Diving Deep into a Quantum-Information-Processing Problem: Quantum Limits on the Performance of Linear Amplifiers

I. What's the problem?

II. Quantum limits on noise in (deterministic) phase-preserving linamps. The whole story

I. Nondeterministic immaculate linamps

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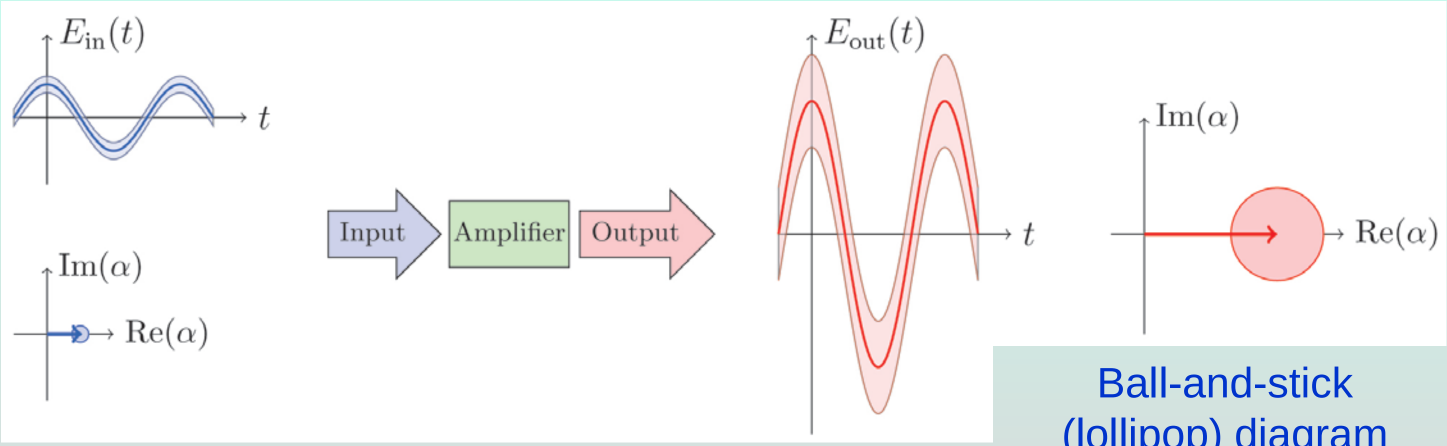
Pinnacles National Park
Central California

(Deterministic) phase-preserving linear amplifiers



Ball-and-stick (lollipop) diagram

Simultaneous measurements of x and p at the input, repeated ν times, can determine the center of a coherent state with uncertainty

$$\frac{\delta x_{\mathrm{in}}}{\sqrt{\nu}} = \frac{\delta p_{\mathrm{in}}}{\sqrt{\nu}} = \frac{1}{\sqrt{\nu}}$$

Simultaneous measurements of x and p at the output, repeated ν times, can determine the center of the input state with uncertainty

$$\frac{\delta x_{\mathrm{out}}}{g\sqrt{\nu}} = \frac{\delta p_{\mathrm{out}}}{g\sqrt{\nu}} \geq \frac{1}{\sqrt{\nu}}$$

where g is the *amplitude* gain.

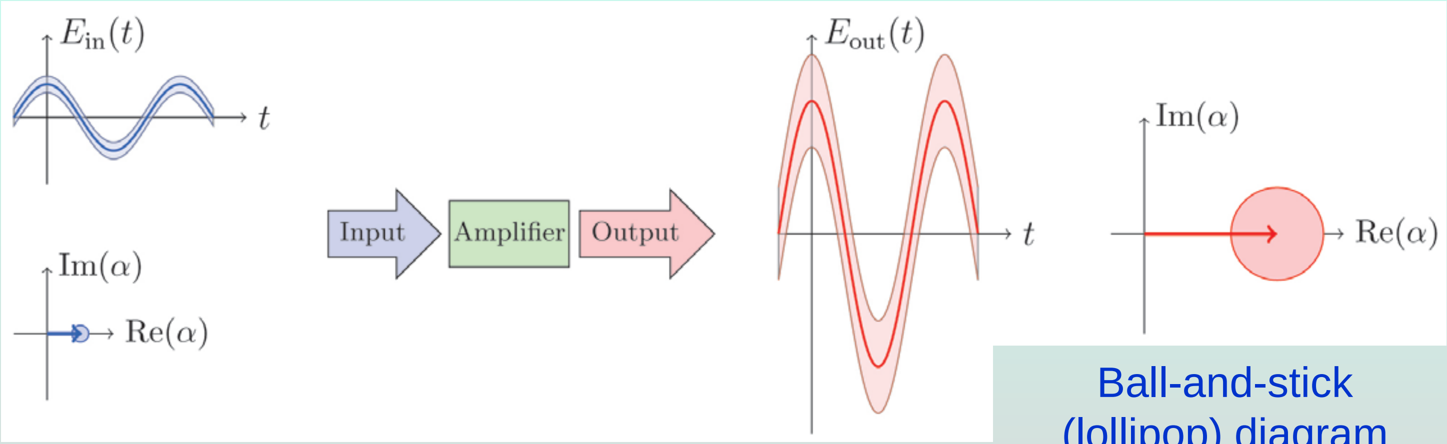
$$(\delta x_{\mathrm{out}})^2 \geq g^2(\delta x_{\mathrm{in}})^2$$
$$(\delta p_{\mathrm{out}})^2 \geq g^2(\delta p_{\mathrm{in}})^2$$

$$\langle(\Delta x_{\mathrm{out}})^2\rangle \geq g^2\langle(\Delta x_{\mathrm{in}})^2\rangle + \frac{1}{2}(g^2 - 1)$$

$$\langle(\Delta p_{\mathrm{out}})^2\rangle \geq g^2\langle(\Delta p_{\mathrm{in}})^2\rangle + \frac{1}{2}(g^2 - 1)$$

$$(\delta x)^2 = \langle(\Delta x)^2\rangle + \frac{1}{2}$$
$$(\delta p)^2 = \langle(\Delta p)^2\rangle + \frac{1}{2}$$

(Nondeterministic) immaculate linear amplifiers



Ball-and-stick (lollipop) diagram

An immaculate linear amplifier takes an input coherent state $|\alpha\rangle$ to an amplified output coherent state $|g\alpha\rangle$, but only works with probability $p(\surd)$.

Simultaneous measurements of x and p at the input, repeated ν times, can determine the center of a coherent state with uncertainty

$$\frac{\delta x_{\text{in}}}{\sqrt{\nu}} = \frac{\delta p_{\text{in}}}{\sqrt{\nu}} = \frac{1}{\sqrt{\nu}}$$

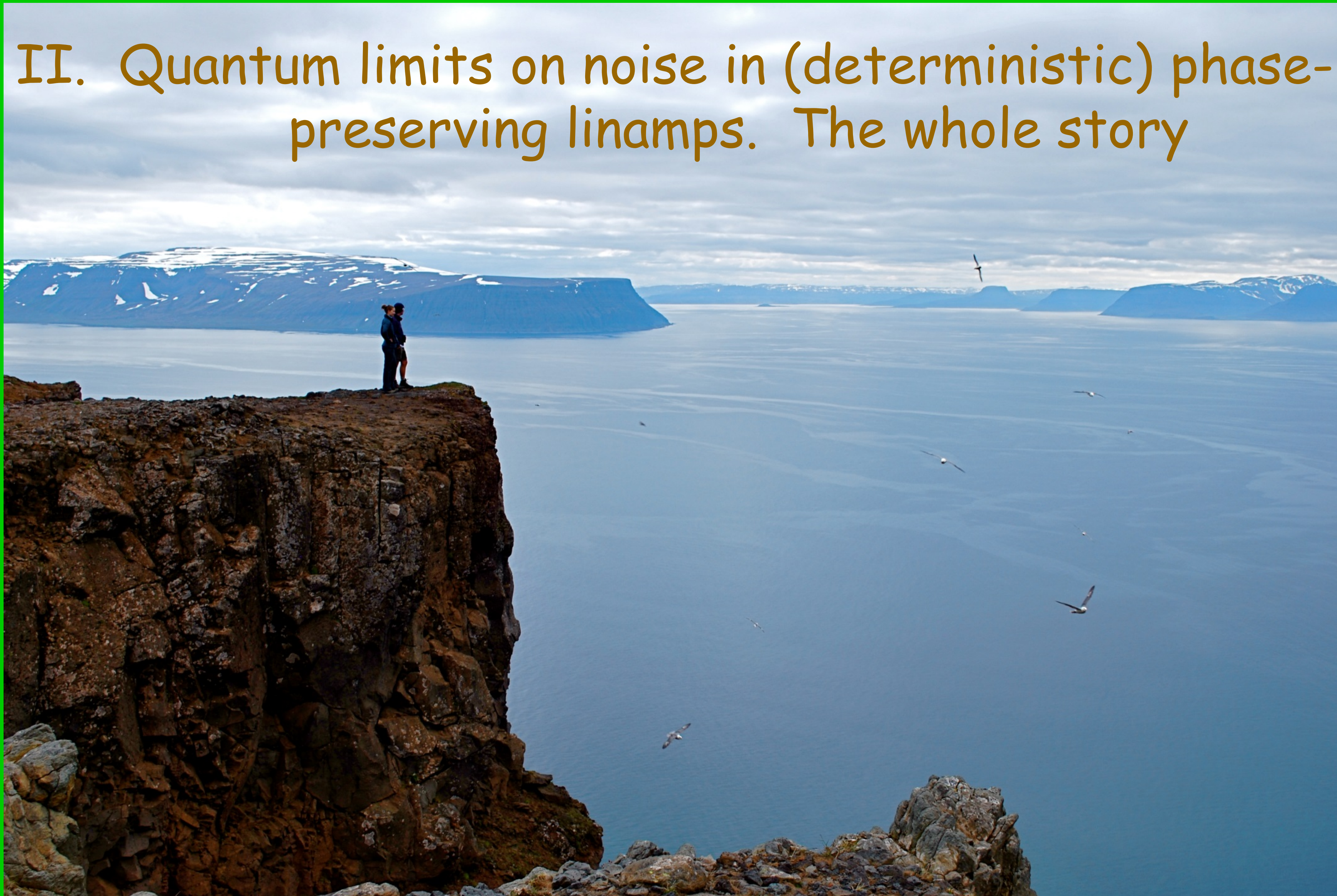
Simultaneous measurements of x and p at the output, repeated ν times, can determine the center of the input state with uncertainty

$$\frac{\delta x_{\text{out}}}{g\sqrt{\nu p(\surd)}} = \frac{\delta p_{\text{out}}}{g\sqrt{\nu p(\surd)}} = \frac{1}{g\sqrt{\nu p(\surd)}} \geq \frac{1}{\sqrt{\nu}}$$

$$p(\surd) \leq \frac{1}{g^2}$$

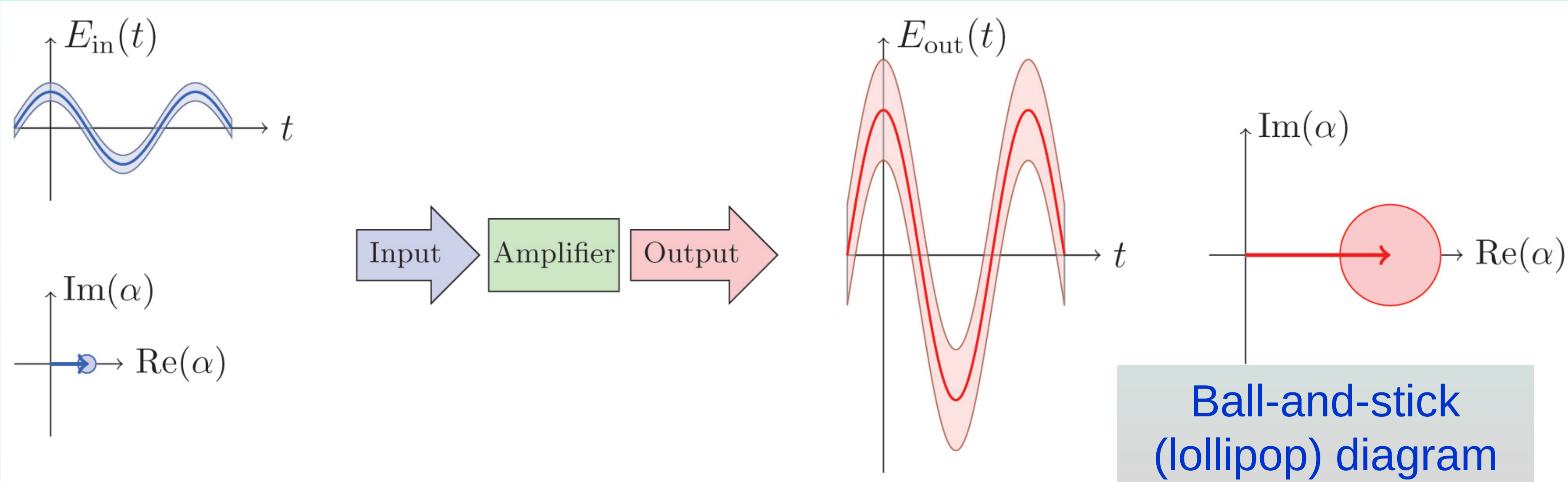
This is a distinguishability argument that turns out to be radically inadequate.

II. Quantum limits on noise in (deterministic) phase-preserving linamps. The whole story



Holstrandir Peninsula overlooking Ísafjarðardjúp
Westfjords, Iceland

Phase-preserving linear amplifiers



$$a_{\rm out} = g a_{\rm in}$$

Inconsistent with unitarity.
Can't maintain canonical commutator, $[a, a^\dagger] = 1$,
at both input and output.

Phase-preserving linear amplifiers

added-noise operator

$$a_{\text{out}} = g a_{\text{in}} + L^\dagger \qquad [a, a^\dagger] = 1 \quad \Longrightarrow \quad [L, L^\dagger] = g^2 - 1$$

output noise — gain — input noise — added noise

$$\langle |\Delta a_{\text{out}}|^2 \rangle = g^2 \langle |\Delta a_{\text{in}}|^2 \rangle + \langle |\Delta L|^2 \rangle$$

$$\geq g^2 - \frac{1}{2} \qquad \geq \frac{1}{2} \qquad \geq \frac{1}{2}(g^2 - 1)$$

Zero-point noise

Refer noise to input

$$\geq 1 - \frac{1}{2g^2} \qquad \geq \frac{1}{2}\left(1 - \frac{1}{g^2}\right)$$

Added noise number $A = \dfrac{\langle |\Delta L|^2 \rangle}{g^2 - 1} \geq \dfrac{1}{2}$

Noise temperature $kT_n \geq \dfrac{\hbar\omega}{\ln 3}$

C. M. Caves, PRD 26, 1817 (1982).
C. M. Caves, J. Combes, Z. Jiang, and S. Pandey, PRA 86, 063802 (2012).

Ideal phase-preserving linear amplifier. Parametric amplifier

primary mode a, ancillary mode b

$$H = i\hbar\kappa(ab - a^\dagger b^\dagger)$$

$$\Longleftrightarrow \qquad U(t,0) = e^{\kappa t(ab - a^\dagger b^\dagger)} \equiv S(r)\ , \qquad r = \kappa t$$

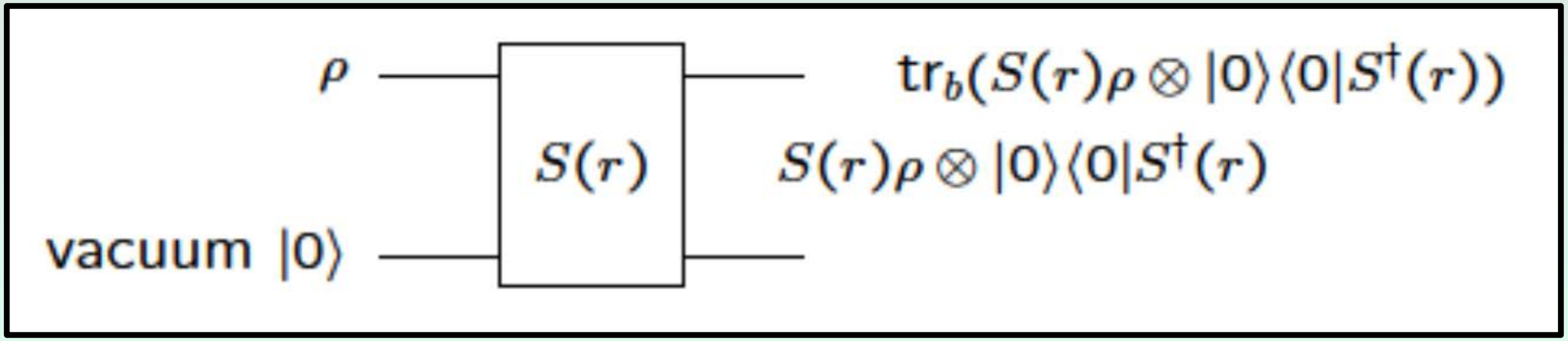
$S(r)$ is the *two-mode squeeze operator*.

$$a_{\text{out}} = a_{\text{in}} \cosh r + b_{\text{in}}^\dagger \sinh r$$

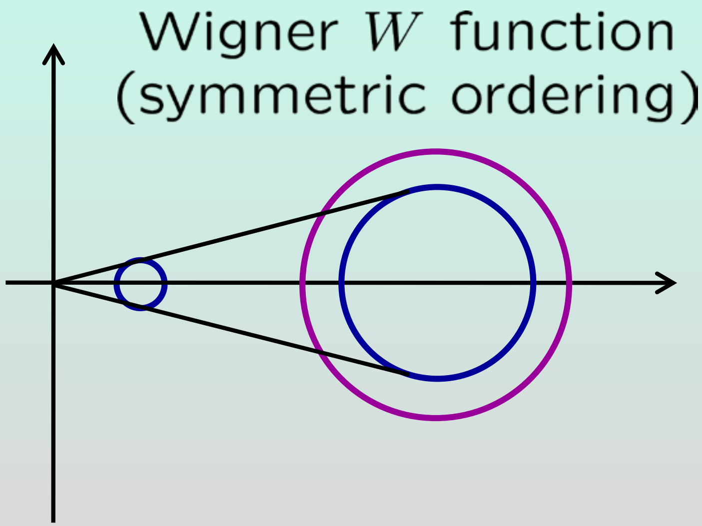
$$\cosh r = g \qquad \sinh r = \sqrt{g^2 - 1}$$

$$L = b_{\text{in}}\sqrt{g^2 - 1}$$

To other models

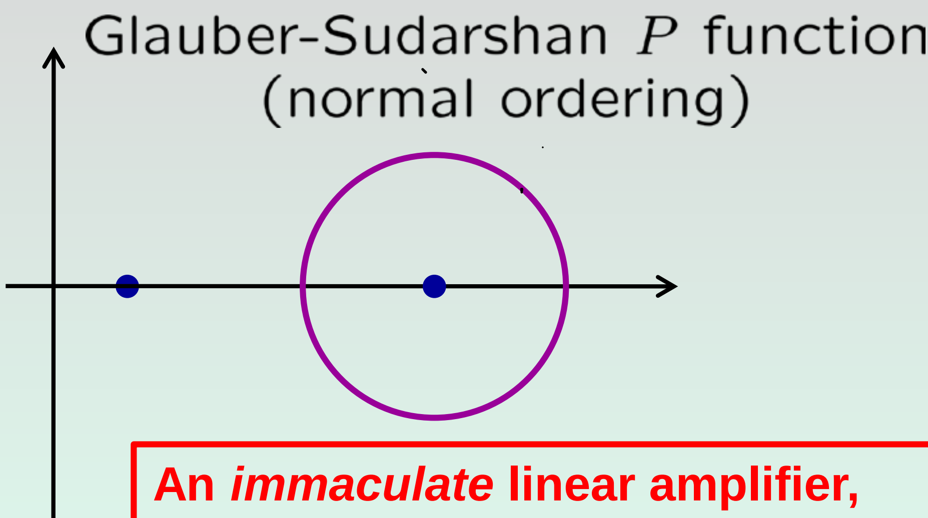


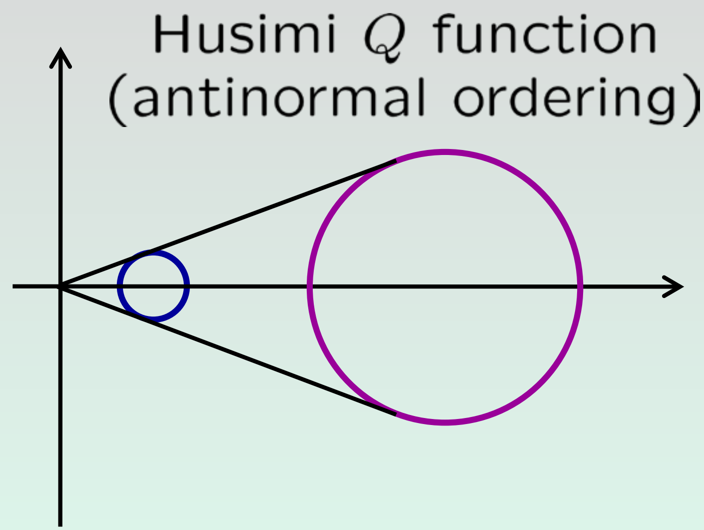
Ideal phase-preserving linear amplifier



The noise is Gaussian. Circles are drawn here at <u>half</u> the standard deviation of the Gaussian.

A *perfect* linear amplifier, which only has the (blue) amplified input noise, is not physical.





An *immaculate* linear amplifier, which only has the (blue) input noise, is not physical.

Phase-preserving linear amplifiers

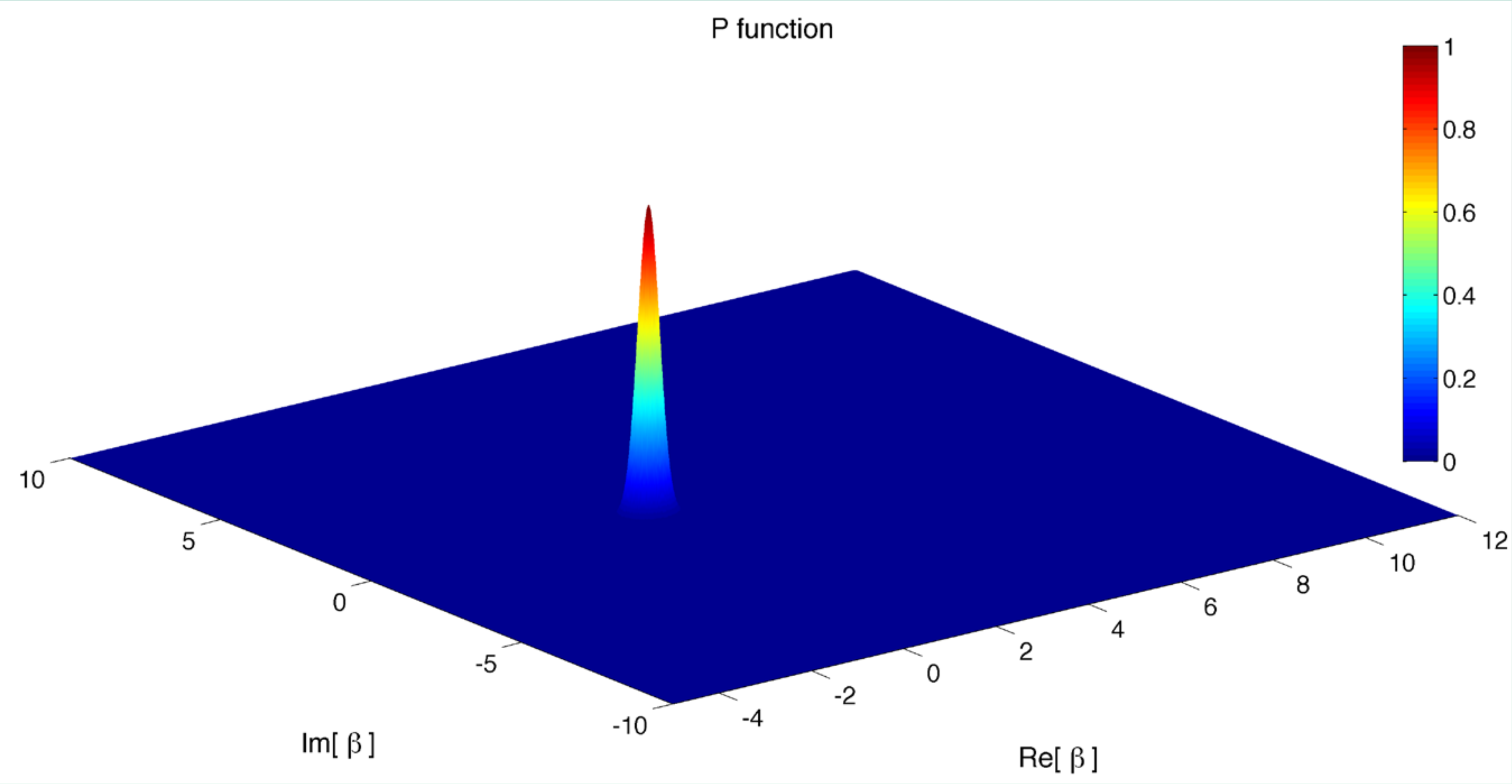
Microwave-frequency amplifiers using superconducting technology are at quantum limits and are being used as linear detectors in photon-coherence experiments. This requires more than second moments of amplifier noise.

What about nonGaussian added noise?
What about higher moments of added noise?

THE BIGGER PROBLEM
What are the quantum limits on the entire distribution of added noise?

Initial coherent state

Input coherent-state amplitude $\alpha = 1$

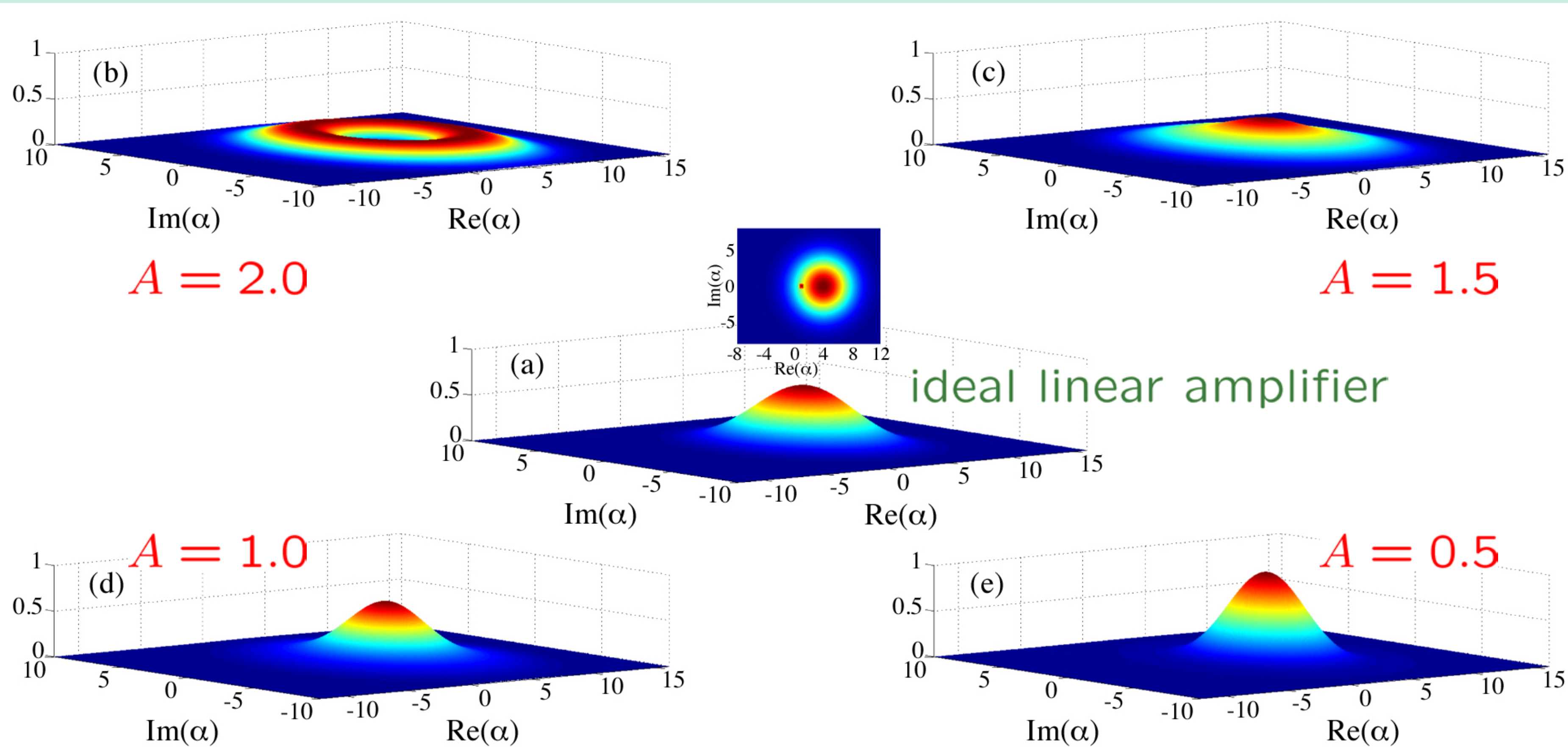


NonGaussian amplification of initial coherent state

Input coherent-state amplitude $\alpha = 1$

Amplitude gain $g = 4$

Output P functions (≥ 0): nonGaussian added noise with



Which of these are legitimate linear amplifiers?

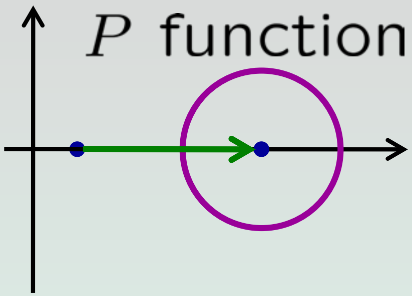
What is a phase-preserving linear amplifier?

$$\mathcal{A} \quad : \quad |\alpha\rangle \rightarrow |g\alpha\rangle$$

Immaculate amplification of input coherent state

$$\mathcal{B}(\rho) \;=\; \int d^2\beta\,\Pi(\beta)\,D(a,\beta)\,\rho\,D^\dagger(a,\beta)$$

Smearing probability distribution. Smears out the amplified coherent state and includes amplified input noise and added noise. For coherent-state input, it is the *P* function of the output.



amplifier map: $\mathcal{E} = \mathcal{B} \circ \mathcal{A}$

THE PROBLEM

What are the restrictions on the smearing probability distribution that ensure that the amplifier map is physical (completely positive)?

Attacking the problem. Tack 1

$$
\begin{aligned}
\mathcal{A} &: |\alpha\rangle \to |g\alpha\rangle \\
\mathcal{B}(\rho) &= \int d^2\beta\, \Pi(\beta) D(a,\beta)\rho D^\dagger(a,\beta) \\
\mathcal{E} &= \mathcal{B}\circ\mathcal{A}
\end{aligned}
$$

Find the "left-right" eigenstates of $\mathcal{E}$ (eigenstates of the Choi-Jamiolkowski state); determine the conditions on $\Pi(\beta)$ to ensure that all the eigenvalues are nonnegative.

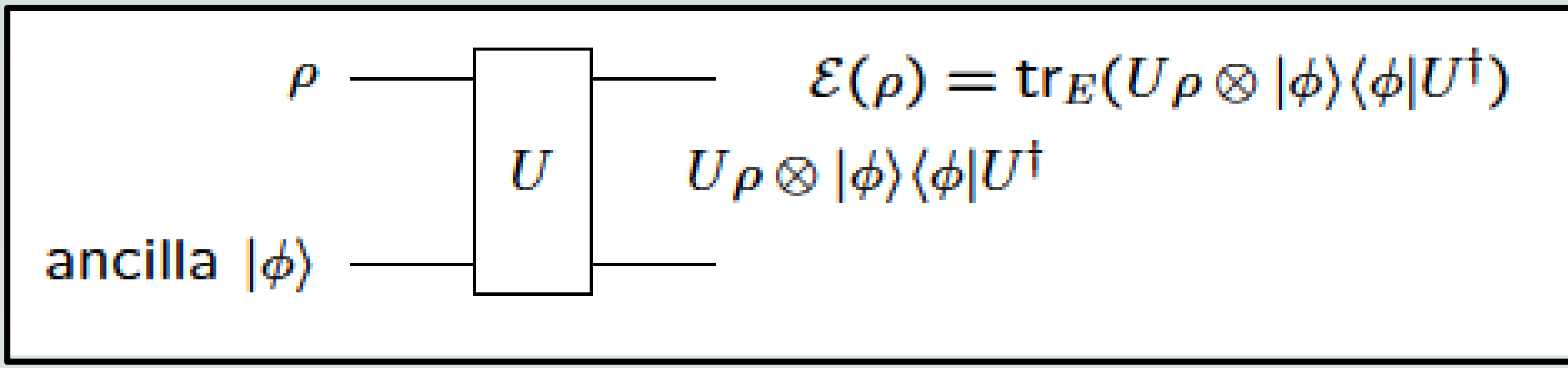
This is hopeless.

If your problem requires a determination of when a class of linear operators is positive, FIND YOURSELF A NEW PROBLEM.

Attacking the problem. Tack 2

$$\begin{aligned} \mathcal{A} &: \quad |\alpha\rangle \to |g\alpha\rangle \\ \mathcal{B}(\rho) &= \int d^2\beta\, \Pi(\beta) D(a,\beta)\rho D^\dagger(a,\beta) \\ \mathcal{E} &= \mathcal{B}\circ\mathcal{A} \end{aligned}$$

KRAUS REPRESENTATION THEOREM (Stinespring extension):
If $\mathcal{E}$ is completely positive, there exists an ancilla E with initial pure state $|\phi\rangle$ and a joint unitary U such that $\mathcal{E}(\rho) = \mathrm{tr}_E(U\rho\otimes|\phi\rangle\langle\phi|U^\dagger)$.



$$\tilde{\Pi}(\beta) = \mathrm{tr}\Big(e^{\beta a^\dagger} e^{-\beta^* a}\mathcal{E}(|0\rangle\langle 0|)\Big) = \mathrm{tr}\Big(U^\dagger e^{\beta a^\dagger} e^{-\beta^* a} U|0\rangle\langle 0|\otimes|\phi\rangle\langle\phi|\Big)$$

But we have no way to get from this to general statements about the smearing distribution, because the joint unitary and ancilla state are too general.

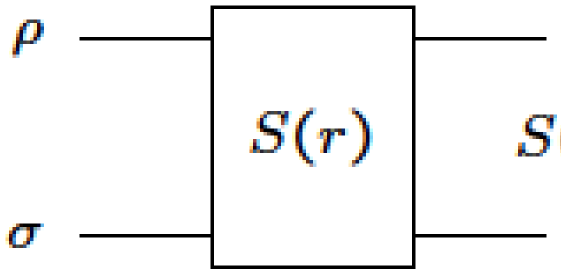
Attacking the problem. Tack 3, the right tack

$$\begin{aligned} \mathcal{A} &: |\alpha\rangle \to |g\alpha\rangle \\ \mathcal{B}(\rho) &= \int d^2\beta\, \Pi(\beta) D(a,\beta)\rho D^\dagger(a,\beta) \\ \mathcal{E} &= \mathcal{B} \circ \mathcal{A} \end{aligned}$$

Define a Hermitian, unit-trace operator σ of an ancillary mode b by

$$\Pi(\beta) = \frac{Q_\sigma\left(-\beta^*/\sqrt{g^2-1}\right)}{g^2-1} = \frac{1}{\pi}\left\langle -\frac{\beta^*}{\sqrt{g^2-1}}\right| \sigma \left| -\frac{\beta^*}{\sqrt{g^2-1}}\right\rangle$$

The smearing distribution is a rescaled Q function of the "state" σ, and

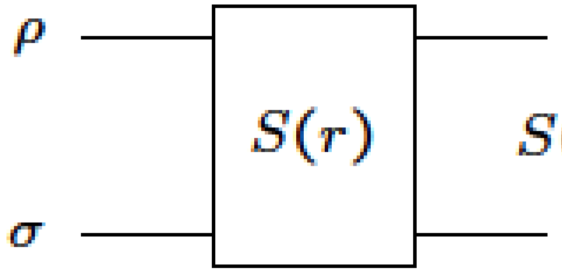


$$\mathcal{E}(\rho) = \mathrm{tr}_b(S(r)\rho \otimes \sigma S^\dagger(r))$$

$$S(r)\rho \otimes \sigma S^\dagger(r)$$

Attacking the problem. Tack 3, the right tack

$$\begin{aligned}\mathcal{A} &: |\alpha\rangle \to |g\alpha\rangle \\ \mathcal{B}(\rho) &= \int d^2\beta\, \Pi(\beta) D(a,\beta)\rho D^\dagger(a,\beta) \\ \mathcal{E} &= \mathcal{B}\circ\mathcal{A}\end{aligned}$$



$$\mathcal{E}(\rho) = \mathrm{tr}_b(S(r)\rho\otimes\sigma S^\dagger(r))$$

$$S(r)\rho\otimes\sigma S^\dagger(r)$$

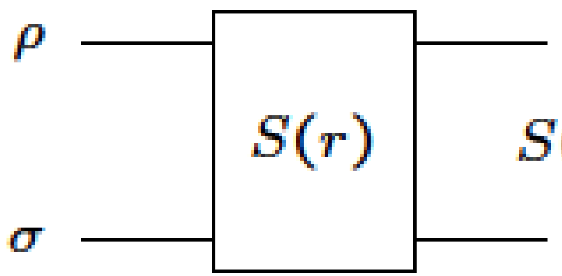
Smearing distribution $\Pi(\beta) = \dfrac{Q_\sigma\left(-\beta^*/\sqrt{g^2-1}\right)}{g^2-1}$

THE PROBLEM TRANSFORMED

Given that the amplifier map must be physical (completely positive), what are the quantum restrictions on the ancillary mode's initial "state" σ?

Attacking the problem. Tack 3, the right tack

$$\begin{aligned}\mathcal{A} &: |\alpha\rangle \to |g\alpha\rangle \\ \mathcal{B}(\rho) &= \int d^2\beta\, \Pi(\beta) D(a,\beta)\rho D^\dagger(a,\beta) \\ \mathcal{E} &= \mathcal{B}\circ\mathcal{A}\end{aligned}$$



$$\mathcal{E}(\rho) = \mathrm{tr}_b(S(r)\rho\otimes\sigma S^\dagger(r))$$

$$S(r)\rho\otimes\sigma S^\dagger(r)$$

Smearing distribution $\Pi(\beta) = \dfrac{Q_\sigma\left(-\beta^*/\sqrt{g^2-1}\right)}{g^2-1}$

THE ANSWER

Any phase-preserving linear amplifier is equivalent to a two-mode squeezing paramp with the smearing function being a rescaled *Q* function of a *physical* initial state σ of the ancillary mode.

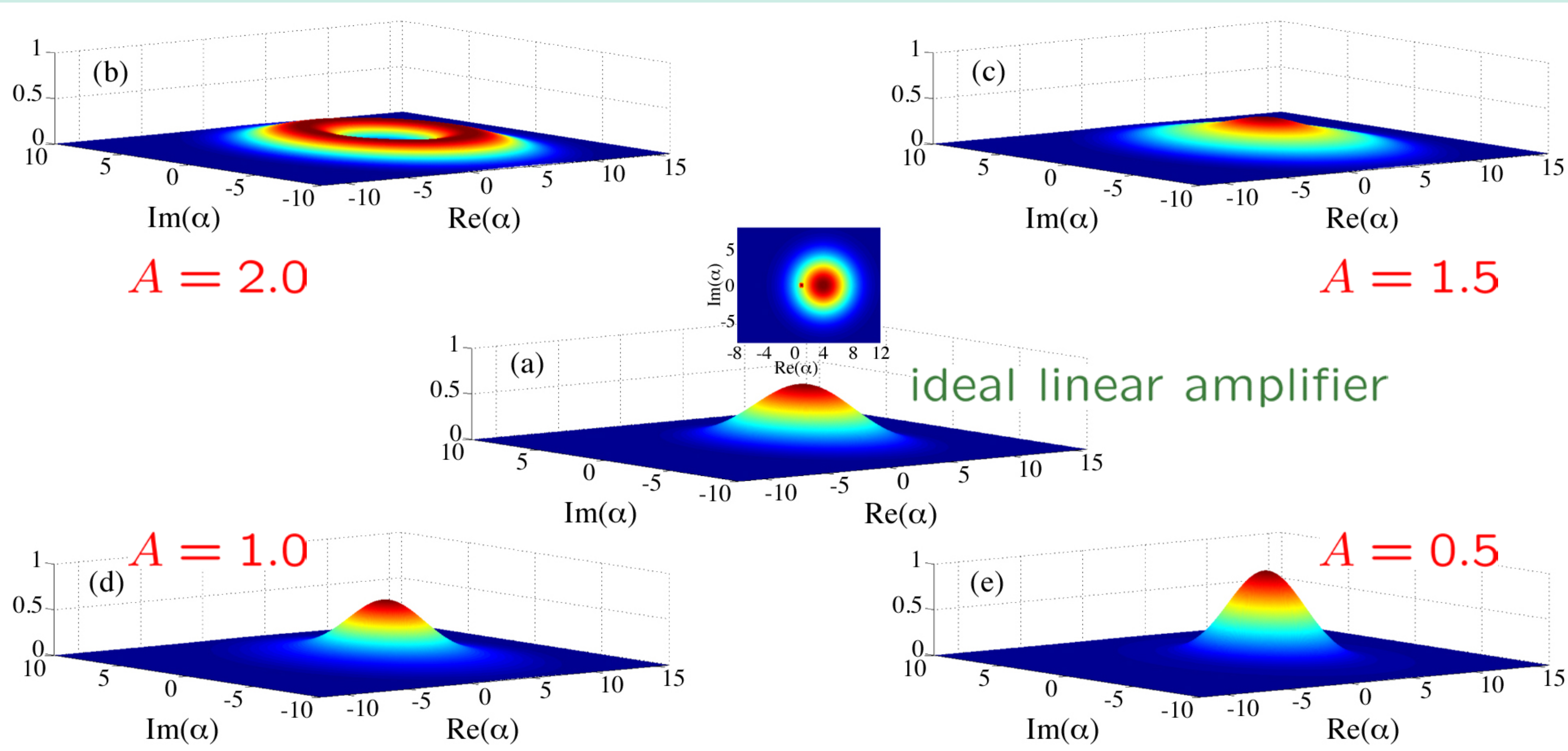
C. M. Caves, J. Combes, Z. Jiang, and S. Pandey, PRA 86, 063802 (2012).

NonGaussian amplification of initial coherent state

Input coherent-state amplitude $\alpha = 1$

Amplitude gain $g = 4$

Output P functions (≥ 0): nonGaussian added noise with



$$\sigma = (\tfrac{1}{2} - \lambda)|0\rangle\langle 0| + \lambda|1\rangle\langle 1| + \tfrac{1}{2}|2\rangle\langle 2|$$

Legit: (b) $\lambda = 0.5$; (c) $\lambda = 0$

Not Legit: (d) $\lambda = -0.5$; (e) $\lambda = -1.0$

Quantum limits on phase-preserving linear amplifiers

The problem of characterizing an amplifier's performance, in absolute terms and relative to quantum limits, becomes a species of "indirect quantum-state tomography" on the effective, but imaginary ancillary-mode state σ.

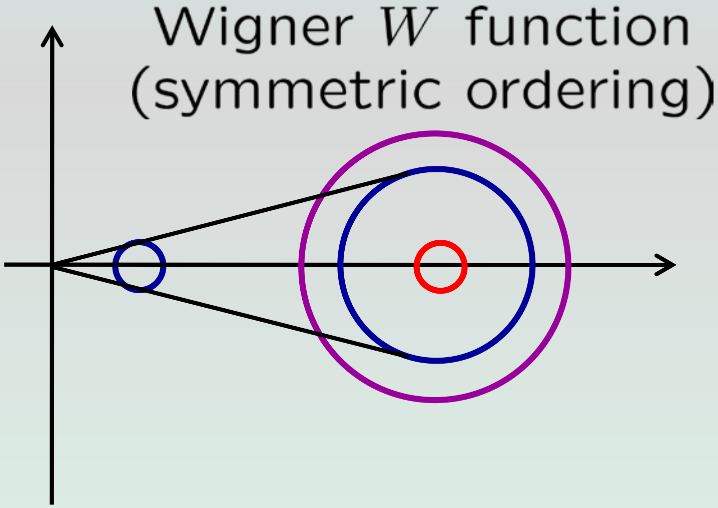
III. Nondeterministic immaculate linamps

Western diamondback rattlesnake
My front yard, Sandia Heights

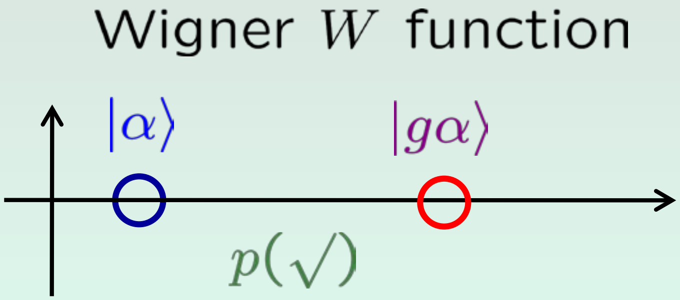
Immaculate linear amplifier

Original idea (Ralph and Lund): When presented with an input coherent state, a nondeterministic linear amplifier amplifies immaculately with probability *p* and punts with probability *1 – p.*

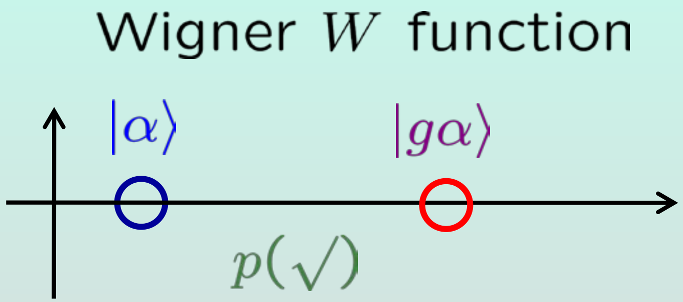
T. C. Ralph and A. P. Lund, in QCMC, edited by A. Lvovsky (AIP, 2009), p. 155.



This is an *immaculate* linear amplifier, more perfect than perfect; it doesn't even have the amplified input noise.



Immaculate linear amplifier



If the amplifier works immaculately on any input circle of coherent states, then its probability of working is zero.

Theoretical tool: Unambiguous state discrimination applied to uniformly spaced states on the circle.

Whoa! What happened to $p(\surd)g^2 \leq 1$?

Probabilistic, approximate, phase-insensitive, immaculate linear amplifier

Determining the center of a coherent state in ν trials

Quantum-limited, simultaneous measurement of x and p
OR
ideal linear amplification

$$\delta x = \delta p = \frac{1}{\sqrt{\nu}}$$

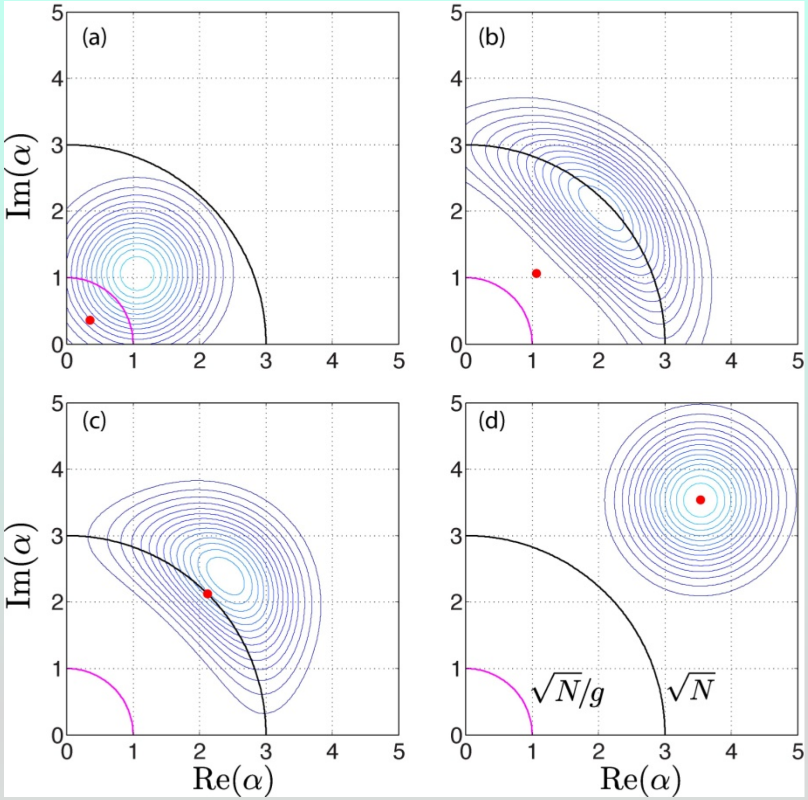
Nondeterministic amplification over a phase-space region of radius $\sqrt{N}/g$, followed by quantum-limited, simultaneous measurement of x and p

$$\delta x = \delta p = \frac{1}{g}\frac{1}{\sqrt{\nu p(\surd)}} \simeq \frac{e^{|\alpha|^2/2} g^{N-1}}{\sqrt{\nu}}$$

Working probability: $p(\surd) \simeq \dfrac{e^{-|\alpha|^2}}{g^{2N}}$

S. Pandey, Z. Jiang, J. Combes, and C. M. Caves, PRA 88, 033852 (2013).

Probabilistic, approximate, phase-insensitive, immaculate linear amplifier



High-fidelity operating region:

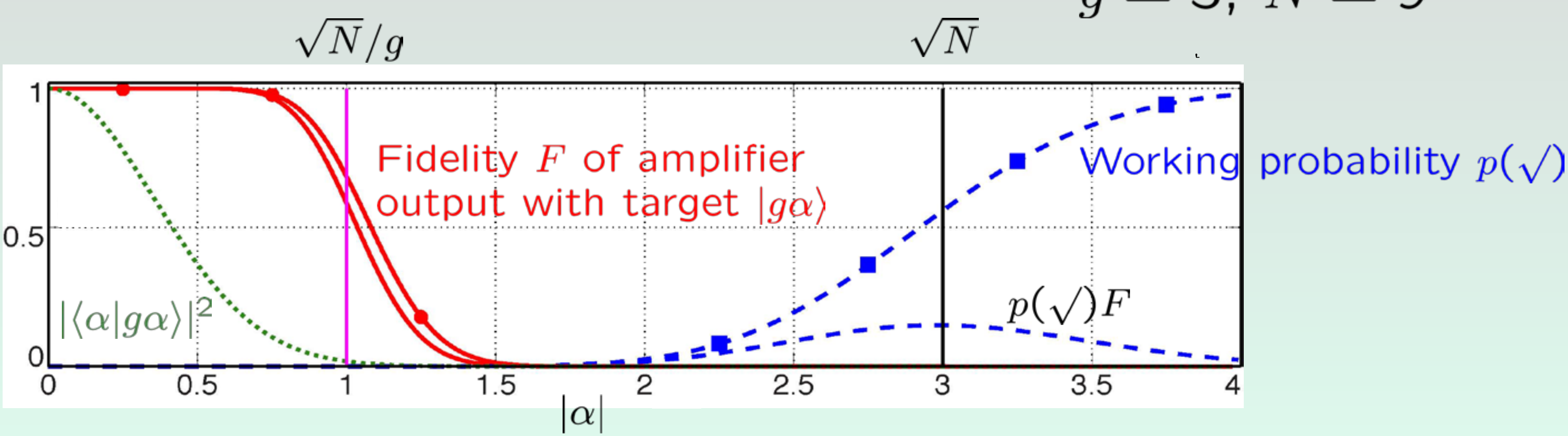
$$|\alpha| \lesssim \sqrt{N}/g$$

Fidelity: $F \simeq 1$

Working probability: $p(\surd) \simeq \dfrac{e^{-|\alpha|^2}}{g^{2N}}$

p-F product: $p(\surd)F \simeq \dfrac{e^{-|\alpha|^2}}{g^{2N}}$

$g = 3,\ N = 9$



Probabilistic, approximate, <u>phase-insensitive</u>, immaculate linear amplifier

A <u>phase-insensitive</u> immaculate amplifier doesn't do the job of linear amplification as well as an ideal linear amplifier or, indeed, even as well as doing nothing.

Interpolating between ideal and immaculate linamps by concatenating the two—such an amplifier adds less noise than an ideal linamp and works more often than an immaculate linamp, perfect linamps being an example—never works as well as an ideal linear amplifier.

J. Combes, N. Walk, A. P. Lund, T. C. Ralph, and C. M. Caves, PRA 93, 052310 (2016).

**That’s it, folks!
Thanks for your
attention.**

**Echidna Gorge
Bungle Bungle Range
Western Australia**

Ideal phase-preserving linear amplifier Models

- **Parametric amplifier with ancillary mode in vacuum**

$$H = \hbar\omega(a^\dagger a + b^\dagger b) + i\hbar\kappa(abe^{2i\omega t} - a^\dagger b^\dagger e^{-2i\omega t}) \;, \qquad g = \cosh\kappa t$$

- **Simultaneous measurement of *x* and *p* followed by creation of amplified state**

E. Arthurs and J. L. Kelly, Jr., Bell Syst. Tech. J. 44, 725 (1965).

- **Negative-mass (inverted-oscillator) ancillary mode in vacuum**

$$H = \hbar\omega(a^\dagger a - b^\dagger b) + i\hbar\kappa(ab - a^\dagger b^\dagger) \;, \qquad g = \cosh\kappa t$$

R. J. Glauber, in New Techniques and Ideas in Quantum Measurement Theory, edited by D. M. Greenberger (NY Acad Sci, 1986), p. 336.

- **Master equation**

$$\frac{d\rho}{dt} = \frac{\gamma}{2}(2a^\dagger\rho a - aa^\dagger\rho - \rho aa^\dagger) \;, \qquad g = e^{\gamma t/2}$$

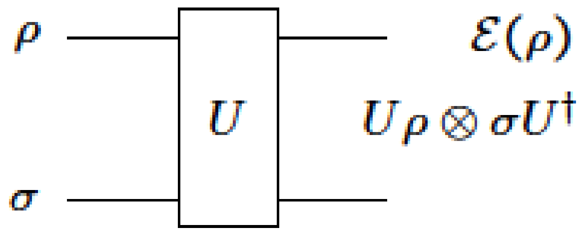
C. W. Gardiner and P. Zoller, Quantum Noise, 3rd Ed. (Springer, 2004).

Back

Completely positive maps and physical ancilla states

Tent Rocks
Kasha-Katuwe National Monument
Northern New Mexico

When does the ancilla state have to be physical?



$$\mathcal{E}(\rho) = \mathrm{tr}_B(U\rho\otimes\sigma U^\dagger)$$

THE GENERAL PROBLEM

What are the restrictions on U such that $\mathcal{E}$ being completely positive implies σ is a physical density operator?

Schmidt decomposition: $U = \sum_n A_n \otimes B_n$

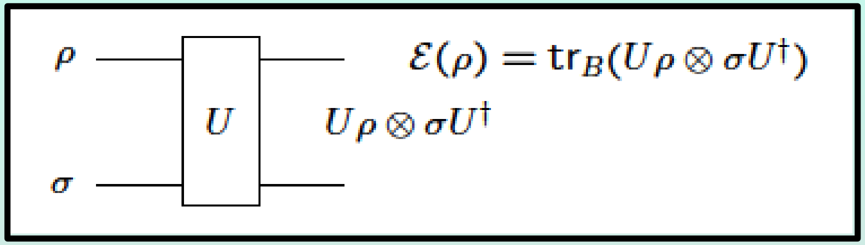
(orthogonal) Schmidt operators

$$\mathcal{O} = \{\,\text{operators } O \text{ on } B \mid \mathrm{tr}_B(U^\dagger O) = 0\,\}$$

$$\mathcal{B} = \begin{pmatrix}\text{orthocomplement}\\ \text{of } \mathcal{O}\end{pmatrix} = \begin{pmatrix}\text{span of Schmidt}\\ \text{operators } B_n\end{pmatrix}$$

$$\mathcal{C}_1 = \{\,C = B^\dagger B \mid B \in \mathcal{B},\ \mathrm{tr}_B(C) = 1\,\} = \begin{pmatrix}\text{subset of density}\\ \text{operators on } B\end{pmatrix}$$

When does the ancilla state have to be physical?



Z. Jiang, M. Piani, and C. M. Caves, Quantum Information Processing 12, 1999 (2013).

THE GENERAL PROBLEM

What are the restrictions on U such that $\mathcal{E}$ being completely positive implies σ is a physical density operator?

THE ANSWER

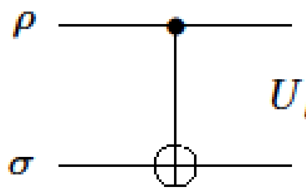
$\mathcal{C}_1$ contains all pure states. If $\mathcal{C}_1$ does not contain all pure states, any pure state not in $\mathcal{C}_1$ can be used as an eigenvector of σ with negative eigenvalue.

$$\mathcal{O} = \{\,\text{operators } O \text{ on } B \mid \mathrm{tr}_B(U^\dagger O) = 0\,\}$$

$$\mathcal{B} = \begin{pmatrix}\text{orthocomplement}\\ \text{of } \mathcal{O}\end{pmatrix} = \begin{pmatrix}\text{span of Schmidt}\\ \text{operators } B_n\end{pmatrix}$$

$$\mathcal{C}_1 = \{\,C = B^\dagger B \mid B \in \mathcal{B},\ \mathrm{tr}_B(C) = 1\,\} = \begin{pmatrix}\text{subset of density}\\ \text{operators on } B\end{pmatrix}$$

When does the ancilla state have to be physical?



$$\mathcal{E}(\rho) = \mathrm{tr}_B(U\rho \otimes \sigma U^\dagger)$$

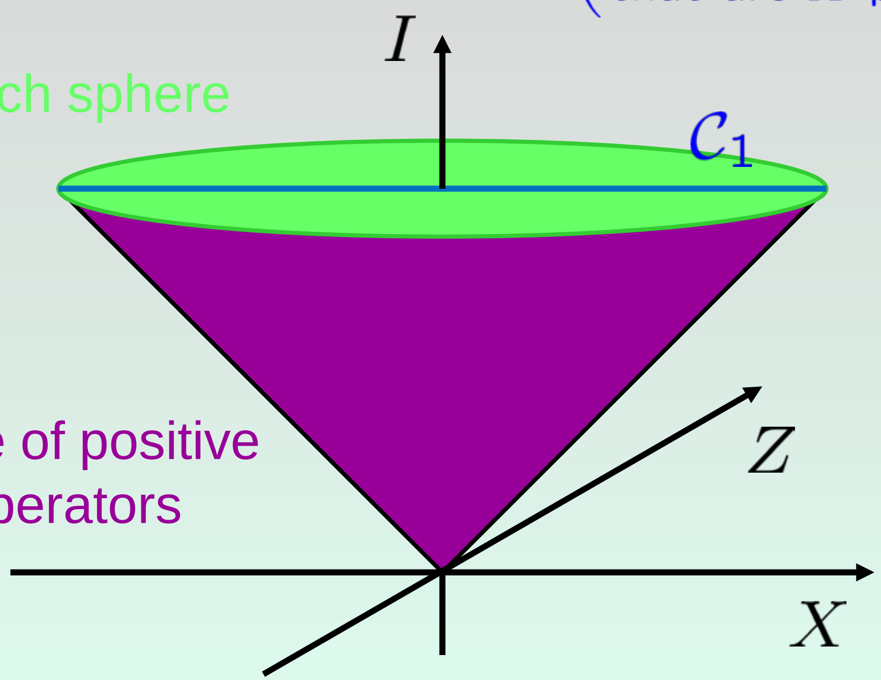
$$U\rho \otimes \sigma U^\dagger\,, \quad U = \text{C-NOT} = P_0 \otimes I + P_1 \otimes X$$

$$\mathcal{B} = (\text{span of } I \text{ and } X) = \{\,\alpha I + \beta X\,\}$$

$$\mathcal{C} = \{\,C = B^\dagger B \mid B \in \mathcal{B}\,\} = \begin{pmatrix}\text{positive operators that} \\ \text{are linear combinations} \\ \text{of } I \text{ and } X\end{pmatrix} = \{\,aI + bX \mid a \ge |b|\,\}$$

$$\mathcal{C}_1 = \{\,C \in \mathcal{C} \mid \mathrm{tr}_B(C) = 1\,\} = \begin{pmatrix}\text{density operators} \\ \text{that are } X\text{-polarized}\end{pmatrix} = \{\tfrac{1}{2}(I + SX) \mid |S| \le 1\,\}$$

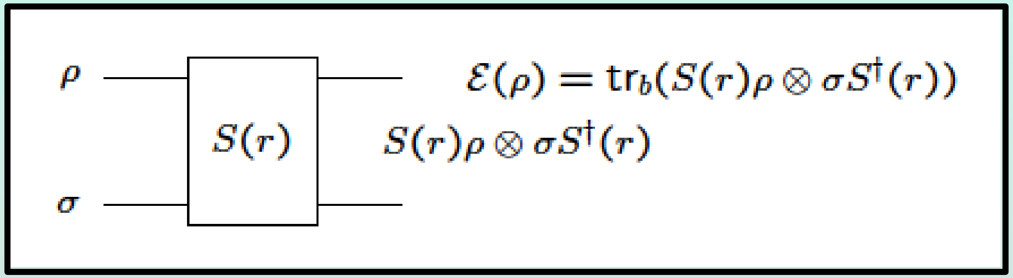
Bloch sphere



Cone of positive operators

In any eigenbasis other than that of X, σ can have a negative eigenvalue, yet give a completely positive $\mathcal{E}$.

Why does the ancilla state for a linear amplifier have to be physical?



THE PROBLEM

If $\mathcal{E}$ is completely positive, does σ have to be a physical density operator?

THE ANSWER

Yes, because the two-mode squeeze operator S is *full rank*, and thus $\mathcal{C}_1$ contains *all* pure states.

$$\mathrm{tr}_B(S^\dagger O) = 0 \quad \Longrightarrow \quad O = 0$$

$$\mathcal{O} = \{\, O \mid \mathrm{tr}_B(S^\dagger O) = 0 \,\} = \text{(the trivial subspace)}$$

$$\mathcal{B} = \begin{pmatrix}\text{orthocomplement} \\ \text{of } \mathcal{O}\end{pmatrix} = \begin{pmatrix}\text{space of all} \\ \text{operators on } B\end{pmatrix}$$

$$\mathcal{C}_1 = \{\, C = B^\dagger B \mid B \in \mathcal{B},\ \mathrm{tr}_B(C) = 1 \,\} = \begin{pmatrix}\text{space of all density} \\ \text{operators on } B\end{pmatrix}$$

Gratuitous advice

Human "reasoning" problems

1. **Making probabilistic judgments**
2. **Reducing problems to dichotomies**
3. **Confusing correlation with causation**
4. **Assuming the converse**
5. **Goal displacement**

Among these tribal idols of human nature itself, we must prominently include both our legendary difficulty in acknowledging, or even conceiving, the concept of probability, and also the motivating theme of this article: our lamentable tendency to taxonomize complex situations as dichotomies of conflicting opposites.

S. J. Gould, Science **287**, 253 (2000).

1. Inadequate risk assessment. Overweighting of anecdotal evidence. Letting ideology, religion, or prejudice trump evidence.
2. In human affairs, there are no hard boundaries—all arguments have a slippery slope—and essentially all problems are more than a bit and generally multi-parameter.

Gratuitous advice

Human "reasoning" problems

1. **Making probabilistic judgments**
2. **Reducing problems to dichotomies**
3. **Confusing correlation with causation**
4. **Assuming the converse**

Almost all discussion in the public sphere founders on some combination of 1 to 4.

As scientists, we can and should do better!

Gratuitous advice

Human “reasoning” problem

5. Goal displacement

Campbell’s Law

The more any quantitative social indicator is used for social decision-making, the more subject it will be to corruption pressures and the more apt it will be to distort and corrupt the social processes it is intended to monitor.

D. T. Campbell, Journal of MultiDisciplinary Evaluation **7**(15), 3 (2011); originally published as Paper #8, Occasional Paper Series, Public Policy Center, Dartmouth College, December 1976.

Gaming and goal displacement are inevitable with any single indicator. For scientists, the result is *high-impact-factor syndrome*, the judging of people in terms of number of publications in high-impact-factor journals.

Gratuitous advice

Human "reasoning" problems

1. **Making probabilistic judgments**
2. **Reducing problems to dichotomies**
3. **Confusing correlations with causation**
4. **Implying or assuming the converse**

Special advice to physicists (from John Wheeler)

1. **Guess, by any method, the answer to a problem before you start.**
2. **Make mistakes as fast as possible.**