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Subject: **Antisymmetric operators on a real vector space**
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A linear operator A on a real vector space is *antisymmetric* if $\langle\phi|A|\psi\rangle = -\langle\psi|A|\phi\rangle$ for all $|\phi\rangle$ and $|\psi\rangle$. In the complexification, $-iA$ is Hermitian and thus has an orthonormal set of eigenvectors $|e_j\rangle$ with real eigenvalues:

$$-iA|e_j\rangle = \lambda_j|e_j\rangle\ , \quad \lambda_j^* = \lambda_j\ . \tag{1}$$

This implies that

$$A|e_j\rangle = i\lambda_j|e_j\rangle\ . \tag{2}$$

Complex conjugating this equation gives

$$A|e_j^*\rangle = -i\lambda_j|e_j^*\rangle\ , \tag{3}$$

The upshot is that the eigenvectors of A come in complex conjugate pairs, with pure imaginary, complex conjugate eigenvalues. Only for zero eigenvalues can the eigenvector be real and thus not have a complex conjugate partner.

We can summarize these results in the following way. Denote the eigenvectors by $|e_{j,\mu}\rangle$, where $\mu = \pm 1$ for eigenvector pairs and $\mu = 0$ for the single eigenvectors that have zero eigenvalue. Then we can write

$$A|e_{j,\mu}\rangle = \mu i\lambda_j|e_{j,\mu}\rangle\ , \qquad |e_{j,-\mu}\rangle = |e_{j,+\mu}^*\rangle\ , \qquad \langle e_{j,\mu}|e_{k,\nu}\rangle = \delta_{jk}\delta_{\mu\nu}\ . \tag{4}$$

Associated with the (complex) eigenvectors are pairs of orthonormal real vectors $|a_{j\sigma}\rangle$, $\sigma = \pm 1$, defined by

$$\begin{aligned} |a_{j,+1}\rangle &= \frac{1}{\sqrt{2}}\big(|e_{j,+1}\rangle + |e_{j,-1}\rangle\big)\ , \\ |a_{j,-1}\rangle &= -\frac{i}{\sqrt{2}}\big(|e_{j,+1}\rangle - |e_{j,-1}\rangle\big)\ , \end{aligned} \qquad |e_{j,\pm 1}\rangle = \frac{1}{\sqrt{2}}\big(|a_{j,+1}\rangle \pm i|a_{j,-1}\rangle\big)\ . \tag{5}$$

Extending this definition to the single eigenvectors by defining $|a_{j,0}\rangle = |e_{j,0}\rangle$, we can summarize as follows:

$$A|a_{j,\sigma}\rangle = -\sigma\lambda_j|a_{j,-\sigma}\rangle\ , \qquad \langle a_{j,\sigma}|a_{k,\tau}\rangle = \delta_{jk}\delta_{\sigma\tau}\ . \tag{6}$$

The antisymmetric operator A has the following forms:

$$\begin{aligned} A &= \sum_{j,\mu}\mu i\lambda_j|e_{j,\mu}\rangle\langle e_{j,\mu}| = \sum_{\{j|\mu\neq 0\}} i\lambda_j\big(|e_{j,+1}\rangle\langle e_{j,+1}| - |e_{j,-1}\rangle\langle e_{j,-1}|\big) \\ &= \sum_{j,\sigma} -\sigma\lambda_j|a_{j,-\sigma}\rangle\langle a_{j,\sigma}| = \sum_{\{j|\sigma\neq 0\}} \lambda_j\big(|a_{j,+1}\rangle\langle a_{j,-1}| - |e_{j,-1}\rangle\langle a_{j,+1}|\big)\ . \end{aligned} \tag{7}$$

Notice that the real, symmetric operator A^2 becomes

$$A^2 = -\sum_{j,\mu} \mu^2 \lambda_j^2 |e_{j,\mu}\rangle\langle e_{j,\mu}| = -\sum_{j,\sigma} \sigma^2 \lambda_j^2 |a_{j,\sigma}\rangle\langle a_{j,\sigma}| \; ; \tag{8}$$

i.e., the quantities λ_j are the square roots of the eigenvalues of the positive operator $-A^2$.

We can write the forms in Eq. (7) as expressions for the matrix elements of A in the two bases:

$$\begin{aligned} A_{j\mu,k\nu} &= \langle e_{j,\mu}|A|e_{k,\nu}\rangle = \lambda_j \delta_{jk} i\mu\delta_{\mu\nu} \\ A_{j\sigma,k\tau} &= \langle a_{j,\sigma}|A|a_{k,\tau}\rangle = \lambda_j \delta_{jk} \sigma \delta_{\sigma,-\tau} = \lambda_j \delta_{jk} \epsilon_{\sigma\tau} \; . \end{aligned} \tag{9}$$

For the eigenvector pairs, $i\mu\delta_{\mu\nu}$ is the two-dimensional diagonal matrix,

$$||i\mu\delta_{\mu\nu}|| = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \; , \tag{10}$$

and $\epsilon_{\sigma\tau} = \sigma\delta_{\sigma,-\tau}$ is the two-dimensional antisymmetric matrix,

$$||\epsilon_{\sigma\tau}|| = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \; . \tag{11}$$

The connection between the matrix elements in the two bases can be summarized as

$$\epsilon_{\sigma\tau} = \sum_{\mu,\nu} \langle a_\sigma|e_\mu\rangle i\mu\delta_{\mu\nu} \langle e_\nu|a_\tau\rangle = \sum_{\mu,\nu} u_{\sigma\mu} i\mu\delta_{\mu\nu} u^\dagger_{\nu\tau} = i\sum_{\mu} \mu u_{\sigma\mu} u^*_{\tau\mu} \; , \tag{12}$$

where

$$u = ||\langle a_\sigma|e_\mu\rangle|| = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \tag{13}$$

is the unitary matrix that connects the two bases. In matrix language, the connection is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = u \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} u^\dagger \; . \tag{14}$$

After multiplying by $-i$, the reader will recognize this matrix equation as

$$\sigma_2 = u\sigma_3 u^\dagger \; . \tag{15}$$

Another way of stating our result is that a real, antisymmetric matrix can be brought to the lower form in Eq. (9) by an orthogonal transformation. Moreover, if two real, antisymmetric matrices commute, then they can be brought to this form by the same orthogonal transformation.