

Quantum information theory

Lectures 6-7

Qubits

Qubits: $D=2$

2-d Hilbert space

Orthonormal basis: $|0\rangle, |1\rangle$

Fiducial basis

Computational basis

Standard basis

Examples:

① Spin- $\frac{1}{2}$ particle: $|0\rangle = |\uparrow\rangle, |1\rangle = |\downarrow\rangle$

② Photon polarization: $|0\rangle = |R\rangle, |1\rangle = |L\rangle$

③ Two-level atom: $|0\rangle = |e\rangle, |1\rangle = |g\rangle$

④ Double-well potential: $|0\rangle = |R\rangle, |1\rangle = |L\rangle$



Arbitrary pure state:

$$|\psi\rangle = a|0\rangle + b|1\rangle, \quad |a|^2 + |b|^2 = 1$$

choose real and positive
by choice of overall phase:

$$a = \langle 0|\psi\rangle = \cos(\theta/2), \quad 0 \leq \theta \leq \pi$$

$$b = \langle 1|\psi\rangle = e^{+i\phi} \sin(\theta/2), \quad 0 \leq \phi \leq 2\pi$$

relative phase

A measurement of $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$ yields ± 1 ($|0\rangle$) with probability $|a|^2$ and -1 ($|1\rangle$) with probability $|b|^2$.

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi} \sin(\theta/2)|1\rangle \equiv |\vec{n}\rangle = |+\vec{n}\rangle$$

$$-\vec{n}: \quad c \leftrightarrow s \quad (\theta \leftrightarrow \pi - \theta)$$

$$\phi \rightarrow \phi + \pi$$

↑
unit vector specified
by θ and ϕ in
polar coordinates

$$|-\vec{n}\rangle = \sin(\theta/2)|0\rangle - e^{i\phi} \cos(\theta/2)|1\rangle = |-\vec{n}\rangle$$

$$\langle \vec{n} | -\vec{n} \rangle = 0$$

$$|+\vec{e}_z\rangle = |\vec{e}_z\rangle = |0\rangle$$

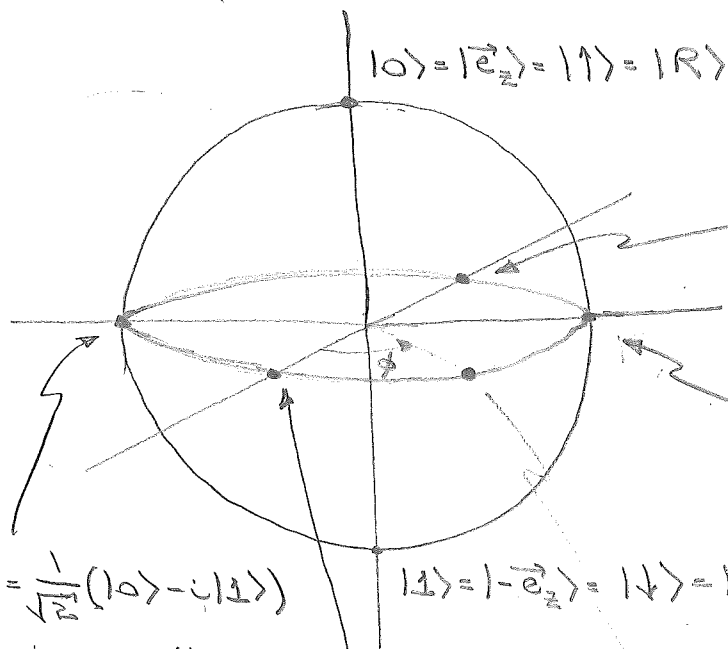
$$|-\vec{e}_z\rangle = |-\vec{e}_z\rangle = |1\rangle$$

$$|x\rangle = |-\vec{e}_x\rangle$$

Bloch sphere (spin- $\frac{1}{2}$)

Poincaré sphere (photon polarization)

(2)



$$|\rightarrow\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

Spin in -x direction
y polarization

$$|\uparrow\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

Spin in y direction
polarization at 45° to x and y

$$|\leftarrow\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Spin in -y direction
polarization at 45°
to y and -x

$$|\downarrow\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$

$$\frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi}|1\rangle) = e^{i\phi/2} (|\rightarrow\rangle \cos(\phi/2) - i|\uparrow\rangle \sin(\phi/2))$$

Spin polarization at angle $\phi/2$

$$|\rightarrow\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

Spin in x direction
x polarization

$|\hat{n}\rangle$ - spin along $\hat{n}$
elliptical polarization

conventional basis for polarization
use $|\rightarrow\rangle, |\leftarrow\rangle$ for state at back of
Poincaré sphere

$$\frac{1}{2}(1 + \cos\theta) = \frac{1}{2}$$

$$\frac{1}{2}(1 - \cos\theta)$$

$$\rho_{\vec{n}} = |\vec{n}\rangle\langle\vec{n}| = \cos^2(\theta/2)|0\rangle\langle 0| + \sin^2(\theta/2)|1\rangle\langle 1| + \frac{\cos(\theta/2)\sin(\theta/2)}{\frac{1}{2}\sin\theta} (e^{-i\phi}|0\rangle\langle 1| + e^{i\phi}|1\rangle\langle 0|)$$

$$\rho_{\vec{n}} = \frac{1}{2} \left(\underbrace{|0\rangle\langle 0| + |1\rangle\langle 1|}_1 + \underbrace{\cos\theta}_{n_z} \underbrace{(|0\rangle\langle 0| - |1\rangle\langle 1|)}_{\equiv \sigma_z} + \underbrace{\sin\theta \cos\phi}_{n_x} \underbrace{(|0\rangle\langle 1| + |1\rangle\langle 0|)}_{\equiv \sigma_x} + \underbrace{\sin\theta \sin\phi}_{n_y} \underbrace{(-i|0\rangle\langle 1| + i|1\rangle\langle 0|)}_{\equiv \sigma_y} \right)$$

$$\vec{P} = |\vec{n}\rangle\langle\vec{n}| = \frac{1}{2}(1 + \vec{n} \cdot \vec{\sigma}) \quad \text{Spin } -\frac{1}{2}: \vec{S} = \frac{1}{2}\hbar\vec{\sigma}$$

Pauli matrices

$$\left[\begin{array}{l} \sigma_x = \sigma_1 = X = |0\rangle\langle 1| + |1\rangle\langle 0| \longleftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma_y = \sigma_2 = Y = -i|0\rangle\langle 1| + i|1\rangle\langle 0| \longleftrightarrow \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \sigma_z = \sigma_3 = Z = |0\rangle\langle 0| - |1\rangle\langle 1| \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{array} \right.$$

Properties of Pauli matrices:

① Hermitian: $\sigma_j = \sigma_j^\dagger$

Unitary: $\sigma_j^\dagger \sigma_j = \sigma_j^2 = 1$

② $\sigma_j \sigma_k = \delta_{jk} + i \epsilon_{jkl} \sigma_l$ ← Sum on repeated indices

↑
antisymmetric symbol

$\sigma_1 \sigma_2 = -\sigma_2 \sigma_1 = i\sigma_3$
 $\sigma_2 \sigma_3 = -\sigma_3 \sigma_2 = i\sigma_1$
 $\sigma_3 \sigma_1 = -\sigma_1 \sigma_3 = i\sigma_2$

All products of Pauli matrices can be reduced to a single matrix

$[\sigma_j, \sigma_k] = 2i \epsilon_{jkl} \sigma_l$
 $[\sigma_j, \sigma_k]_+ = \sigma_j \sigma_k + \sigma_k \sigma_j$
 $\quad \quad \quad \nearrow = 2\delta_{jk} 1$

anticommutator

$[\sigma_1, \sigma_2] = 2i\sigma_3$
 $[\sigma_2, \sigma_3] = 2i\sigma_1$
 $[\sigma_3, \sigma_1] = 2i\sigma_2$

③ $\text{tr}(\sigma_j) = 0 \implies \text{tr}(\vec{n} \cdot \vec{\sigma}) = 0$

④ $\text{tr}(\sigma_j^\dagger \sigma_k) = \text{tr}(\sigma_j \sigma_k) = 2\delta_{jk}$ ← orthogonality

⑤ The operators $1, \sigma_1, \sigma_2, \sigma_3$ are a basis for the 4-d space operators (2×2 matrices), so any operator can be written as

$$A = A_0 1 + A_j \sigma_j = A_0 1 + \vec{A} \cdot \vec{\sigma} = A_d \sigma_d$$

$$A^\dagger = A_d^* \sigma_d$$

$$A = A^\dagger \Rightarrow A_d \text{ real}$$

$$\sigma_0 = 1$$

$$d = 0, 1, 2, 3$$

⑥ Orthogonality: $\text{tr}(\sigma_d^\dagger \sigma_\beta) = \text{tr}(\sigma_d \sigma_\beta) = 2\delta_{d\beta}$

$$A = A_d \sigma_d \Leftrightarrow A_d = \frac{1}{2} \text{tr}(\sigma_d A)$$

⑦ $A = A_d \sigma_d$ and $B = B_d \sigma_d$

$$[A, B] = 2i \vec{A} \times \vec{B} \cdot \vec{\sigma}$$

$$[A, A^\dagger] = 2i \vec{A} \times \vec{A} \cdot \vec{\sigma}$$

$$A \text{ normal} \Leftrightarrow \vec{A} \times \vec{A} = 0$$

$$\Rightarrow \vec{A} \cdot \vec{\sigma} = e^{i\vec{A} \cdot \vec{\sigma}} \vec{S} = e^{i|\vec{A}| \vec{n} \cdot \vec{\sigma}}$$

↑
real

$$AB = (A_0 B_0 + \vec{A} \cdot \vec{B}) 1$$

$$+ (A_0 \vec{B} + B_0 \vec{A} + i \vec{A} \times \vec{B}) \cdot \vec{\sigma}$$

$$\Rightarrow \text{tr}(AB) = 2(A_0 B_0 + \vec{A} \cdot \vec{B}) = 2A_d B_d$$

Special case: $(\vec{n} \cdot \vec{\sigma})(\vec{m} \cdot \vec{\sigma}) = \vec{n} \cdot \vec{m} 1 + i \vec{n} \times \vec{m} \cdot \vec{\sigma}$

⑧ $P_{\vec{n}} = \frac{1}{2}(1 + \vec{n} \cdot \vec{\sigma})$

$$1 = P_{\vec{n}} + P_{-\vec{n}} = |\vec{n}\rangle\langle\vec{n}| + |-\vec{n}\rangle\langle-\vec{n}|$$

$$\vec{n} \cdot \vec{\sigma} = P_{\vec{n}} - P_{-\vec{n}} = \underbrace{|\vec{n}\rangle\langle\vec{n}| - |-\vec{n}\rangle\langle-\vec{n}|}_{\text{eigendecomposition of } \vec{n} \cdot \vec{\sigma}}$$

$$\vec{n} \cdot \vec{\sigma} |\vec{n}\rangle = |\vec{n}\rangle$$

$$\vec{n} \cdot \vec{\sigma} |-\vec{n}\rangle = -|-\vec{n}\rangle$$

⑨ If $A = A_0 \mathbb{1} + \vec{A} \cdot \vec{\sigma}$ is Hermitian, define the unit vector $\vec{n} \equiv \vec{A}/|\vec{A}|$, giving

$$A = A_0 \mathbb{1} + |\vec{A}| \vec{n} \cdot \vec{\sigma}$$

$$= (A_0 + |\vec{A}|) |\vec{n}\rangle\langle\vec{n}| + (A_0 - |\vec{A}|) |-\vec{n}\rangle\langle-\vec{n}|$$

eigen-decomposition

Normal operator:
 $\vec{A} = e^{i\chi} |\vec{A}| \vec{n}$
 $(A_0 \pm e^{i\chi} |\vec{A}|)$ are eigenvalues

eigenvalues

eigenvectors

⑩ Raising and lowering operators:

$$\sigma_{\pm} = \frac{1}{2} (\sigma_1 \pm i\sigma_2) \iff$$

$$\sigma_1 = \sigma_+ + \sigma_-$$

$$\sigma_2 = -i(\sigma_+ - \sigma_-)$$

$$\sigma_+ = |0\rangle\langle 1| \iff \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

raising operator

$$\sigma_- = |1\rangle\langle 0| \iff \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

lowering operator

$$\sigma_+^\dagger = \sigma_-$$

$$\sigma_{\pm}^2 = 0$$

$$[\sigma_{\pm}, \sigma_{\mp}] = \pm \sigma_3$$

$$\sigma_{\pm} \sigma_{\mp} = \frac{1}{2} (1 \pm \sigma_3)$$

$$[\sigma_{\pm}, \sigma_{\mp}]_{\pm} = 1$$

$$\sigma_{\pm} \sigma_3 = \mp \sigma_{\pm}$$

$$[\sigma_{\pm}, \sigma_3] = \mp 2\sigma_{\pm}$$

$$\sigma_3 \sigma_{\pm} = \pm \sigma_{\pm}$$

$$[\sigma_{\pm}, \sigma_3]_{\pm} = 0$$

⑪ $e^{i\vec{n} \cdot \vec{\sigma} \chi} = 1 \cos \chi + i\vec{n} \cdot \vec{\sigma} \sin \chi$

Observables. If the system is in the state $P_{\vec{n}} = |\vec{n}\rangle\langle\vec{n}|$, a measurement of $\vec{m} \cdot \vec{\sigma} = |\vec{m}\rangle\langle\vec{m}| - |-\vec{m}\rangle\langle-\vec{m}|$ yields $+1$ ($|\vec{m}\rangle$) with probability

$$|\langle\vec{m}|\vec{n}\rangle|^2 = \text{tr}(P_{\vec{m}}P_{\vec{n}}) = \frac{1}{2}(1 + \vec{m} \cdot \vec{n})$$

and -1 ($|-\vec{m}\rangle$) with probability

$$|\langle-\vec{m}|\vec{n}\rangle|^2 = \text{tr}(P_{-\vec{m}}P_{\vec{n}}) = \frac{1}{2}(1 - \vec{m} \cdot \vec{n})$$

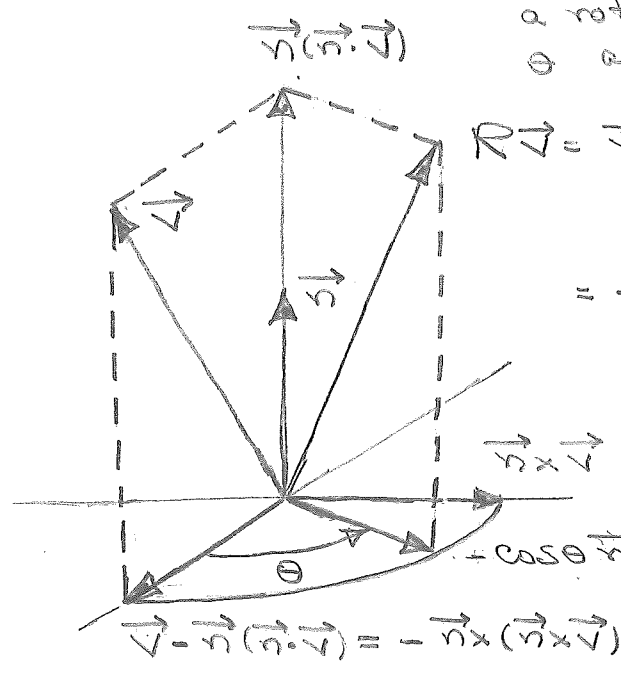
Unitary dynamics. Any unitary operator can be written as

$$U = e^{i(\delta 1 - \vec{n} \cdot \vec{\sigma} \theta/2)} = e^{i\delta} e^{-i\vec{n} \cdot \vec{\sigma} \theta/2}$$

$U_R = \exp\left(-\frac{i}{\hbar} \vec{n} \cdot \vec{S} \theta\right)$
 ↑
 global phase ↑
 canonical form of rotation operator

Key fact: $U_R^\dagger \vec{\sigma} U_R = \underbrace{R_{\vec{n}}(\theta)}_{\text{orthogonal matrix for a rotation by angle } \theta \text{ about } \vec{n}} \vec{\sigma}$

orthogonal matrix for a rotation by angle θ about $\vec{n}$ $R^T R = I$



$$\begin{aligned} R\vec{V} &= \vec{n}(\vec{n} \cdot \vec{V}) + \cos\theta \vec{n} \times (\vec{n} \times \vec{V}) + \sin\theta \vec{n} \times \vec{V} \\ &= +\vec{V} \cos\theta + (1 - \cos\theta) \vec{n}(\vec{n} \cdot \vec{V}) + \sin\theta \vec{n} \times \vec{V} \end{aligned}$$

$$-\cos\theta \vec{n} \times (\vec{n} \times \vec{V}) + \sin\theta \vec{n} \times \vec{V}$$

$$\vec{V} - \vec{n}(\vec{n} \cdot \vec{V}) = -\vec{n} \times (\vec{n} \times \vec{V})$$

$\sum_{i,j} b_{ij}^x \sigma_{ij}^x = \sum_{i,j} b_{ij}^y \sigma_{ij}^y = \sum_{i,j} b_{ij}^z \sigma_{ij}^z = \sum_{i,j} b_{ij}^0 \sigma_{ij}^0$

Consequence: $U_R \vec{\sigma} \cdot \vec{m} U_R^\dagger = R^T \vec{\sigma} \cdot \vec{m} = \vec{\sigma} \cdot R\vec{m}$

$$(\vec{\sigma} \cdot R\vec{m}) U_R |\vec{m}\rangle = U_R \underbrace{\vec{\sigma} \cdot \vec{m}}_{+|\vec{m}\rangle} |\vec{m}\rangle = U_R |\vec{m}\rangle$$

$$\Rightarrow U_R |\vec{m}\rangle = e^{i\phi(R, \vec{m})} |R\vec{m}\rangle$$

Unitary dynamics rotates states on the Bloch sphere.

Example: Hamiltonian $H = -\vec{m} \cdot \vec{B} = -\frac{1}{2} \gamma B \hbar Z$
 (spin-1/2)
 $\uparrow$ magnetic field
 $\uparrow$ magnetic moment

Choose $\vec{B} = B \vec{e}_z$

$$\vec{m} = -\gamma \vec{S} = -\frac{1}{2} \gamma \hbar \vec{\sigma}$$

$$-\vec{m} \cdot \vec{B} = \frac{1}{2} \gamma B \hbar Z$$

Unitary evolution operator:

$$U(t) = e^{-iHt/\hbar} = e^{-i(\gamma B t/2)Z}$$

rotation about $\vec{e}_z$
by angle $\gamma B t$

Comment: Why put $|0\rangle$ at the top of the Bloch sphere ($\sigma_z|0\rangle = |0\rangle$), when $|0\rangle$ suggests the ground state, which conventionally goes at the bottom?

$$\sigma_z|x\rangle = (-1)^x|x\rangle \leftarrow$$

$\uparrow$
 $0 \leftrightarrow 1$

State at top has eigenvalue $+1$, whereas state at bottom has eigenvalue -1 .

This is all because of the two ways of doing mod-2 arithmetic — multiplicative $(-1)^x$ or additive (x) .

Tricky point:

$$U_R = e^{-i\vec{n} \cdot \vec{\sigma} \theta/2} = e^{-i\theta/2} |\vec{n}\rangle\langle\vec{n}| + e^{+i\theta/2} |-\vec{n}\rangle\langle-\vec{n}|$$

$$\Rightarrow \det U_R = e^{-i\theta/2} e^{+i\theta/2} = 1$$

$$\det U = e^{2i\delta}$$

$$U = e^{i\delta} U_R$$

Requiring U to have unit determinant, i.e., to be in $SU(2)$, means that $\delta = 0$ or π . But we don't need the π case, which changes the sign of U , because advancing θ by 2π also changes the sign of U (double-valued representation of $SO(3)$).