$\boxed{\text{CMC}}$
Abq
11-5-30

Lecture 1

Introduction to quantum information

Canadian Summer School on Quantum Information

2011 June 6

http://info.phys.unm.edu/~caves/qistutorial/lecture1.pdf

States, dynamics, and measurements

Classical bit $\begin{matrix}0\\1\end{matrix}$ $\quad a, x$

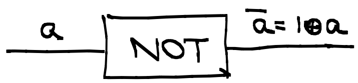
Bit string $a = a_1 \ldots a_N$

State space: 2^N strings

Information Theory (IT) or Physics (P)

Dynamics (gates)

NOT



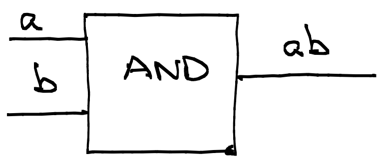
Truth table

0	1
1	0

Matrix

$$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & \multicolumn{2}{c}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \\ 1 & & \end{array}$$

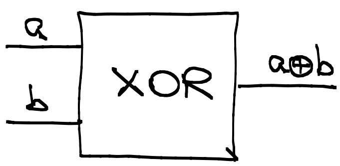
AND



0 0	0
0 1	0
1 0	0
1 1	1

$$\begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{matrix} 00 & 01 & 10 & 11 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

XOR



0 0	0
0 1	1
1 0	1
1 1	0

$$\begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{matrix} 00 & 01 & 10 & 11 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

For a gate with N inputs and M outputs, each output is a polynomial function, up to Nth order, of the inputs.

NAND, COPY, SWAP, plus the ability to summon ancilla bits, are universal for classical computation.

Quantum mechanics: qubits (two-state quantum systems)

$|0\rangle$ $\quad |a\rangle$

$|1\rangle$

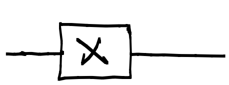
kets

Strings of qubits $\quad |a\rangle = |a_1 \ldots a_N\rangle$

IT: QM is an IT framework for physical law.

Dynamics (gates)

NOT



Matrix

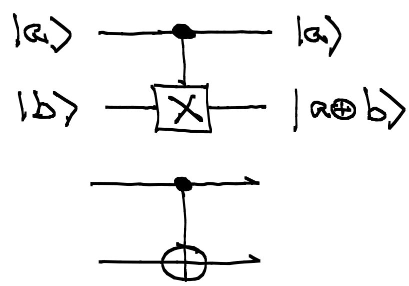
$$\begin{array}{c|cc} & |0\rangle & |1\rangle \\ \hline |0\rangle & 0 & 1 \\ |1\rangle & 1 & 0 \end{array}$$

~~AND~~

P: Irreversibility, dissipation, erasure
Quantum gates must be reversible.

IT: All classical computations can be made reversible.
The circuit model needs a 3-bit reversible gate.

XoR $\to$ CNOT



00	00
01	01
10	11
11	10

$$\begin{array}{c|cccc} & |00\rangle & |01\rangle & |10\rangle & |11\rangle \\ \hline |00\rangle & 1 & 0 & 0 & 0 \\ |01\rangle & 0 & 1 & 0 & 0 \\ |10\rangle & 0 & 0 & 0 & 1 \\ |11\rangle & 0 & 0 & 1 & 0 \end{array}$$

$\underbrace{\qquad\qquad\qquad}_{\text{permutation matrix}}$

Great leap: Superpositions

[Probabilities. Convex combinations. State space.]

$$\langle a|b\rangle = (|a\rangle, |b\rangle) = \delta_{ab}$$

$|0\rangle$ and $|1\rangle$ are <u>ortho</u> <u>normal</u> vectors in a 2D complex vector space. The general <u>state</u> is a normalized vector

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle \longleftrightarrow \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}, \quad |c_0|^2 + |c_1|^2 = 1,$$

overall phase does not change the state

ket

N qubits: $|\psi\rangle = \sum_{a_1,\dots,a_N} c_{a_1\dots a_N}|a_1\dots a_N\rangle = \sum_a c_a|a\rangle, \quad \sum_a |c_a|^2 = 1$

Size of state space: $\sim 2^{2^N}$

Digression: inner product

① complex symmetric: $(|\phi\rangle, |\psi\rangle) = (|\psi\rangle, |\phi\rangle)^*$

② complex bilinear: $(|\chi\rangle, \alpha|\phi\rangle + \beta|\xi\rangle) = \alpha(|\chi\rangle, |\phi\rangle) + \beta(|\chi\rangle, |\xi\rangle)$

$(\alpha|\phi\rangle + \beta|\xi\rangle, |\chi\rangle) = \alpha^*(|\phi\rangle, |\chi\rangle) + \beta^*(|\xi\rangle, |\chi\rangle)$

③ $(|\psi\rangle, |\psi\rangle) \geq 0, \quad = \iff |\psi\rangle = 0$

Bra-ket notation: $\langle\psi| = c_0^*\langle 0| + c_1^*\langle 1| \longleftrightarrow \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}^\dagger = (c_0^* \;\; c_1^*)$

bra

$(|\phi\rangle, |\psi\rangle) = \langle\phi|\psi\rangle$

1. $\langle \phi | \psi \rangle = \langle \psi | \phi \rangle^*$
2. $|\psi\rangle = \alpha|\phi\rangle + \beta|\xi\rangle$: $\langle \chi|\psi\rangle = \alpha\langle\chi|\phi\rangle + \beta\langle\chi|\xi\rangle$
   $\langle\psi| = \alpha^*\langle\phi| + \beta^*\langle\xi|$ $\quad \langle\psi|\chi\rangle = \alpha^*\langle\phi|\chi\rangle + \beta^*\langle\xi|\chi\rangle$
3. $\langle\psi|\psi\rangle \geq 0$, $= \Leftrightarrow |\psi\rangle = 0$

Dynamics (gates):

Single-qubit gates $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ —$\boxed{U}$— $U|\psi\rangle = \begin{array}{l} c_0(U_{00}|0\rangle + U_{10}|1\rangle) \\ + c_1(U_{01}|0\rangle + U_{11}|1\rangle) \end{array}$

$$\begin{aligned} U &= U_{00}|0\rangle\langle 0| + U_{01}|0\rangle\langle 1| \\ &\quad + U_{10}|1\rangle\langle 0| + U_{11}|1\rangle\langle 1| \\ &= \sum_{a,b} U_{ab} \underbrace{|a\rangle\langle b|}_{\text{outer product}} \end{aligned} \longleftrightarrow \underbrace{\begin{pmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{pmatrix}}_{\text{columns are orthonormal}}$$

U is a unitary operator.
Classical reversible gates are permutation matrices.

Popular single-qubit gates

identity $\quad I = |0\rangle\langle 0| + |1\rangle\langle 1| = \sum_a |a\rangle\langle a| \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Pauli operators:

bit flip $\quad X = |0\rangle\langle 1| + |1\rangle\langle 0| = \sum_a |a\rangle\langle \bar{a}| \longleftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

bit-sign flip $\quad Y = -i|0\rangle\langle 1| + i|0\rangle\langle 1| \longleftrightarrow \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

sign flip $\quad Z = |0\rangle\langle 0| - |1\rangle\langle 1| = \sum_a (-1)^a |a\rangle\langle a| \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$X^2 = Y^2 = Z^2 = I$

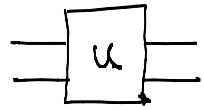
$XY = iZ$

$Y = -iXZ = iZX$

phase $\quad S = |0\rangle\langle 0| + i|1\rangle\langle 1| = \sum_a i^a |a\rangle\langle a| \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ $\quad S^2 = Z$
$\quad = e^{i\pi/4} e^{-iZ\pi/4}$

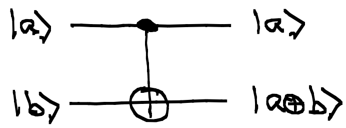
$T = |0\rangle\langle 0| + e^{i\pi/4}|1\rangle\langle 1| = \sum_a e^{ia\pi/4}|a\rangle\langle a| \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$
$\quad = e^{i\pi/8} e^{-iZ\pi/8}$ $\quad T^2 = S$

Hadamard $\quad H = \frac{1}{\sqrt{2}}(X + Z) = \frac{1}{\sqrt{2}} \sum_{a,b} (-1)^{ab} |a\rangle\langle b| \longleftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $\quad H^2 = I$, $\quad HXH = Z$

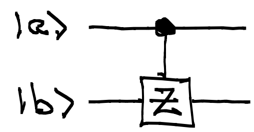
Two-qubit gates: 

Popular two-qubit gates

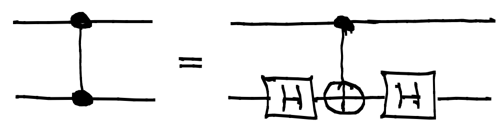
$$\text{CNOT} = \sum_{a,b} |a, a\oplus b\rangle\langle a,b|$$

CNOT 

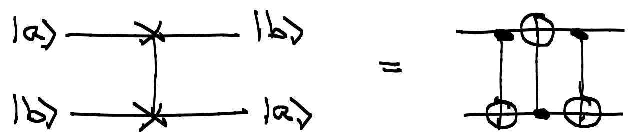
$$\longleftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

CSIGN 

$(-1)^{ab}|a,b\rangle$



$$\text{CSIGN} = \sum_{a,b} (-1)^{ab} |a,b\rangle\langle a,b| \longleftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

SWAP 

Measurements

IT: Analog vs. digital in quantum mechanics

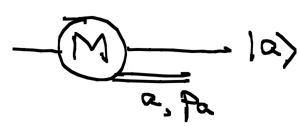
$|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ 

$0,\ p_0 = |c_0|^2 = |\langle 0|\psi\rangle|^2$

$1,\ p_1 = |c_1|^2 = |\langle 1|\psi\rangle|^2$

measurement in the standard basis

If the system survives the measurement,



Observables

① $Z = |0\rangle\langle 0| - |1\rangle\langle 1| = \sum_a (-1)^a |a\rangle\langle a|$

$Z|a\rangle = (-1)^a|a\rangle$

eigenvalues eigenvectors

IT: Eigenvalues are just labels for the eigenvectors.

P: Eigenvalues are possible values of a physical quantity.

$|\psi\rangle$ 

$(-1)^a$

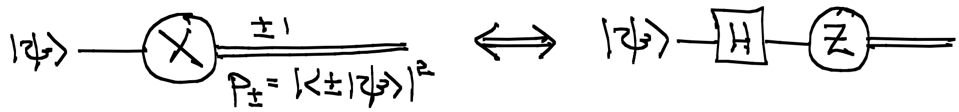
$p_a = |\langle a|\psi\rangle|^2$

(2) $X = |0\rangle\langle 1| + |1\rangle\langle 0| = \sum_a |\bar{a}\rangle\langle a| = |+\rangle\langle +| - |-\rangle\langle -|$

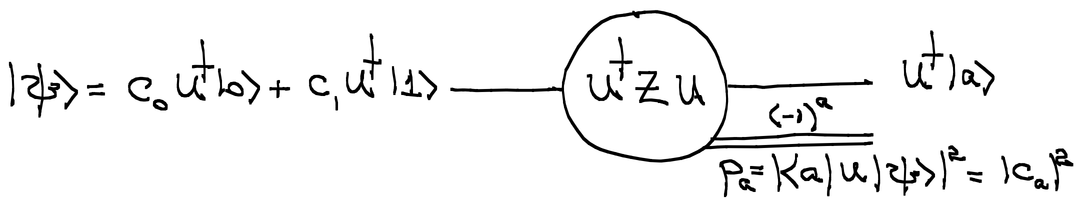
$|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$

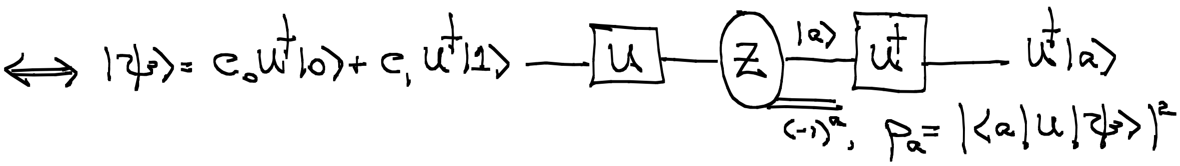
$X|\pm\rangle = \pm|\pm\rangle$

$H|0\rangle = |+\rangle, \quad H|1\rangle = |-\rangle$



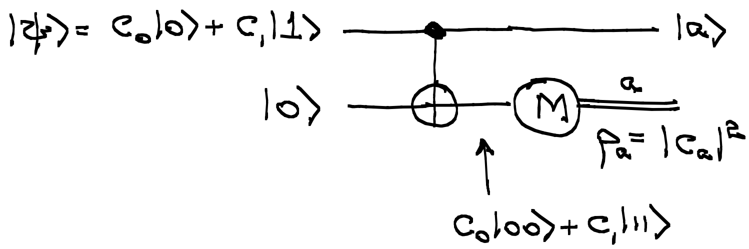
Measurement in another basis: $(U^\dagger Z U)\, U^\dagger|a\rangle = (-1)^a\, U^\dagger|a\rangle$





To make a measurement in the basis $U^\dagger|a\rangle$, transform to the standard basis $|a\rangle$ using U, and then measure in the standard basis. To correct the post-measurement state, apply $U^\dagger$.

Measurement with ancilla



$\langle a_2|\psi\rangle = c_a|a_1\rangle$

Principle of deferred measurement