\,(\sqrt{\sigma}\rho\sqrt{\sigma})^{-1/2}\sqrt{\sigma}}$$

$$= \sum_j \mu_j |e_j\rangle\langle e_j|$$

spectral decomposition of Hermitian operator

Choose $E_j = |e_j\rangle\langle e_j|$ (projection-valued measurement),

and it works with $\lambda_j = \mu_j^{-1}$.

(8) Marginalization: $F(\rho_A,\sigma_A) \geq F(\rho_{AB},\sigma_{AB})$

Let $|\rho\rangle$, $|\sigma\rangle$ be optimal purifications of ρ_{AB}, σ_{AB}, i.e., $F(\rho_{AB},\sigma_{AB}) = \langle\rho|\sigma\rangle$. Since they are also purifications of ρ_A and σ_A, we have

$$F(\rho_A,\sigma_A) \geq \langle\rho|\sigma\rangle = F(\rho_{AB},\sigma_{AB}).$$

For probabilities, this is the property

$$\sum_j \sqrt{p_j q_j} \geq \sum_{j,k} \sqrt{p_{jk} q_{jk}},$$

which is a trivial consequence of

$$\sum_k \sqrt{p_{jk} q_{jk}} = \sqrt{p_j q_j} \underbrace{\sum_k \sqrt{p_{k|j} q_{k|j}}}_{\leq 1} \leq \sqrt{p_j q_j}$$

(9) Monotonicity: $F(\mathcal{Q}(\rho),\mathcal{Q}(\sigma)) \geq F(\rho,\sigma)$.

$$\begin{aligned} F(\mathcal{Q}(\rho),\mathcal{Q}(\sigma)) &= F\big(\mathrm{tr}_A(U\rho\otimes|0\rangle\langle 0|U^\dagger), \mathrm{tr}_A(U\sigma\otimes|0\rangle\langle 0|U^\dagger)\big) \\ &\geq F(U\rho\otimes|0\rangle\langle 0|U^\dagger, U\sigma\otimes|0\rangle\langle 0|U^\dagger) \\ &= F(\rho\otimes|0\rangle\langle 0|, \sigma\otimes|0\rangle\langle 0|) \\ &= F(\rho,\sigma) \end{aligned}$$

(10) Convexity properties

<u>Strong concavity</u>: $F\left(\sum_j p_j \rho_j, \sum_j q_j \sigma_j\right) \geq \sum_j \sqrt{p_j q_j}\, F(\rho_j,\sigma_j)$

Let $|\rho_j\rangle$ and $|\sigma_j\rangle$ be optimal purifications of ρ_j and σ_j, i.e., $F(\rho_j,\sigma_j) = \langle\rho_j|\sigma_j\rangle$. Define

$$|\rho\rangle = \sum_j \sqrt{p_j}\,|\rho_j\rangle\otimes|j\rangle,$$
$$|\sigma\rangle = \sum_j \sqrt{q_j}\,|\sigma_j\rangle\otimes|j\rangle,$$

which are purifications of $\rho = \sum_j p_j\rho_j$ and $\sigma = \sum_j q_j\sigma_j$, so

First show strong concavity for probabilities: $\vec{p}_j$ and $\vec{q}_j$ are vectors of conditional probabilities, $p_{\alpha|j}$ and $q_{\alpha|j}$, with joint probabilities $p_{\alpha j} = p_{\alpha|j}p_j$ and $q_{\alpha j} = q_{\alpha|j}q_j$ and unconditioned probabilities $p_\alpha = \sum_j p_{\alpha|j}p_j$ and $q_\alpha = \sum_j q_{\alpha|j}q_j$.

$$F\left(\sum_j p_j\vec{p}_j, \sum_j q_j\vec{q}_j\right) = \sum_\alpha \sqrt{p_\alpha q_\alpha}$$

$$F(\rho,\sigma) \ge |\langle \rho|\sigma\rangle|$$

$$= \Big|\sum_j \sqrt{p_j q_j}\,\langle \rho_j|\sigma_j\rangle\Big|$$

$$= \sum_j \sqrt{p_j q_j}\,F(\rho_j,\sigma_j)$$

$$\sum_j \sqrt{p_j q_j}\,F(\vec{p}_j,\vec{q}_j) = \sum_{j,\alpha} \sqrt{p_j p_{\alpha|j}}\sqrt{q_j q_{\alpha|j}}$$

Thus strong concavity for probabilities is just the marginalization property.

Now assume E_α is an optimal POVM for $\rho = \sum_j p_j \rho_j$ and $\sigma = \sum_j q_j \sigma_j$, i.e.,

$$F(\rho,\sigma) = F(\vec{p},\vec{q})$$

$$q_\alpha = \mathrm{tr}(\sigma E_\alpha) = \sum_j q_j \underbrace{\mathrm{tr}(\sigma_j E_\alpha)}_{q_{\alpha|j}}$$

$$p_\alpha = \mathrm{tr}(\rho E_\alpha) = \sum_j p_j \underbrace{\mathrm{tr}(\rho_j E_\alpha)}_{p_{\alpha|j}}$$

We have

$$F(\rho,\sigma) = F(\vec{p},\vec{q})$$

$$\ge \sum_j \sqrt{p_j q_j}\,\underbrace{F(\vec{p}_j,\vec{q}_j)}_{\ge F(\rho_j,\sigma_j)}$$

$$\ge \sum_j \sqrt{p_j q_j}\,F(\rho_j,\sigma_j)$$

<u>Double concavity</u>: $F\Big(\sum_j p_j \rho_j, \sum_j p_j \sigma_j\Big) \ge \sum_j p_j F(\rho_j,\sigma_j)$

<u>Concavity</u>: $F\Big(\sum_j p_j \rho_j, \sigma\Big) \ge \sum_j p_j F(\rho_j,\sigma)$

Relations between trace distance and fidelity:

① Pure states: $D(|\psi\rangle, |\phi\rangle) = \sqrt{1-|\langle\psi_1|\psi_2\rangle|^2} = \sqrt{1-F^2(|\psi\rangle,|\phi\rangle)}$

② Mixed states:

ⓐ Let $|\rho\rangle, |\sigma\rangle$ be purifications such that $F(\rho,\sigma) = \langle\rho|\sigma\rangle$

$$D(\rho,\sigma) \underset{\substack{\uparrow \\ \text{marginalization}}}{\leq} D(|\rho\rangle,|\sigma\rangle) = \sqrt{1-|\langle\rho|\sigma\rangle|^2} = \sqrt{1-F^2(\rho,\sigma)}$$

ⓑ Let E_α be a POVM such that

$$F(\rho,\sigma) = F(\vec{p},\vec{q}) = \sum_\alpha \sqrt{p_\alpha q_\alpha}$$

We know that

$$D(\rho,\sigma) \geq \sum_\alpha D(\vec{p},\vec{q}) = \frac{1}{2}\sum_\alpha |p_\alpha - q_\alpha|$$

$$= \frac{1}{2}\sum_\alpha |\sqrt{p_\alpha}-\sqrt{q_\alpha}|\underbrace{|\sqrt{p_\alpha}+\sqrt{q_\alpha}|}_{\geq |\sqrt{p_\alpha}-\sqrt{q_\alpha}|}$$

$$\geq \frac{1}{2}\sum_\alpha \left(\sqrt{p_\alpha}-\sqrt{q_\alpha}\right)^2$$

$$= \frac{1}{2}\sum_\alpha p_\alpha + q_\alpha - 2\sqrt{p_\alpha q_\alpha}$$

$$= 1 - F(\vec{p},\vec{q})$$

$$= 1 - F(\rho,\sigma)$$

$$\boxed{1 - F(\rho,\sigma) \leq D(\rho,\sigma) \leq \sqrt{1-F^2(\rho,\sigma)}}$$

Entanglement fidelity:

arbitrary Kraus decomposition

State ρ; operation $\mathcal{A} = \sum_\alpha A_\alpha \odot A_\alpha^\dagger$

Purify ρ to $|\Phi_{RQ}\rangle$.

Squared fidelity for transmitting purification through operation $\mathcal{A}$

$$F_e(\rho,\mathcal{A}) \equiv F^2\Big(|\Phi_{RQ}\rangle, \mathcal{I}\otimes\mathcal{A}(|\Phi_{RQ}\rangle\langle\Phi_{RQ}|)\Big)$$

$$= \langle\Phi_{RQ}|\,\mathcal{I}\otimes\mathcal{A}(|\Phi_{RQ}\rangle\langle\Phi_{RQ}|)|\Phi_{RQ}\rangle$$

$$= \sum_\alpha \left|\langle\Phi_{RQ}|I\otimes A_\alpha|\Phi_{RQ}\rangle\right|^2$$

$$= \sum_\alpha \left|\mathrm{tr}(\rho A_\alpha)\right|^2$$

$$= (\rho|\mathcal{A}|\rho)$$

independent of Kraus decomposition and purification

Suppose $|\Phi_{RQ}\rangle = |\Phi_{\sqrt{\rho}}\rangle = I\otimes\sqrt{\rho}\,|\Psi\rangle$

Then

$$F_e(\rho,\mathcal{A}) = \langle\Phi_{\sqrt{\rho}}|\,\mathcal{I}\otimes\mathcal{A}\circ\sqrt{\rho}\odot\sqrt{\rho}\,(|\Psi\rangle\langle\Psi|)|\Phi_{\sqrt{\rho}}\rangle$$

$$= (\sqrt{\rho}|\mathcal{A}\circ\sqrt{\rho}\odot\sqrt{\rho}|\sqrt{\rho})$$

$$= \sum_\alpha \left|\mathrm{tr}(\sqrt{\rho}A_\alpha\sqrt{\rho})\right|^2$$

$$= \sum_\alpha \left|\mathrm{tr}(\rho A_\alpha)\right|^2$$

$$= (\rho|\mathcal{A}|\rho)$$

Properties of entanglement fidelity

① $\rho = |\psi\rangle\langle\psi| = \pi$ pure: $F_e(\pi, \mathcal{Q}) = (\pi|\mathcal{Q}|\pi)$

$$= \langle\psi|\,\mathcal{Q}(\pi)\,|\psi\rangle$$

$$= F^2(\pi, \mathcal{Q}(\pi))$$

② Convexity of F_e: $F_e(\rho, \mathcal{Q}) \le \sum_j p_j F_e(\rho_j, \mathcal{Q})$

Proof is simply the proof for x^2.

$$\sum_j p_j F_e(\rho_j, \mathcal{Q}) = \sum_j p_j (\rho_j|\mathcal{Q}|\rho_j)$$

$$= \sum_j p_j (\rho + \Delta\rho_j|\mathcal{Q}|\rho + \Delta\rho_j)$$

$$= (\rho|\mathcal{Q}|\rho) + \sum_j p_j (\Delta\rho_j|\mathcal{Q}|\Delta\rho_j) + 2\,\mathrm{Re}\Big(\underbrace{\sum_j p_j (\rho|\mathcal{Q}|\Delta\rho_j)}_{(\rho|\mathcal{Q}|\sum_j p_j \Delta\rho_j) = 0}\Big)$$

$$\ge (\rho|\mathcal{Q}|\rho)$$

$$= F_e(\rho, \mathcal{Q})$$

$\rho_j = |\psi_j\rangle\langle\psi_j| = \pi_j$ pure: $F_e(\rho, \mathcal{Q}) \le \sum_j p_j F^2(\pi_j, \mathcal{Q}(\pi_j))$

(8) $\rho = \sum_j p_j \pi_j$: $F_e(\rho, \mathcal{A}) \le \left(\sum_j p_j F(\pi_j, \mathcal{A}(\pi_j))\right)^2 \equiv \bar{F}^2$ (c)

$$\le \sum_j p_j F^2(\pi_j, \mathcal{A}(\pi_j)) \equiv \overline{F^2}$$

$$F_e(\rho, \mathcal{A}) = (\rho|\mathcal{A}|\rho) = \sum_{j,k} p_j p_k (\pi_j|\mathcal{A}|\pi_k)$$

$$\le \sum_{j,k} p_j p_k \underbrace{|(\pi_j|\mathcal{A}|\pi_k)|}_{\le (\pi_j|\mathcal{A}|\pi_j)^{1/2}(\pi_k|\mathcal{A}|\pi_k)^{1/2}}$$

$$= \left(\sum_j p_j \underbrace{(\pi_j|\mathcal{A}|\pi_j)^{1/2}}_{F(\pi_j, \mathcal{A}(\pi_j))}\right)^2$$

$$= \left(\sum_j p_j F(\pi_j, \mathcal{A}(\pi_j))\right)^2$$