To: *R. Schack and D. Steinbach*
From: *C. M. Caves*
Subject: **Quantum hypersensitivity to perturbation: A crude, but informative picture**
1996 June 20

The purpose of this report is to develop a crude picture of how hypersensitivity works in quantum systems. The picture, which is based on the quantized baker's map formulated by Rüdiger and me last summer, explains the phenomena found in previous numerical work, particularly the features displayed in Dierk's work on the kicked rotor. The arguments are both crude and subtle, a potentially explosive combination. Nonetheless, I think the picture captures the essence of what is going on in our numerical simulations. It might be worthwhile trying to make the arguments more rigorous within the context of a model like the quantized baker's map.

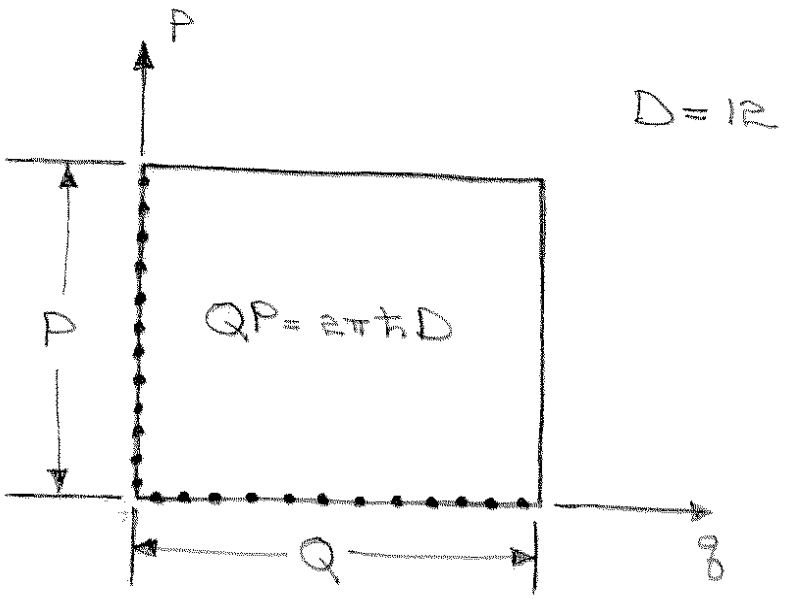

Figure 1

Consider a two-dimensional map on the torus. The torus can be represented as a rectangular phase space that has periodic boundary conditions (see Figure 1). Let Q be the extent of the rectangle in the q direction and P the extent in the p direction. By appropriate choice of units, Q and P can both be set to unity; this not always being the most convenient choice, however, we leave Q and P arbitrary. When quantized, the system has a D-dimensional Hilbert space $\mathcal{H}$, where

$$D = \frac{QP}{2\pi\hbar} . \tag{1}$$

The D position eigenstates, separated by a position discreteness $Q/D = 2\pi\hbar/P$, and the D momentum eigenstates, separated by a momentum discreteness $P/D = 2\pi\hbar/Q$, together form a grid of D^2 points on phase space.

It is useful to able to think in terms of an orthonormal basis consisting of D localized states on phase space (see Figure 2). These states are centered at the $D = rs$ grid points of an $r \times s$ grid. A localized state occupies a quantum cell of area $2\pi\hbar$, with extent Q/r in position and extent P/s in momentum. The position eigenstates correspond to $r = D$ and $s = 1$, and the momentum eigenstates to $r = 1$ and $s = D$. Crudely speaking, a localized state is a linear combination of the $D/r = s$ position eigenstates that lie within its column in the grid or a linear combination of the $D/s = r$ momentum eigenstates that lie within its row. Indeed, the s localized states in a column of the grid are roughly the "momentum" states corresponding to the s position eigenstates within the column, and the r localized states in a row are roughly the "position" states corresponding to the r momentum eigenstates within the row.

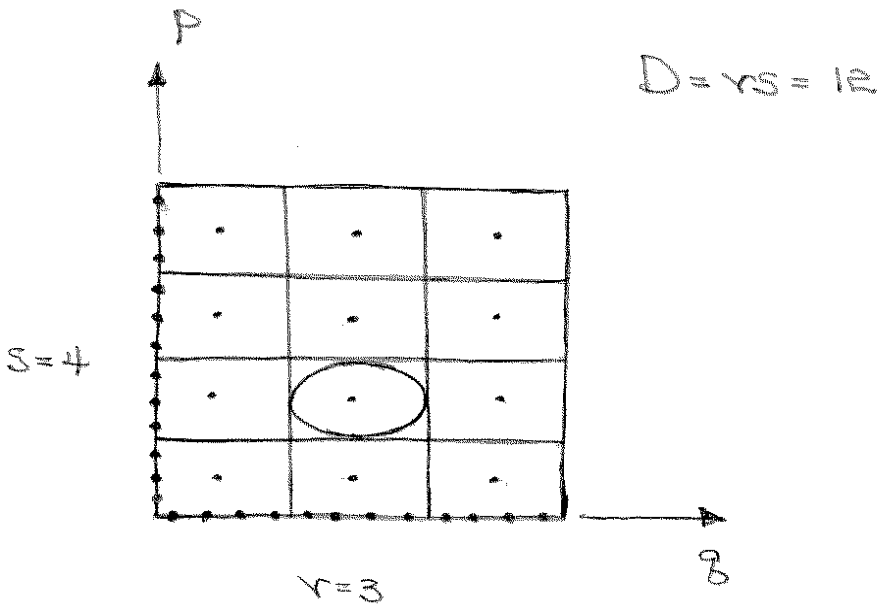

Figure 2

A case of particular interest occurs when the grid of localized states has the same number of states in the q and p directions, i.e., $r = s = \sqrt{D}$. We call these states *coherent states*, even though they generally look squeezed in the original phase-space units; crudely speaking, a coherent state is a linear combination of $\sqrt{D}$ position eigenstates or $\sqrt{D}$ momentum eigenstates. We use a coherent state as the initial state in the following.

It is convenient throughout the discussion to put position and momentum on the same footing by measuring all quantities in units of the position and momentum discreteness scales. In these units the area of phase space is D^2, and the area occupied by a localized state—the quantum cell size—is D.

Suppose now that the two-dimensional map has classical KS entropy K (in bits). Our quantum model of the map is that a coherent state is taken under one iteration of the map to a localized state that has extent $2^{-K}\sqrt{D}$ in the contracting dimension and extent $2^K\sqrt{D}$ in the expanding dimension (see Figure 3). This localized state is a linear combination of 2^K coherent states, with roughly equal probabilities for all 2^K states. This evolved state is a more general sort of localized state than the ones introduced above, because generally it is not aligned with the q-p axes. The unitarity of the quantum map means that the localized states generated from all coherent states form an orthonormal basis. After n steps an initial coherent state evolves into a linear combination of 2^{Kn} coherent states; again the collection of all such linear combinations forms an orthonormal basis.

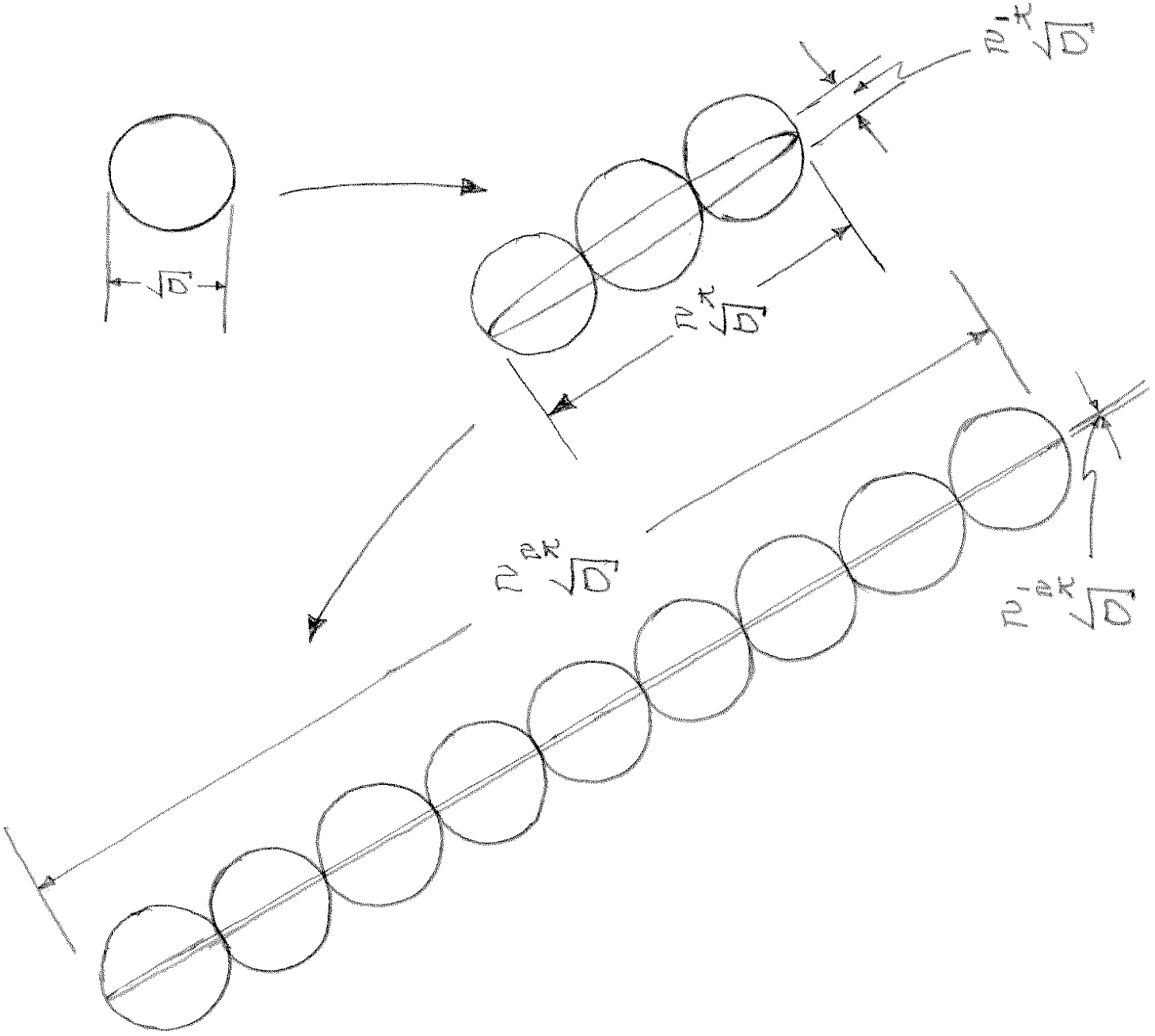

Figure 3

We assume that the folding of the phase-space pattern is due wholly to the finite size of phase space, i.e., to the periodic boundary conditions at the boundaries of the rectangular phase space. Given this assumption, there is an Ehrenfest time n_E, defined to be the number of steps that it takes for an initial coherent state to evolve into a single stripe across phase space. This Ehrenfest time is given approximately by

$$2^{Kn_E}\sqrt{D} \sim D \qquad \Longleftrightarrow \qquad n_E \sim \frac{1}{2K}\log D\,. \tag{2}$$

For times up to the Ehrenfest time n_E, the quantum evolution mimics the classical evolution. Beyond the Ehrenfest time n_E the folding leads to quantum interference between the multiple stripes in the classical pattern. Though this quantum interference is displayed clearly on phase space, there is no evidence of it in Hilbert space. Indeed, the Hilbert-space evolution takes no notice of the passage through n_E; it continues to be described as a state that is a linear combination of an exponentially increasing number of coherent states. The quantum interference is a consequence of the folding that is required to map the coherent states onto phase space; Hilbert space is oblivious to this mapping. For a particular quantum map described in terms of coherent states, the form of the quantum interference depends on how the coherent states are mapped onto phase space. We see below that n_E is important because our perturbation is defined on phase space and thus is aware of the folding of the classical dynamics.

There is a second Ehrenfest time n'_E, defined to be the number of steps that it takes an initial coherent state to spread over all coherent states. This second Ehrenfest time is given approximately by

$$2^{Kn'_E} \sim D \qquad \Longleftrightarrow \qquad n'_E \sim \frac{1}{K} \log D \sim 2n_E \; . \tag{3}$$

Notice that as $\hbar$ decreases, both Ehrenfest times increase logarithmically. After the second Ehrenfest time the evolution displays genuinely quantum-mechanical effects, even on Hilbert space, for there is interference between multiple paths to go from the initial coherent state to a final coherent state. As a consequence, the quantum evolution loses contact entirely with the classical evolution. The interference between different paths, not the interference on phase space, is responsible for quantum recurrences. Unless noted otherwise, we always assume $n \lesssim n'_E$ in the following.

Consider now a stochastic perturbation that generates a displacement on phase space. The important part of the perturbation is the displacement orthogonal to the classical stripes. Assume that the perturbation is L-fold, with the L potential displacements (all of equal probability) spaced uniformly and located symmetrically about zero. The perturbation strength is characterized by a parameter α, where the *maximum* positive or negative displacement orthogonal to the classical stripes has magnitude $\alpha/2$. Each of the L realizations of the perturbation is described by a unitary operator.

Classically the perturbation becomes effective at step n_*,[1] the first step after the classical pattern shrinks to width α in the contracting dimension.[2] Thus n_* is given approximately by

$$\alpha \sim 2^{-K(n_*-1)}\sqrt{D} \sim 2^{K(n_E-n_*+1)} \qquad \Longleftrightarrow \qquad n_* \sim 1 + \frac{1}{K} \log \frac{\sqrt{D}}{\alpha} \sim n_E + 1 - \frac{1}{K} \log \alpha \; . \tag{4}$$

We assume throughout that $\alpha \lesssim \sqrt{D}$, so that the perturbation does not become effective before the first step; otherwise, the perturbation produces dramatic effects at the first step. The perturbation becomes effective before the first Ehrenfest time if $\alpha \gtrsim 2^K$ and after the first Ehrenfest time if $\alpha \lesssim 2^K$. These two cases exhibit a striking difference in behavior, to which we now turn.

Assume first that α is somewhat larger than 2^K (but not bigger than $\sqrt{D}$), so that the perturbation becomes effective before n_E, i.e., $1 \lesssim n_* \lesssim n_E$.[3] In quantum-mechanical terms we can say that before the perturbation becomes effective, it is too weak to push the system into orthogonal states; the resulting nearby states persist through the perturbation histories and produce the structure of pairs, quartets, octets, and so forth that we see in our numerical simulations. We ignore this structure in the following discussion, proceeding as if the perturbation does nothing at all until it becomes effective. After the perturbation becomes effective, it is strong enough to push the system into orthogonal states.

To analyze the perturbation in the case $1 \lesssim n_* \lesssim n_E$, it is convenient to use as a basis the localized states that are produced from coherent states after $n_* - 1$ iterations of the map, i.e., at the step just before the perturbation becomes effective[4] (see Figure 4). These localized basis states have extent $2^{-K(n_*-1)}\sqrt{D} = \alpha \gtrsim 2^K$ in the contracting dimension and extent $2^{K(n_*-1)}\sqrt{D} = D/\alpha \lesssim 2^{-K} D$ in the expanding dimension.

[1] In order for the perturbation to turn on abruptly, as we assume here, we must have $K \gtrsim 1$. If K is less than 1, one way to proceed is to define a new map, each step of which consists of enough steps of the original map to make the new KS entropy bigger than 1.

[2] There are subtleties at step $n_* - 1$ which we are ignoring.

[3] For this to hold n_E must be somewhat greater than 1; i.e., 2^K must be smaller than $\sqrt{D}$. Combined with the condition in footnote 1, this means that we always assume $2 \lesssim 2^K \lesssim \sqrt{D}$.

[4] Notice that if $\alpha \sim \sqrt{D}$, then $n_* \sim 1$, and these localized basis states are the coherent states.

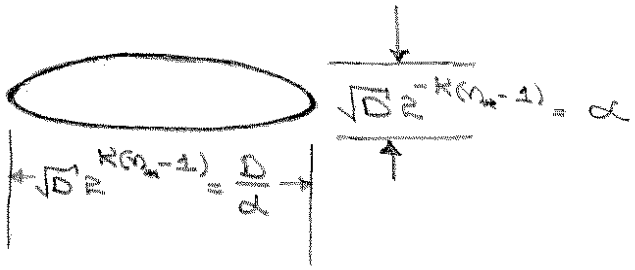

Figure 4

Classically the unperturbed state after step n has extent $2^{-Kn}\sqrt{D} = 2^{-K(n-n_*+1)}\alpha$ in the contracting dimension and extent $2^{Kn}\sqrt{D} = 2^{K(n-n_*+1)}D/\alpha$ in the expanding dimension; quantum mechanically the unperturbed state $|\psi_n\rangle$ is a linear combination of $D_n = 2^{K(n-n_*+1)}$ localized basis states, with roughly equal probabilities for each of these localized states (see Figure 5). Indeed, it is instructive to consider these D_n states as the position states in a D_n-dimensional Hilbert space $\mathcal{H}_n$; we label these position states by $|\phi_{nj}\rangle$, $j = 1, \ldots, D_n$. Conjugate to these position states are D_n orthogonal momentum states. The extent α of the position states in the contracting dimension gives the range of possible values of the corresponding momentum, which therefore ranges from $-\alpha/2$ to $\alpha/2$, just covering the range of possible perturbation strengths. This is why we use the states $|\phi_{nj}\rangle$ as as a basis: the Hilbert space they span, $\mathcal{H}_n$, has just the right extent in momentum to accommodate the perturbation. The unperturbed state after step n can be expanded as

$$|\psi_n\rangle = \sum_{j=1}^{D_n} c_j |\phi_{nj}\rangle \ , \qquad \text{all expansion coefficients } c_j \text{ having roughly the same magnitude.} \tag{5}$$

The unperturbed state can be defined to be the zero-momentum state in $\mathcal{H}_n$.

The strength of the perturbation applied at step n is such that all the perturbed states after step n remain in the Hilbert space $\mathcal{H}_n$. Indeed, as we discuss further below, the perturbation can be effected by making appropriate phase changes in the expansion coefficients in the position-state basis. Thus all the perturbed states after the perturbation at step n have expansions of the form (5).

We can say more, however. The expansion and contraction of the unperturbed unitary evolution at step n maps the D_{n-1} orthogonal momentum states in $\mathcal{H}_{n-1}$ into the D_{n-1} orthogonal momentum states in $\mathcal{H}_n$ that are closest (on both sides) to the unperturbed state. Let us denote by $\mathcal{H}_{n0}$ the D_{n-1}-dimensional subspace spanned by these momentum states. Formally we can say that the nth unperturbed step maps $\mathcal{H}_{n-1}$ unitarily into $\mathcal{H}_{n0}$. Moreover, since the perturbed states after step $n-1$ all lie in $\mathcal{H}_{n-1}$, they are mapped by the nth unperturbed step into $\mathcal{H}_{n0}$.

In analyzing the effect of the perturbation at step n, it is convenient to decompose $\mathcal{H}_n$ into 2^K subspaces similar to $\mathcal{H}_{n0}$; these subspaces are mutually orthogonal, and each is spanned by a contiguous set of $D_{n-1} = D_n/2^K = 2^{K(n-n_*)}$ momentum states. We denote these subspaces by $\mathcal{H}_{nk}$, where k is an index that labels the subspace and runs over an appropriate set of 2^K integers. The formal statement is that $\mathcal{H}_n$ is the direct sum of the subspaces $\mathcal{H}_{nk}$:

$$\mathcal{H}_n = \bigoplus_k \mathcal{H}_{nk} \ . \tag{6}$$

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Rüdiger's work shows that statements here are too strong. The D_{n-1} momentum states in $\mathcal{H}_{n-1}$ do not get mapped to the D_{n-1} momentum states in $\mathcal{H}_{no}$; the ramping of the phases does not work out correctly for this. On the other hand, the phases of the mapped states can only go through a number of 2π rotations such that they do lie in $\mathcal{H}_{no}$, even though they are not the momentum states. This is all that is required for the argument.

Indeed, this is a boon, because it means that the chaotic dynamics randomizes the phases, rather than giving uniform ramping, thus leading to hypersensitivity when $L \geq 2^K$.

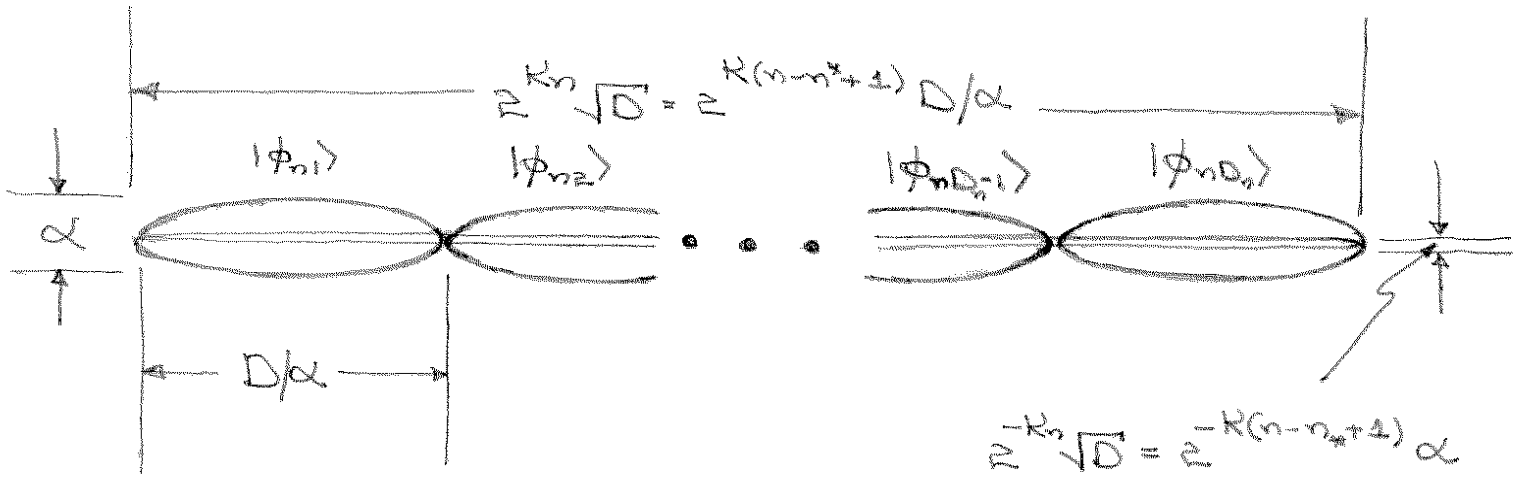

Figure 5

Consider now the effect of the perturbation at times before the first Ehrenfest time, i.e., $n_* - 1 \lesssim n \lesssim n_E$. The phase-space displacement produced by the perturbation can be effected by ramping the phases of the position states $|\phi_{nj}\rangle$; i.e., a displacement by α', where $-\alpha/2 \le \alpha' \le \alpha/2$, is produced by[5]

$$\sum_{j=1}^{D_n} c_j |\phi_{nj}\rangle \longrightarrow \sum_{j=1}^{D_n} c_j e^{2\pi i j \alpha'/\alpha} |\phi_{nj}\rangle \,, \tag{7}$$

which is a momentum displacement in $\mathcal{H}_n$.[6] To see what is going on, it is instructive to consider the special case $L = 2^K$. In this case we argue that the $L^{n-n_*+1} = 2^{K(n-n_*+1)} = D_n$ perturbation histories (recall that we are ignoring the perturbations that occur before the perturbation becomes effective) lead roughly to orthogonal momentum states in $\mathcal{H}_n$. Suppose that this is true for the D_{n-1} perturbed states after the perturbation is applied at step $n-1$. Then the unperturbed evolution at step n maps the perturbed states to the momentum states in the subspace $\mathcal{H}_{n0}$. The 2^K-fold perturbation at step n maps momentum states to momentum states; moreover, it is set up in just such a way that each realization of the perturbation maps the subspace $\mathcal{H}_{n0}$ to a different orthogonal subspace $\mathcal{H}_{nk}$. Thus, for a 2^K-fold perturbation the perturbed states are the D_n orthogonal momentum states in $\mathcal{H}_n$.

[5] One way to obtain the phase factors here is to notice that for a momentum displacement by $\alpha' P/D$ (in ordinary phase-space units), the phase between successive position states (separated by Q/D in ordinary phase-space units) is $(i/\hbar)(\alpha' P/D)(Q/D) = 2\pi i \alpha'/D$; thus the cumulative phase between position states that are separated by D/α position discreteness units is $2\pi i(\alpha'/D)(D/\alpha) = 2\pi i \alpha'/\alpha$.

[6] The perturbation (7) approximates a displacement on phase space only when $\alpha'/\alpha \ll 1$. As noted, what the perturbation (7) really describes is a momentum displacement on $\mathcal{H}_n$. Classically this perturbation is a displacement within a phase-space cell, with periodic boundary conditions at the top and bottom boundaries of the cell. This is not such a bad perturbation to use—we use it classically—but it is not what we do in the numerical simulations. Notice that within a Hilbert space of dimension D_n, the actual maximum range of displacements is $\tilde{\alpha} = \alpha(1 - D_n^{-1})$. The quantity $\tilde{\alpha}$ is a sort of renormalized perturbation strength. We ignore the difference between α and $\tilde{\alpha}$ for the present, but return to it later.

We want to generalize this property to times after the first Ehrenfest time, i.e., $n_E \lesssim n \lesssim n'_E$. The generalization is complicated by the fact that we no longer deal directly with momentum states, because a phase-space displacement is no longer given by a monotonic ramping of position-state phases, as in (7), and thus no longer corresponds to a momentum displacement. The reason is that the perturbation recognizes the folding caused by the boundaries of phase space. The classical pattern consists of $2^{K(n-n_E)}$ stripes across phase space, and the perturbation displaces each stripe by the same amount. Quantum mechanically we must do the same displacement within each stripe.

The quantum-mechanical description of the stripes is obtained from the expansion (5): group the $D_n = 2^{K(n-n_*+1)} = 2^{K(n-n_E)}\alpha$ position states $|\phi_{nj}\rangle$ into $D_n/D_{n_E} = 2^{K(n-n_E)} = 2^{Kn}/\sqrt{D}$ groups, each consisting of $D_{n_E} = 2^{K(n_E-n_*+1)} = \alpha \gtrsim 2^K$ contiguous position states; relabel the position states as $|\phi_{nlj}\rangle$, where the index l labels the group and the index j runs from 1 to D_{n_E} within each group; and rewrite the unperturbed state (5) in the form

$$|\psi_n\rangle = \sqrt{\frac{D_{n_E}}{D_n}} \sum_l |\psi_{nl}\rangle \ , \qquad \text{where} \quad |\psi_{nl}\rangle = \sum_{j=1}^{D_{n_E}} d_{lj}|\phi_{nlj}\rangle \quad \text{is a normalized state for the } l\text{th stripe.} \tag{8}$$

Each stripe state has extent D in the expanding dimension and extent 1 in the contracting dimension. All the perturbed states after the unperturbed evolution at step n lie in $\mathcal{H}_{n0}$ and have an expansion of the form (8).

To analyze the effect of the perturbation, it is instructive to decompose the Hilbert space $\mathcal{H}_n$ in ways that match the decomposition of $|\psi_n\rangle$ into stripes. For a particular l the position states $|\phi_{nlj}\rangle$ span a "stripe" subspace $\mathcal{H}_{nl}$ of dimension $D_{n_E} = \alpha \gtrsim 2^K$. The Hilbert space $\mathcal{H}_n$ is a direct sum of these stripe subspaces:

$$\mathcal{H}_n = \bigoplus_l \mathcal{H}_{nl} \ . \tag{9}$$

Within a stripe subspace $\mathcal{H}_{nl}$ it is convenient to regard the position states as *position* states; conjugate to the *position* states are *momentum* states, which we recognize from our previous discussion as localized states in $\mathcal{H}_n$. The stripe subspace $\mathcal{H}_{nl}$ can be further decomposed into 2^K orthogonal subspaces $\mathcal{H}_{nlk}$, i.e.,

$$\mathcal{H}_{nl} = \bigoplus_k \mathcal{H}_{nlk} \ , \tag{10}$$

where the subspaces $\mathcal{H}_{nlk}$ are spanned by $D_{n_E}/2^K = D_{n_E-1} = \alpha/2^K \gtrsim 1$ contiguous *momentum* states. Our earlier discussion suggests that it is roughly true that

$$\mathcal{H}_{nk} = \bigoplus_l \mathcal{H}_{nlk} \ . \tag{11}$$

We can identify a special subspace $\mathcal{H}_{nl0}$ of $\mathcal{H}_{nl}$, which is the subspace spanned by the D_{n_E-1} *momentum* states that are closest (on both sides) to the zero-*momentum* state. The subspace $\mathcal{H}_{n0}$, which contains all the perturbed states after the unperturbed evolution at step n, can be written roughly as the direct sum

$$\mathcal{H}_{n0} = \bigoplus_l \mathcal{H}_{nl0} \ . \tag{12}$$

The phase-space displacement produced by the perturbation can now be effected by ramping the phases of the *position* states *within each stripe*, with the ramping being the same from stripe to stripe (there is a discontinuity in the ramping between stripes); i.e., a displacement by α', where $-\alpha/2 \le \alpha' \le \alpha/2$, is produced by

$$\sum_{j=1}^{D_{n_E}} d_{lj}|\phi_{nlj}\rangle \longrightarrow \sum_{j=1}^{D_{n_E}} d_{lj}e^{2\pi ij\alpha'/\alpha}|\phi_{nlj}\rangle \ , \tag{13}$$

which is a *momentum* displacement in $\mathcal{H}_{nl}$.[7] To see what is going on, it is again instructive to consider the case $L = 2^K$. In this case we contend that the $L^{n-n_*+1} = 2^{K(n-n_*+1)} = D_n$ perturbed states after the perturbation at step n are roughly orthogonal states in $\mathcal{H}_n$. Suppose that this is true for the D_{n-1} perturbed states after the perturbation is applied at step $n-1$. Then the unperturbed evolution at step n maps the perturbed states to orthogonal states in $\mathcal{H}_{n0} = \oplus_l \mathcal{H}_{nl0}$. The 2^K-fold perturbation at step n is set up in just such a way that each realization of the perturbation maps $\mathcal{H}_{nl0}$ to a different orthogonal subspace $\mathcal{H}_{nlk}$ and thus maps $\mathcal{H}_{n0}$ to a different orthogonal subspace $\mathcal{H}_{nk} = \oplus_l \mathcal{H}_{nlk}$.

Our conclusion from these arguments is the following: if the perturbation becomes effective before the first Ehrenfest time, i.e., $n_* \lesssim n_E$, then for all times n between the time when the perturbation becomes effective and the second Ehrenfest time, i.e., for $n_* \lesssim n \lesssim n'_E$, a perturbation with $L = 2^K$ produces $D_n = 2^{K(n-n_*+1)}$ roughly orthogonal perturbed states, which are a basis for the Hilbert space $\mathcal{H}_n$. The average entropy increases as $\Delta H_S \sim \log D_n = K(n-n_*+1)$, but there is no hypersensitivity to perturbation because all the perturbed states are orthogonal. If $L \lesssim 2^K$, then the perturbed states are orthogonal, but the number of them, L^{n-n_*+1}, is insufficient to provide a basis for $\mathcal{H}_n$, so the average entropy increase is reduced to $\Delta H_S \sim \log L(n-n_*+1)$, and there is no hypersensitivity to perturbation. If $L > 2^K$, then there are enough perturbed states to start to fill $\mathcal{H}_n$ randomly. The average entropy increases as $\Delta H_S \sim K(n-n_*+1)$, and there is hypersensitivity to perturbation, becoming more pronounced as L increases because that increase leads to an increasing number of nonorthogonal states. We summarize these conclusions by the following:

$$\begin{array}{lll} & & L \lesssim 2^K \;: \quad \begin{array}{l} \Delta H_S \sim \log L(n-n_*+1) \\ \textit{no} \text{ hypersensitivity} \end{array} \\ 2 \lesssim 2^K \lesssim \sqrt{D}\,, \quad \begin{array}{l} 1 \lesssim n_* \lesssim n_E \\ \sqrt{D} \gtrsim \alpha \gtrsim 2^K \end{array}, \quad n_* - 1 \lesssim n \lesssim n'_E\,, & L \sim 2^K \;: \quad \begin{array}{l} \Delta H_S \sim K(n-n_*+1) \\ \textit{no} \text{ hypersensitivity} \end{array} \\ & & L > 2^K \;: \quad \begin{array}{l} \Delta H_S \sim K(n-n_*+1) \\ \text{hypersensitivity} \end{array} \end{array} \qquad (14)$$

In comparing these conclusions with numerical simulations, it is a good idea to set the perturbation strength of the simulation equal to the renormalized perturbation strength $\tilde{\alpha} = \alpha - 1$.

Turn now to the case where α is somewhat less than 2^K, so that classically the perturbation becomes effective after the first Ehrenfest time, i.e., $n_* \gtrsim n_E$. It is instructive to deal first with the case where $1 \lesssim \alpha \lesssim 2^K$; this is a subtle transitional case, in which the perturbation varies from strong to weak and the time when the perturbation becomes effective is defined somewhat vaguely as being between n_E and $n_E + 1$. We are interested in times greater than the first Ehrenfest time, i.e., $n_* - 1 \lesssim n \lesssim n'_E$.

It is still convenient to use the position states $|\phi_{nlj}\rangle$ because their extent D/α in the expanding dimension is smaller than the size of phase space. The Hilbert space $\mathcal{H}_n$ can still be decomposed into $2^{K(n-n_E)}$ stripe subspaces $\mathcal{H}_{nl}$ of dimension $D_{n_E} = \alpha \lesssim 2^K$, as in (9). The difference arises when we decompose the stripe subspaces into *momentum* subspaces, because the stripe subspaces cannot accommodate 2^K *momentum* subspaces. What we do instead is to decompose each stripe subspace $\mathcal{H}_{nl}$ into α one-dimensional subspaces $\mathcal{H}_{nlk}$, each containing a single *momentum* state. We then construct subspaces

$$\mathcal{H}_{nk} = \bigoplus_l \mathcal{H}_{nlk}\,, \qquad (15)$$

of which there are α, each of dimension $2^{K(n-n_E)} = D_n/\alpha$. The Hilbert space $\mathcal{H}_n$ is the direct sum of the subspaces $\mathcal{H}_{nk}$, as in (6). The special subspace $\mathcal{H}_{n0}$ is now a space of dimension $2^{K(n-n_E)} = D_n/\alpha$, the space spanned by the zero-*momentum* states in each of the stripe subspaces. The unperturbed evolution at step n takes the space $\mathcal{H}_{n-1}$, of dimension $D_{n-1} = D_n/2^K$, into a *subspace* of $\mathcal{H}_{n0}$.

[7] Notice that the renormalized perturbation strength becomes constant at $\tilde{\alpha} = \alpha(1 - D_{n_E}^{-1}) = \alpha - 1$ for $n \gtrsim n_E$.

The effect of the perturbation is described by (13). To clarify what is going on here, it is best to consider a perturbation with $L = \alpha$. Using the same argument that we used above, we can conclude that the $L^{n-n_*+1} = \alpha^{n-n_*+1}$ perturbed states after step n are orthogonal states in $\mathcal{H}_n$. Thus the average entropy increases as $\Delta H_{\mathcal{S}} \sim \log \alpha(n - n_* + 1)$, and there is no hypersensitivity to perturbation. For $L \lesssim \alpha$, the perturbed states are orthogonal, the average entropy increases goes as $\Delta H_{\mathcal{S}} \lesssim \log L(n - n_* + 1)$, and there is no hypersensitivity. For $L > \alpha$, the perturbed states begin to fill randomly a space of dimension α^{n-n_*+1}, the average entropy increases as $\Delta H_{\mathcal{S}} \sim \log \alpha(n - n_* + 1)$, and there is hypersensitivity, increasing as L increases. We summarize these conclusions by the following:

$$
\begin{array}{lll}
 & & L \lesssim \alpha : \quad \begin{array}{l} \Delta H_{\mathcal{S}} \sim \log L(n - n_* + 1) \\ \textit{no} \text{ hypersensitivity} \end{array} \\
2 \lesssim 2^K \lesssim \sqrt{D}\,, \quad \begin{array}{l} n_E \lesssim n_* \lesssim n_E + 1 \\ 2^K \gtrsim \alpha \gtrsim 1 \end{array} \quad , \quad n_* - 1 \lesssim n \lesssim n'_E\,, & L \sim \alpha : \quad \begin{array}{l} \Delta H_{\mathcal{S}} \sim \log \alpha(n - n_* + 1) \\ \textit{no} \text{ hypersensitivity} \end{array} \\
 & & L > \alpha : \quad \begin{array}{l} \Delta H_{\mathcal{S}} \sim \log \alpha(n - n_* + 1) \\ \text{hypersensitivity} \end{array}
\end{array}
\qquad (16)
$$

In comparing these conclusions with numerical simulations, it is a good idea to set the perturbation strength of the simulation equal to the renormalized perturbation strength $\tilde{\alpha} = \alpha - 1$ and to realize that we really require $\tilde{\alpha} \gtrsim 1$.

Consider now a very weak perturbation, with $\alpha \lesssim 1$, for which the perturbation becomes effective classically after the first Ehrenfest time. This is the case dealt with in many of our numerical simulations. We consider times after n_E, i.e., $n_E \lesssim n \lesssim n'_E$. Though it is no longer convenient to use as a basis the position states introduced above, since these states now extend over more than one stripe,[8] we can still write the unperturbed state after n steps as a superposition of $2^{K(n-n_E)} = 2^{Kn}/\sqrt{D}$ quantum-mechanical stripes, as in (8), i.e.,

$$
|\psi_n\rangle = \sqrt{\frac{D^{1/2}}{2^{Kn}}} \sum_l |\psi_{nl}\rangle\,, \qquad (17)
$$

where each (normalized) stripe state $|\psi_{nl}\rangle$ is a superposition of $2^{Kn_E} = \sqrt{D}$ coherent states. The extent of a stripe state in the expanding dimension is D, and its extent in the contracting dimension is 1.

A phase-space displacement produces a new state

$$
|\psi'_n\rangle = \sqrt{\frac{D^{1/2}}{2^{Kn}}} \sum_l |\psi'_{nl}\rangle\,, \qquad (18)
$$

where the perturbed stripe state $|\psi'_{nl}\rangle$ is obtained by ramping the phases of its coherent states, the ramping being the same from stripe to stripe. The inner product of the perturbed and unperturbed states is given by

$$
\langle\psi'_n|\psi_n\rangle = \frac{\sqrt{D}}{2^{Kn}} \sum_l \langle\psi'_{nl}|\psi_{nl}\rangle\,. \qquad (19)
$$

Each of the $2^{Kn}/\sqrt{D}$ terms in the sum is approximately the same, since the ramping is the same in each stripe. The upshot is that the overall inner product (19) is essentially the same as the inner product in a typical stripe. If the perturbation strength were somewhat bigger than 1, then the perturbation would be strong enough to push the system into orthogonal states within a stripe and thus into orthogonal states overall, as we have already seen. In the present case, however, where the perturbation strength is somewhat smaller than 1, the perturbation is too weak to push the system into orthogonal states within a stripe, and thus it is always too weak to push the system into orthogonal states overall. The conclusion is the following: *if the perturbation is weak enough that it becomes effective classically after the first Ehrenfest time, then it* **never** *becomes effective quantum mechanically.*

[8] Another way of saying this is that our previous decomposition into stripe subspaces spanned by α position states cannot be carried out when α is less than 1.

For a perturbation with strength $\alpha \lesssim 2^K$, the picture developed here and Dierk's results for the kicked rotor reveal an important quantum-mechanical consequence of phase-space folding. If the perturbation were a monotonic ramping of the phases of all the coherent states that contribute to $|\psi_n\rangle$—as far as Hilbert space is concerned, this is a reasonable perturbation—then the perturbation would become strong enough to push the system into orthogonal states after it became effective classically. In the inner product (19) the terms from the various stripes would acquire phases so that they summed to zero. A phase-space displacement, however, is aware of the folding because of the boundaries of phase space; it must be described by the same ramping within each stripe. For a perturbation in the transition region, where $1 \lesssim \alpha \lesssim 2^K$, the result is a reduction below 2^K in the number of orthogonal states available to the perturbation within each stripe and hence overall. For a weak perturbation, with $\alpha \lesssim 1$, the result is that there is only state within each stripe; for the mechanism considered so far, which relies on the perturbation's becoming effective, there is no way for the perturbation to have any effect at all.

There is, of course, another mechanism for exploring Hilbert space dimensions, the slow diffusion in Hilbert space. This diffusion, which is ignored in all of the preceding discussion, produces the structure of pairs, quartets, and so forth in our numerical simulations. For a weak perturbation, diffusion cannot be ignored, diffusion being the *only* mechanism for exploring Hilbert-space dimensions and thus increasing the average entropy. We make no attempt here to quantify the increase in entropy in the case where diffusion is important, but we do note that there is always hypersensitivity in this case, regardless of the value of L, because the perturbation produces far more states than orthogonal states. This weak-perturbation quantum hypersensitivity is the kind found in most of our numerical simulations. Indeed, we often go to times larger than n'_E to make the hypersensitivity more apparent.

In this regard it is important to stress the following. Quantum hypersensitivity is generally different from classical hypersensitivity because it exists for a perturbation that is correlated across all of phase space. The quantum hypersensitivity for perturbations with $\alpha \lesssim 2^K$, is still more different, because it is not tied to the classical KS entropy. Classically one always sees hypersensitivity at the KS rate, provided there are many correlation cells, because the classical pattern inevitably becomes fine enough in the contracting dimensions to "feel" the perturbation, no matter how weak it is. Quantum mechanically one sees hypersensitivity at the KS rate if the perturbation is strong enough to become effective while the quantum evolution is still essentially classical; for weaker perturbations, one sees a quantum-mechanical hypersensitivity that is not tied to the KS entropy.

The classical argument for a linear increase of average entropy is simple. Once the perturbation becomes effective, the width of the average pattern stays constant at α, while the length grows as $2^{K(n-n_*+1)}$. The resulting exponential increase in phase-space area gives an average entropy increase $\Delta H_{\mathcal{S}} \sim K(n - n_* + 1)$. A quantum-mechanical version of this argument can be given as long as $\alpha \gtrsim 1$. The perturbation can access a width α in the contracting dimension, so it is appropriate to expand the perturbed states in terms of the position states, which have extent α in the contracting dimension. The analogue of the exponential increase in classical phase-space area is the exponential increase in the number $D_n = 2^{K(n-n_*+1)}$ of position states in this expansion. The question is whether the perturbation can access D_n orthogonal states within the Hilbert space $\mathcal{H}_n$. What the picture developed here shows is that the factor by which the number of accessed states can increase as a result of one application of the perturbation is limited by the dimension α of the stripe subspaces. For $\alpha \gtrsim 2^K$, the perturbation can access the necessary number of orthogonal states for the classical argument to go through unchanged, but for $\alpha \lesssim 2^K$, there is a reduction in the number of states available to the perturbation and thus a reduction in the rate at which the average entropy increases. For a weak perturbation, with $\alpha \lesssim 1$, the classical argument has no quantum analogue because there is no quantum description of a pattern with width α.

All this reveals a serious defect in work of Paz and Zurek, who argue for a linear increase in entropy at the KS rate. They consider the evolution of the Wigner distribution in the presence of coupling to an environment and focus on the difference between the Moyal bracket, which governs the evolution of the Wigner distribution, and the Poisson bracket, which governs the classical evolution. Their argument is essentially the classical argument. Once the coupling to the environment becomes effective, it halts the contraction of the Wigner pattern in the contracting dimension. The finite width in the contracting dimension, Paz and Zurek argue, renders the quantum corrections in the Moyal bracket negligible. Under the resulting classical evolution, the constant-width Wigner pattern lengthens exponentially in the expanding dimension, thus producing a linear entropy increase at the KS rate.

It is easy to see where this argument goes wrong. Suppose the coupling to the environment does not become effective till after there is some folding. The quantum interference that accompanies the folding is described by the quantum corrections in the Moyal bracket. The quantum corrections to the Moyal bracket are important before the coupling to the environment halts the contraction, and so they remain important afterward. The Wigner pattern does lengthen exponentially after the contraction is halted, but since the Wigner distribution is not classical, one cannot use classical arguments about phase-space area to determine the entropy. Paz and Zurek miss this effect in their numerical work because they consider a linear system that has no folding.

The picture developed here, it seems to me, is worth exploring in some detail, both in analytical models, if possible, and by numerical simulations. If it holds up, as I expect it will, then it provides a framework for taking the next step, which is the study of decoherence when one replaces the stochastic Hamiltonian with a coupling to a real environment that entangles the system and environment.