Quantum metrology: dynamics vs. entanglement

I. Introduction
II. Ramsey interferometry and cat states
III. Quantum and classical resources
IV. Quantum information perspective
V. Beyond the Heisenberg limit
VI. Two-component BECs

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Quantum circuits in this presentation were set using the LaTeX package Qcircuit, developed at the University of New Mexico by Bryan Eastin and Steve Flammia. The package is available at http://info.phys.unm.edu/Qcircuit/ .

I. Introduction

Oljeto Wash
Southern Utah

Quantum information science

A new way of thinking

Computer science

Computational complexity depends on physical law.

New physics

Quantum mechanics as liberator.

What can be accomplished with quantum systems that can't be done in a classical world?

Explore what can be done with quantum systems, instead of being satisfied with what Nature hands us.

Quantum engineering

~~**Old physics**~~

~~*Quantum mechanics as nag.*~~

~~*The uncertainty principle restricts what can be done.*~~

Metrology

Taking the measure of things
The heart of physics

New physics

Quantum mechanics as liberator.

Explore what can be done with quantum systems, instead of being satisfied with what Nature hands us.

Quantum engineering

Old physics

Quantum mechanics as nag.

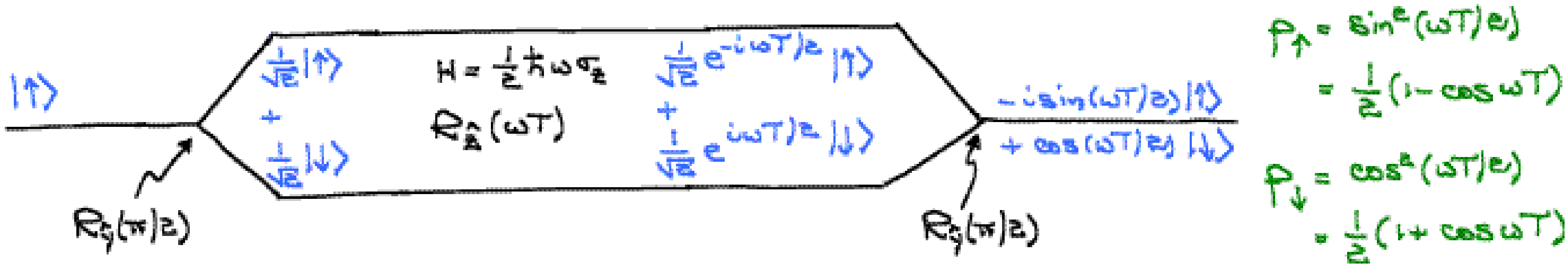
The uncertainty principle restricts what can be done.

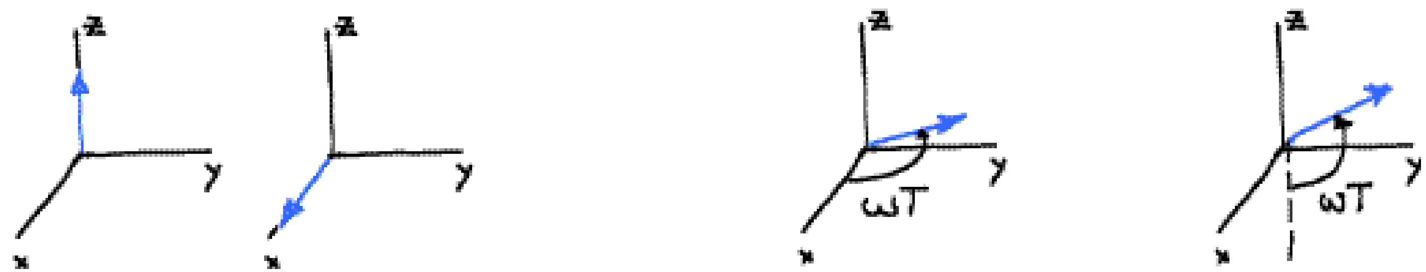
Old conflict in new guise

II. Ramsey interferometry and cat states

Herod's Gate/King David's Peak
Walls of Jerusalem NP
Tasmania

Ramsey interferometry





***N* independent "atoms"**

$$\text{(signal)} = \langle\sigma_z\rangle = -\cos\omega T$$

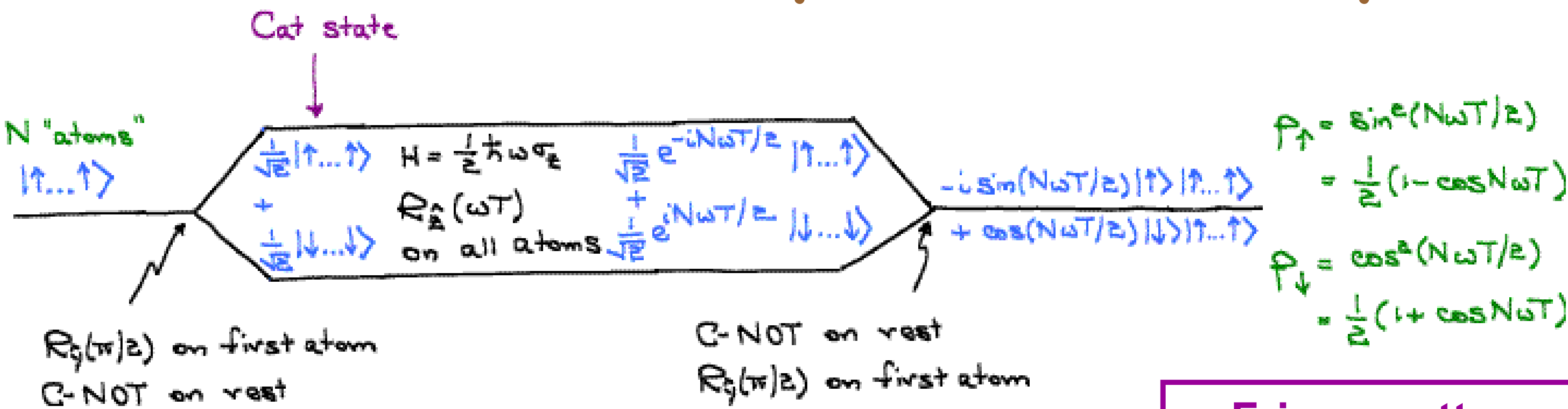
$$\text{(noise)} = \Delta\sigma_z = \sqrt{1-\cos^2\omega T} = |\sin\omega T|$$

$$\Delta(\omega T) = \frac{1}{\sqrt{N}}\frac{\text{(noise)}}{|d(\text{signal})/d(\omega T)|} = \frac{1}{\sqrt{N}}$$

Shot-noise limit

Frequency measurement
Time measurement
Clock synchronization

Cat-state Ramsey interferometry



J. J. Bollinger, W. M. Itano, D. J. Wineland, and D. J. Heinzen, Phys. Rev. A **54**, R4649 (1996).

Fringe pattern with period 2π/*N*

$$\text{(signal)} = \langle\sigma_z\rangle = -\cos N\omega T$$

$$\text{(noise)} = \Delta\sigma_z = \sqrt{1-\cos^2 N\omega T} = |\sin N\omega T|$$

$$\Delta(\omega T) = \frac{1}{\sqrt{\nu}}\frac{\text{(noise)}}{|d(\text{signal})/d(\omega T)|} = \frac{1}{\sqrt{\nu}}\frac{1}{N}$$

Heisenberg limit

$\nu = \text{(number of trials)}$

***N* cat-state atoms**

It's the entanglement, stupid.

III. Quantum and classical resources

View from Cape Hauy
Tasman Peninsula
Tasmania

Making quantum limits relevant

Optimal sensitivity: $\Delta\omega \sim \frac{1}{TN}$

The serial resource, *T*, and the parallel resource, *N*, are equivalent and interchangeable, *mathematically.*

The serial resource, *T*, and the parallel resource, *N*, are not equivalent and not interchangeable, *physically.*

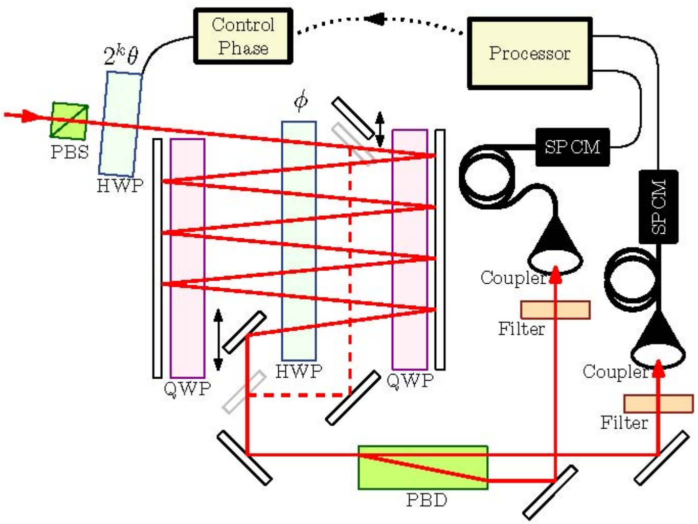
Information science perspective

Platform independence

Physics perspective

Distinctions between different physical systems

Working on T and N



B. L. Higgins, D. W. Berry, S. D. Bartlett, M. W. Mitchell, H. M. Wiseman, and G. J. Pryde, "Heisenberg-limited phase estimation without entanglement or adaptive measurements," arXiv:0809.3308 [quant-ph].

Laser Interferometer Gravitational Observatory (LIGO)

Advanced LIGO

$$\begin{pmatrix}\text{differential}\\ \text{strain}\\ \text{sensitivity}\end{pmatrix} \simeq 3\times 10^{-23}$$

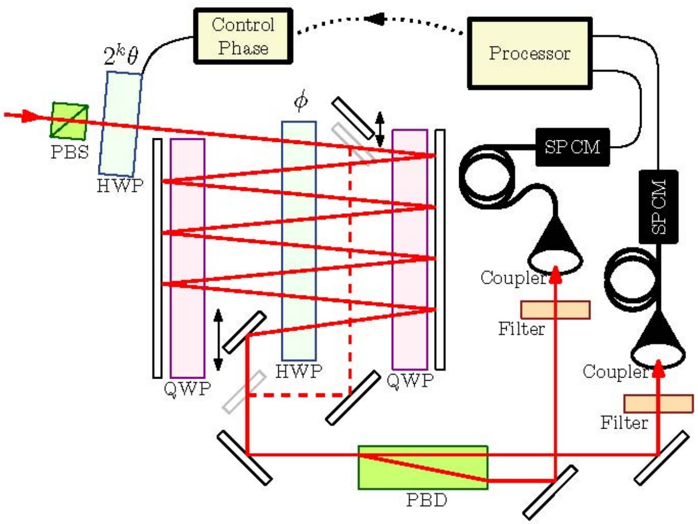
from $10\,\text{Hz}$ to $10^3\,\text{Hz}$.

High-power, Fabry-Perot cavity (multipass), recycling, squeezed-state (?) interferometers

Livingston, Louisiana

Hanford, Washington

Working on T and N



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Livingston, Louisiana

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Making quantum limits relevant. One metrology story

Resources

- Overall measurement time τ or inverse bandwidth (the "*classical* serial resource")
- Coherent interaction time T for an individual probe (the "*quantum* serial resource")
- Rate R at which systems can be deployed [$R(\tau - T) = n$ is the "*classical* parallel resource"]
- Entanglement within each probe consisting of N systems (N is the "*quantum* parallel resource")

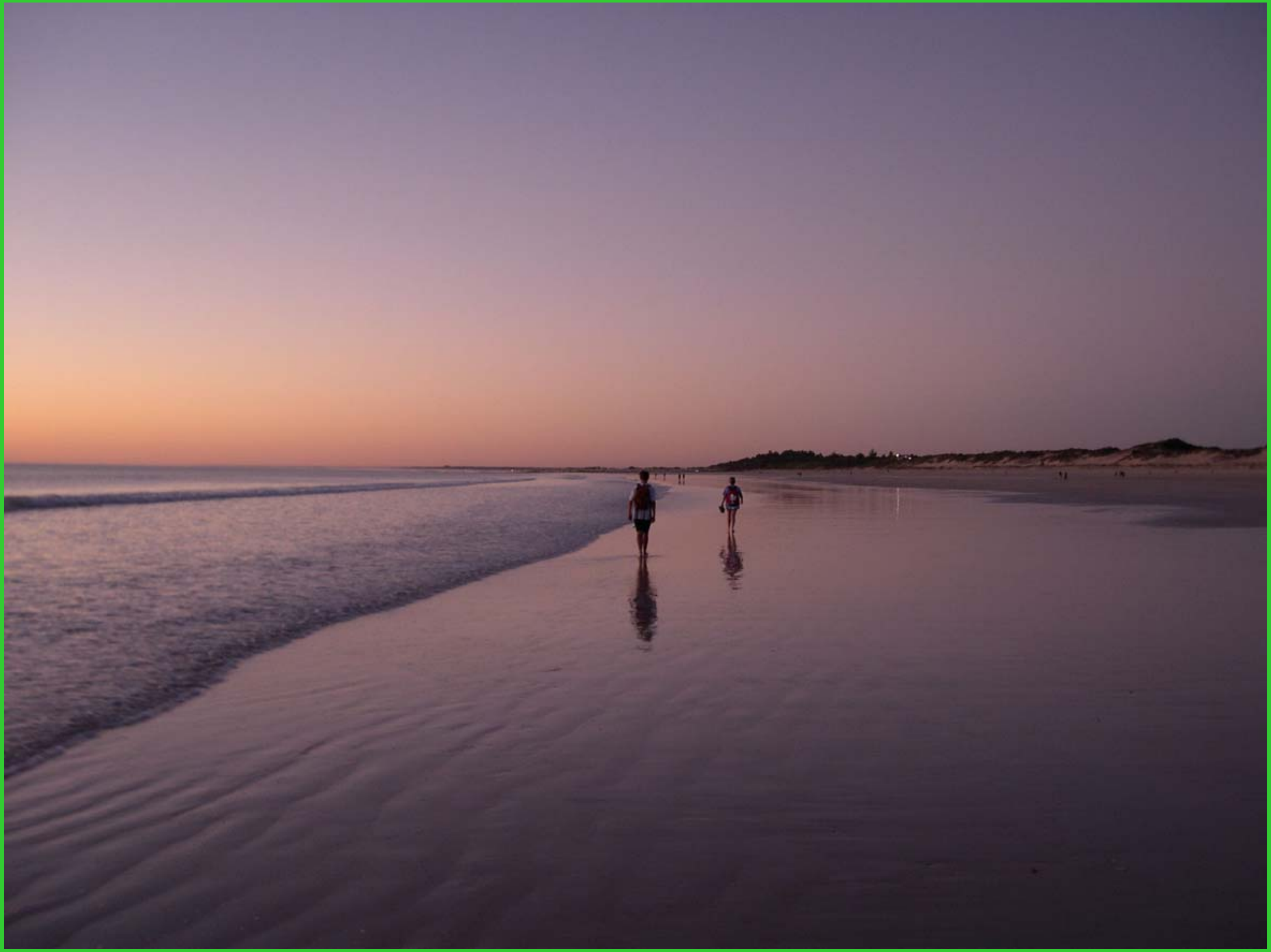
Problem

Given τ and R, a decoherence rate Γ, and a marginal "cost" c for each nonclassical photon, what is the best strategy for estimating frequency $\omega = \phi/T$?

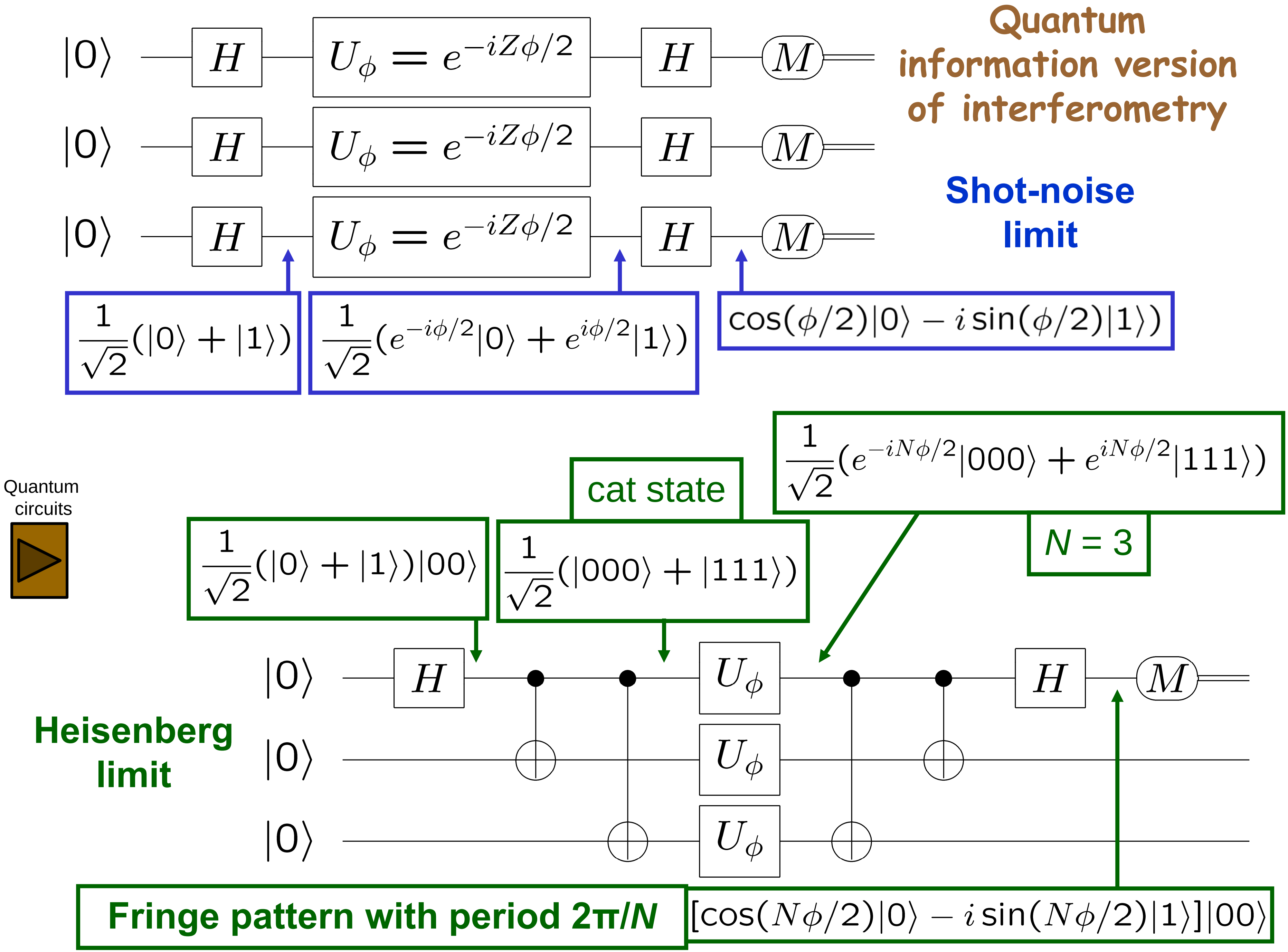
The answer has been worked out (in the case $c = 0$) for squeezed-state optical interferometry and for Ramsey interferometry with phase decoherence: The quantum resources—extended coherent evolution and entanglement—are useful only if $\Gamma\tau \lesssim 1$ and $R\tau \gg 1$. Other situations await analysis.

A. Shaji and C. M. Caves, PRA **76**, 032111 (2007).

IV. Quantum information perspective

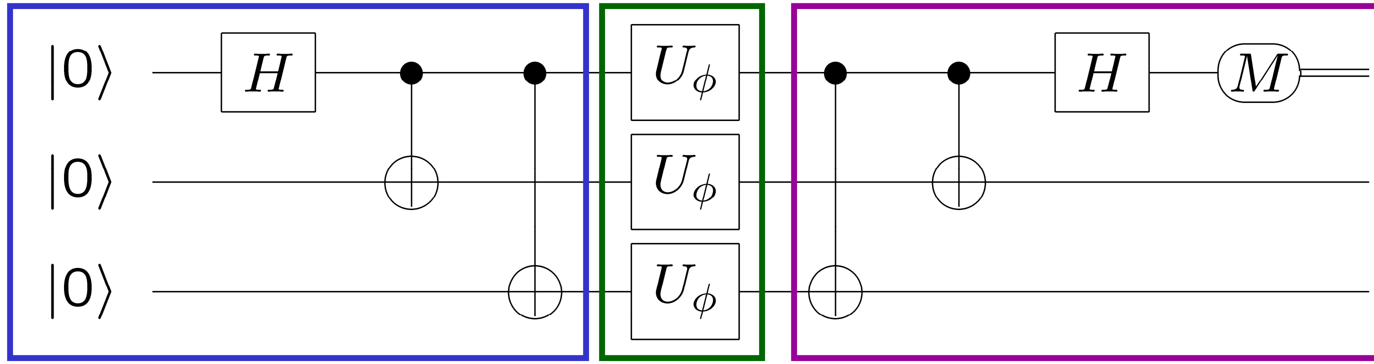


Cable Beach
Western Australia



Quantum information version of interferometry
|0⟩
H
U_φ = e^{−iZφ/2}
H
M
Shot-noise limit
1/√2 (|0⟩ + |1⟩)
1/√2 (e^{−iφ/2}|0⟩ + e^{iφ/2}|1⟩)
cos(φ/2)|0⟩ − i sin(φ/2)|1⟩
Quantum circuits
cat state
1/√2 (e^{−iNφ/2}|000⟩ + e^{iNφ/2}|111⟩)
N = 3
1/√2 (|0⟩ + |1⟩)|00⟩
1/√2 (|000⟩ + |111⟩)
Heisenberg limit
U_φ
Fringe pattern with period 2π/N
[cos(Nφ/2)|0⟩ − i sin(Nφ/2)|1⟩]|00⟩

Cat-state interferometer



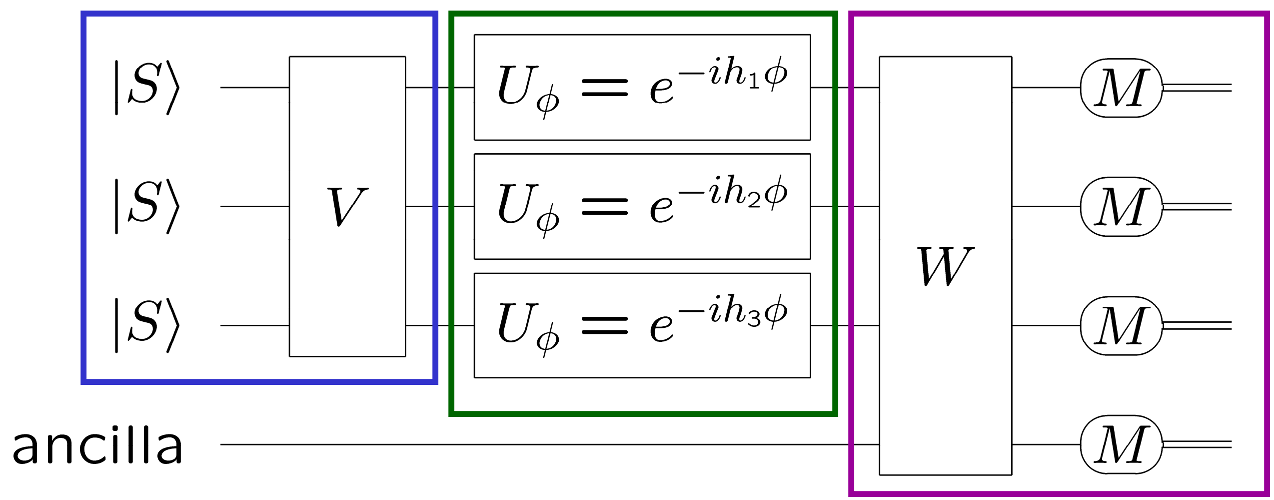
State preparation

$$U = e^{-ih\phi}$$

$$h = \sum_{j=1}^{N} h_j$$

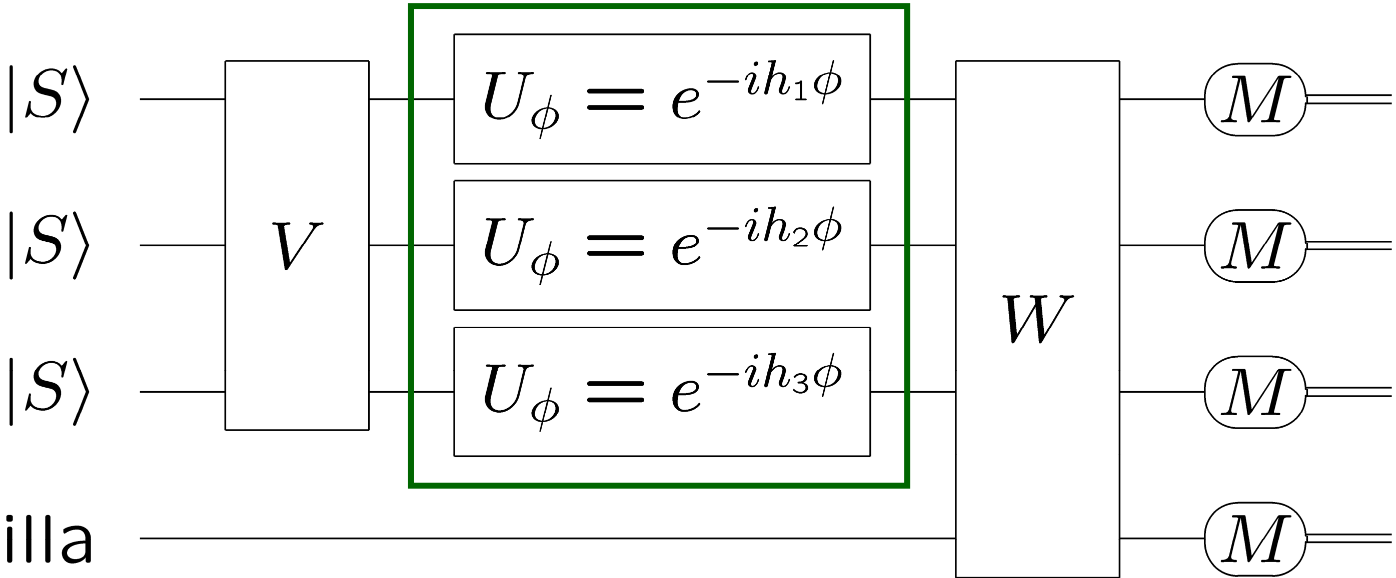
Measurement

Single-parameter estimation



Heisenberg limit

S. L. Braunstein, C. M. Caves, and G. J. Milburn, Ann. Phys. **247**, 135 (1996).
V. Giovannetti, S. Lloyd, and L. Maccone, PRL **96**, 041401 (2006).



W, U_ϕ, and ancilla measurements can be interleaved.

$$U = e^{-ih\phi}\,, \qquad h = \sum_{j=1}^{N} h_j$$

Separable inputs

$$\Delta h \le \frac{1}{2}\sqrt{N}(\Lambda - \lambda)$$

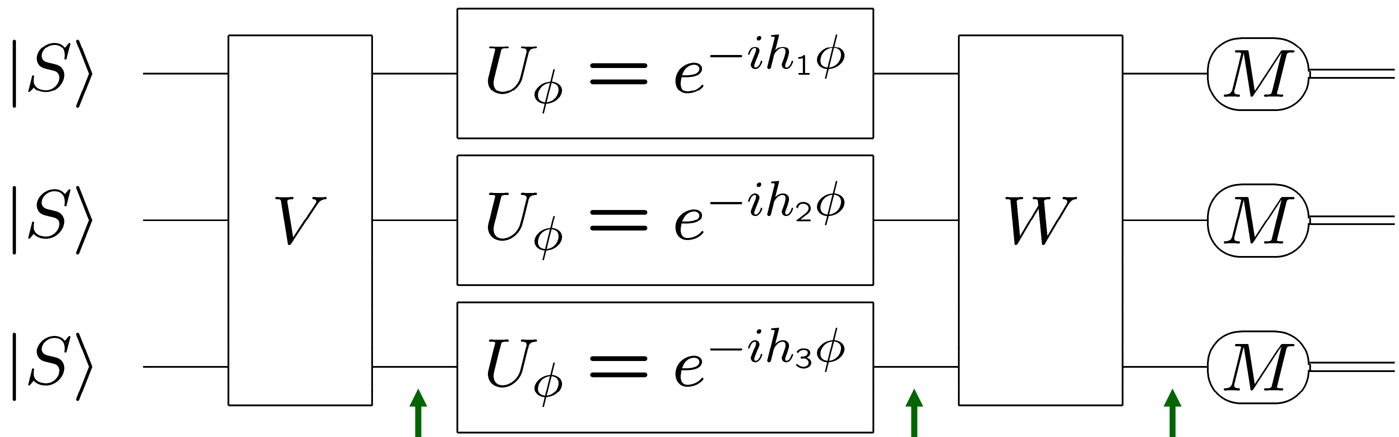
$$\Delta \phi \ge \frac{1}{\sqrt{N}(\Lambda - \lambda)}$$

$$\Delta \phi \ge \frac{1}{2\Delta h} \ge \frac{1}{N(\Lambda - \lambda)}$$

Generalized uncertainty principle (Cramér-Rao bound)

$$\Delta h \le \frac{1}{2}||h|| = \frac{1}{2}N(\Lambda - \lambda)$$

Achieving the Heisenberg limit



cat state

$$\frac{1}{\sqrt{2}}(|\Lambda,\ldots,\Lambda\rangle + |\lambda,\ldots,\lambda\rangle)$$

$$\frac{1}{\sqrt{2}}\left(e^{-iN\Lambda\phi}|\Lambda,\ldots,\Lambda\rangle + e^{-iN\lambda\phi}|\lambda,\ldots,\lambda\rangle\right)$$

$$e^{-iN(\Lambda+\lambda)\phi/2}\Big(\cos[N(\Lambda-\lambda)\phi/2]|\Lambda,\ldots,\Lambda\rangle - i\sin[N(\Lambda-\lambda)\phi/2]|\lambda,\ldots,\lambda\rangle\Big)$$

Fringe pattern with period $2\pi/N(\Lambda-\lambda)$

$$\Delta\phi = \frac{1}{N(\Lambda-\lambda)}$$

Is it entanglement?

It's the entanglement, stupid.

But what about?

- Flip half the spins in a cat state, and you get a state with the same amount of entanglement, but one that is worthless for metrology.
- There are states with far more bipartite entanglement than the cat state—up to about $N/2$ e-bits for equal bipartite splits—yet they are useless for metrology.
- Measurement sensitivity and optimal initial state depend on local Hamiltonians h_j, but entanglement measures are usually constructed to be independent of such mundane details.

We need a generalized notion of entanglement /resources that includes information about the physical situation, particularly the relevant Hamiltonian.

V. Beyond the Heisenberg limit

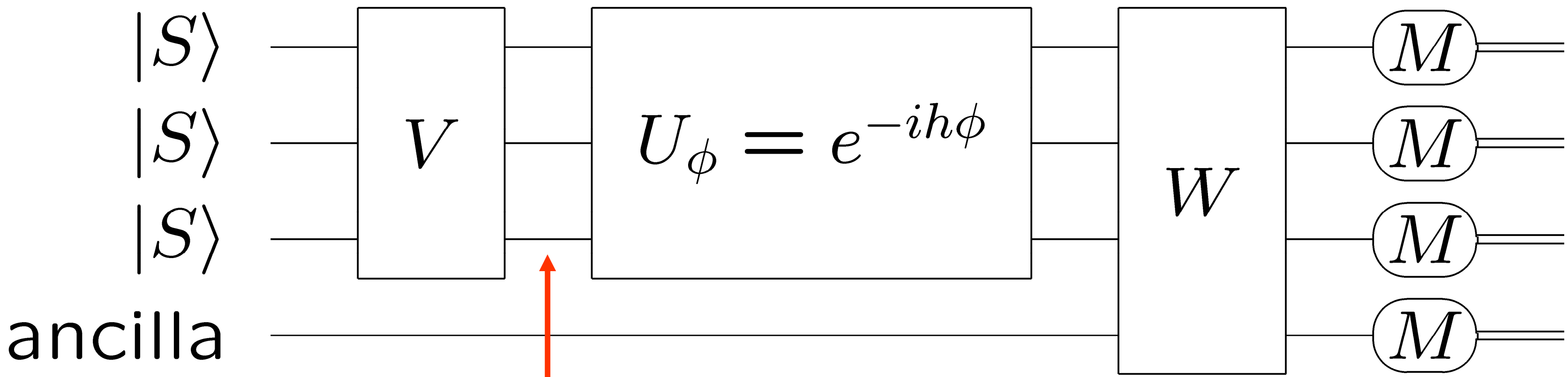
Echidna Gorge
Bungle Bungle Range
Western Australia

Beyond the Heisenberg limit

The purpose of theorems in physics is to lay out the assumptions clearly so one can discover which assumptions have to be violated.

Improving the scaling with N

S. Boixo, S. T. Flammia, C. M. Caves, and JM Geremia, PRL **98**, 090401 (2007).



Cat state does the job.

Nonlinear Ramsey interferometry

$$\Delta\phi \geq \frac{1}{2\Delta h} \geq \frac{1}{||h||} = \frac{1}{N^k(\Lambda^k - \lambda^k)}$$

Metrologically relevant *k*-body coupling

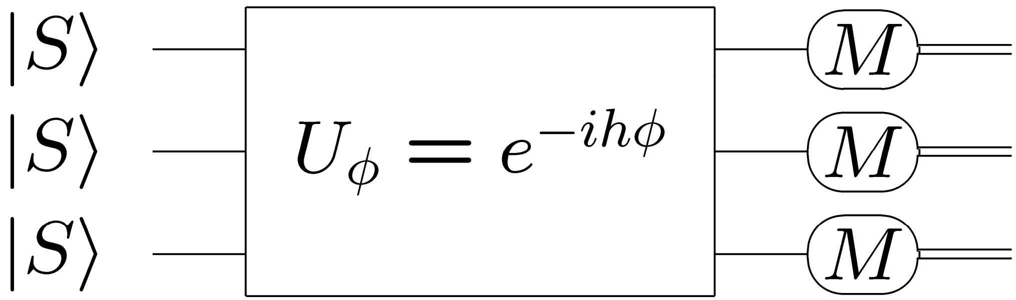
$$h = \left(\sum_{j=1}^{N} h_j\right)^k = \underbrace{\sum_{j_1,\ldots,j_k} h_{j_1} h_{j_2} \cdots h_{j_k}}_{N^k \text{ terms in sum}}$$

$$||h|| = N^k(\Lambda^k - \lambda^k)$$

Improving the scaling with N without entanglement

S. Boixo, A. Datta, S. T. Flammia, A. Shaji, E. Bagan, and C. M. Caves, PRA **77**, 012317 (2008).

Product input



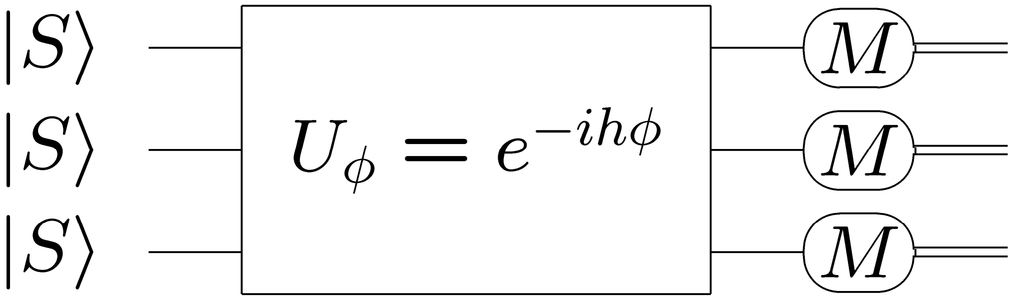
Product measurement

$$h = \left(\sum_{j=1}^{N} Z_j/2 \right)^k = J_z^k$$

$$\Delta\phi \sim \frac{1}{N^{k-1/2}}$$

Improving the scaling with N without entanglement. Two-body couplings

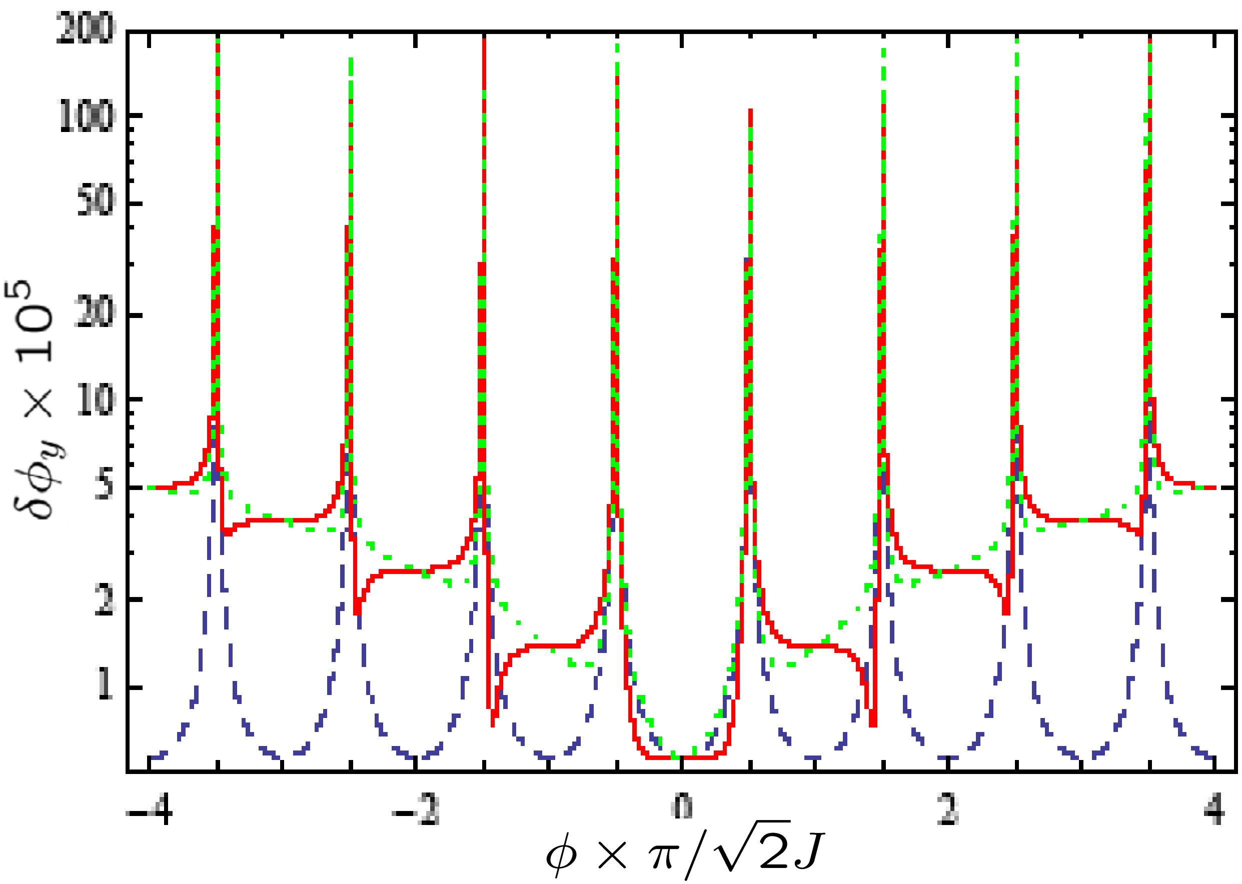
Product input



Product measurement

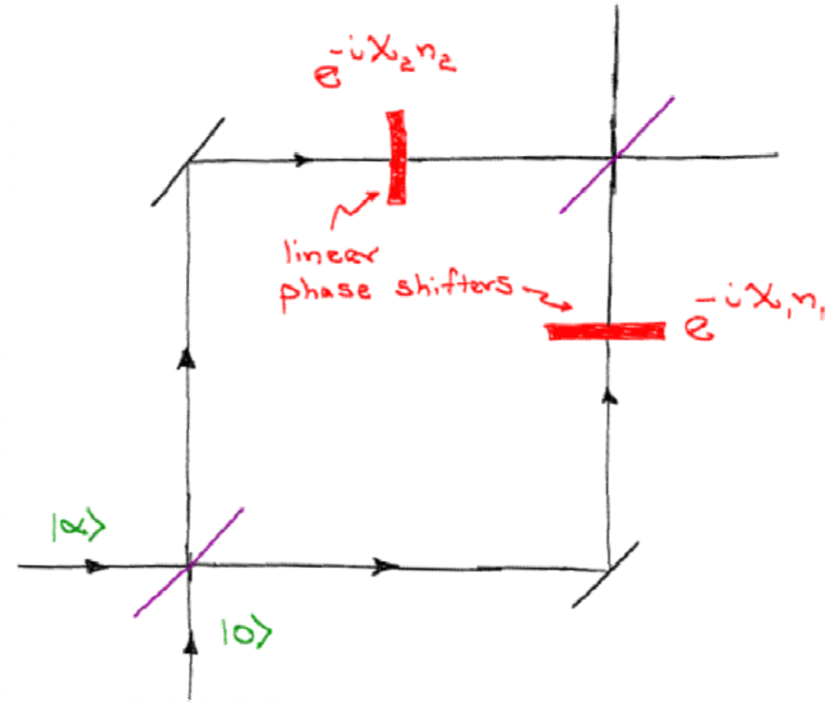
$$h = \left(\sum_{j=1}^{N} Z_j/2 \right)^2 = J_z^2$$

- Prepare system with all spins up.
- Rotate all spins by $\beta = \pi/4$ about y axis.
- For short times, $\phi \ll 1/\sqrt{J}$, nonlinear coupling rotates all spins with angular velocity $\langle J_z \rangle = 2J \cos\beta$ about z axis.
- Measure an equatorial component of $\mathbf{J}$.



Improving the scaling with N without entanglement. Two-body couplings

Linear interferometer

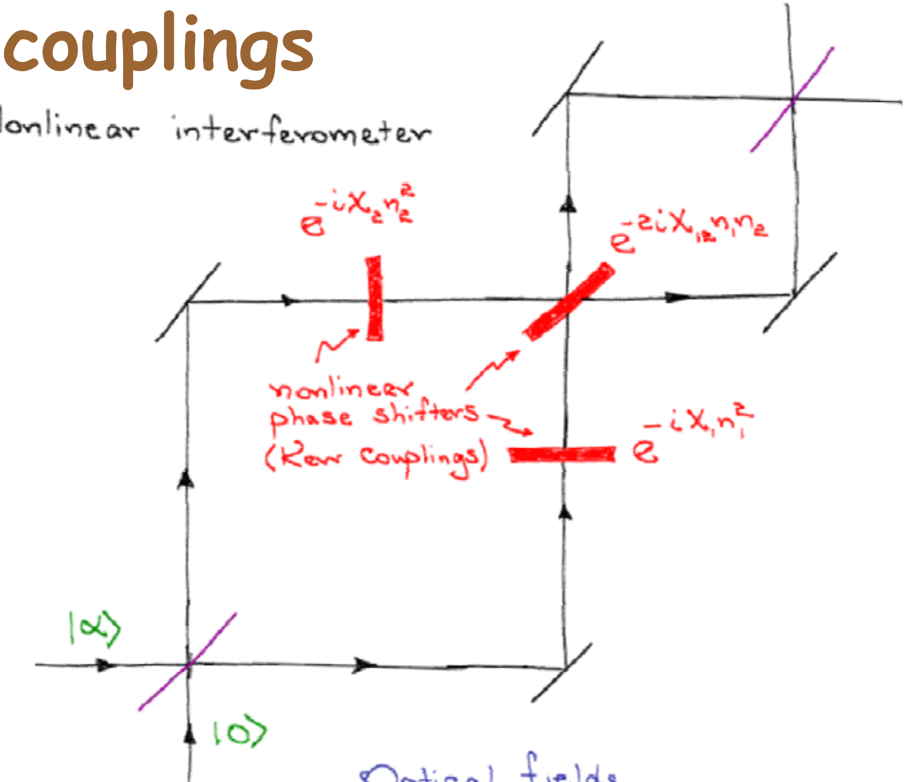


$$\chi_1 n_1 + \chi_2 n_2 = \tfrac{1}{2}(\chi_1+\chi_2)N + \underbrace{(\chi_1-\chi_2)}_{\equiv\,\phi} J_z$$

$$N = n_1 + n_2\,, \qquad J_z = \tfrac{1}{2}(n_1 - n_2)$$

$$\Delta\phi = 1/\sqrt{N}$$

Nonlinear interferometer



Optical fields

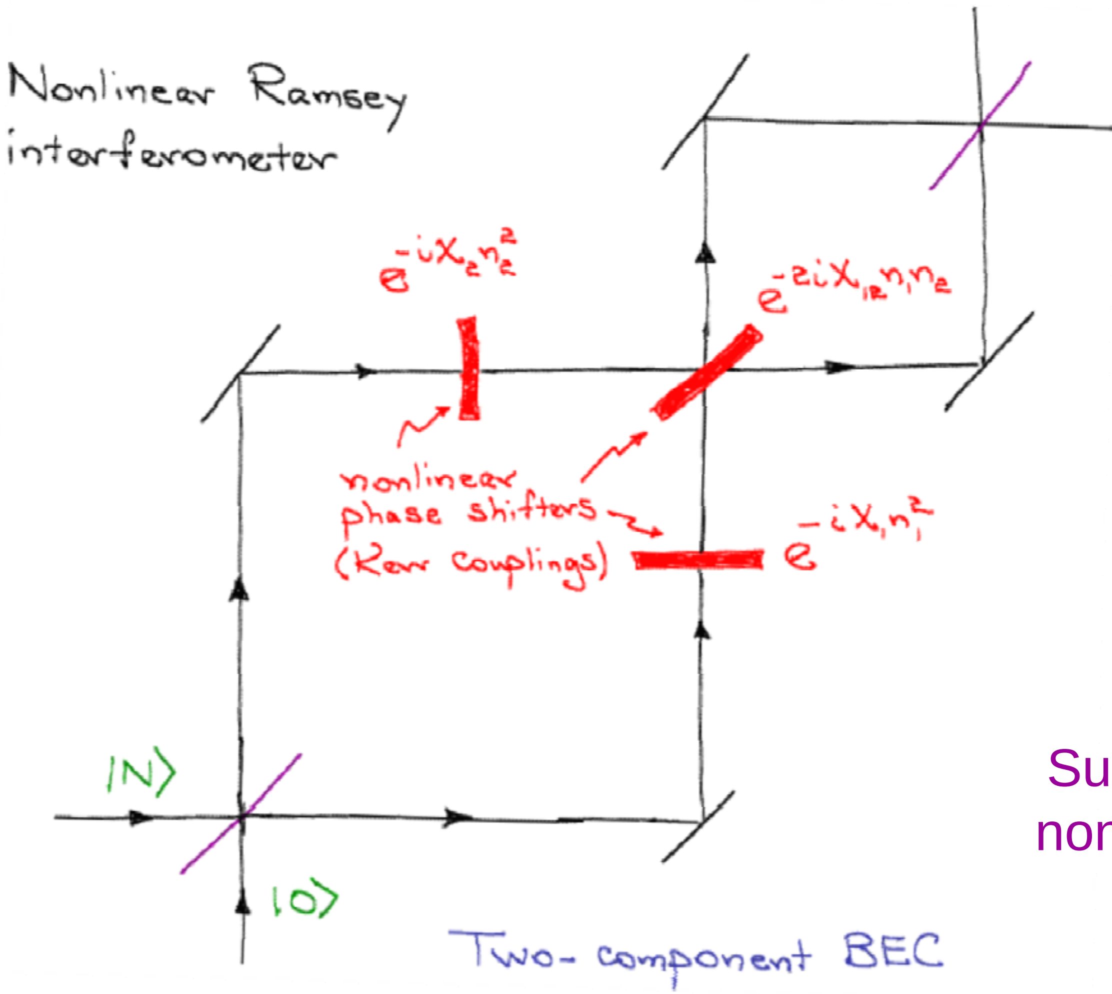
Nanomechanical resonators

$$\begin{aligned}\chi_1 n_1^2 + \chi_2 n_2^2 + 2\chi_{12} n_1 n_2 &= \tfrac{1}{4}(\chi_1+\chi_2+2\chi_{12})N^2 \\ &\quad + (\chi_1-\chi_2)NJ_z \\ &\quad + (\chi_1+\chi_2-2\chi_{12})J_z^2\end{aligned}$$

$$\Delta\phi = 1/N^{3/2}$$

S. Boixo, A. Datta, S. T. Flammia, A. Shaji, E. Bagan, and C. M. Caves, PRA **77**, 012317 (2008); M. J. Woolley, G. J. Milburn, and C. M. Caves, arXiv:0804.4540 [quant-ph].

Improving the scaling with N without entanglement. Two-body couplings



$$\chi_1 n_1^2 + \chi_2 n_2^2 + 2\chi_{12} n_1 n_2$$
$$= \tfrac{1}{4}(\chi_1 + \chi_2 + 2\chi_{12})N^2 + \underbrace{(\chi_1 - \chi_2)}_{\equiv \phi} N J_z + \underbrace{(\chi_1 + \chi_2 - 2\chi_{12})}_{=0} J_z^2$$

$$\Delta\phi = 1/N^{3/2}$$

Super-Heisenberg scaling from nonlinear dynamics, without any particle entanglement

Scaling robust against decoherence

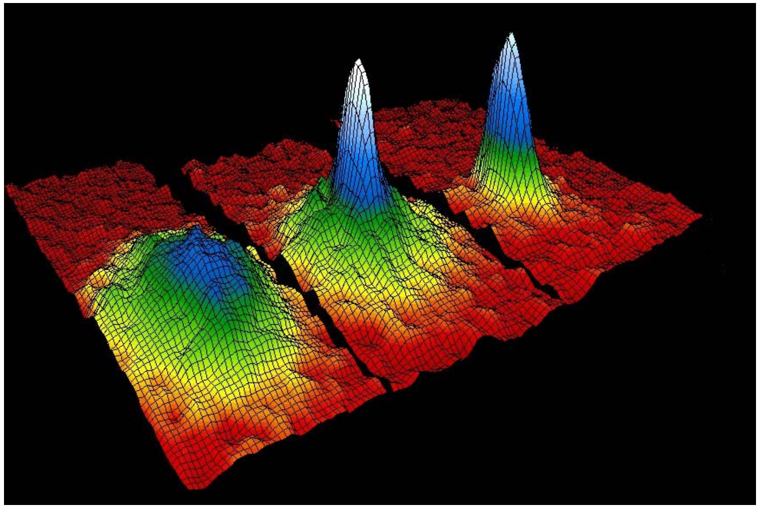
S. Boixo, A. Datta, M. J. Davis, S. T. Flammia, A. Shaji, and C. M. Caves, PRL **101**, 040403 (2008).

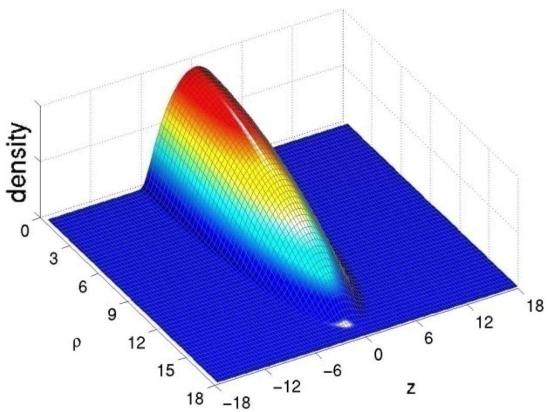
VI. Two-component BECs

Pecos Wilderness
Sangre de Cristo Range
Northern New Mexico

Two-component BECs

S. Boixo, A. Datta, M. J. Davis, S. T. Flammia, A. Shaji, and C. M. Caves, PRL 101, 040403 (2008.





Nonlinear BEC Ramsey interferometer

^{87}Rb atoms cooled to spatial ground state in hyperfine level $|F = 1; M_F = -1\rangle$. Other relevant hyperfine level is $|F = 2; M_F = 1\rangle$, which sees the same trapping potential.

- $\pi/2$ transition.
- Atoms in $|1\rangle$ see nonlinear phase shift $\frac{1}{2}(g_{11}n_1^2 + g_{12}n_1n_2)$, and atoms in $|2\rangle$ see nonlinear phase shift $\frac{1}{2}(g_{12}n_1n_2 + g_{22}n_2^2)$, where $g_{jk} = 4\pi\hbar^2 a_{jk}/m$.
- $\pi/2$ transition.
- Measure number of atoms in $|1\rangle$ and $|2\rangle$.

$$a_{11} = 100.40a_0\ , \quad a_{22} = 95.00a_0\ , \quad a_{12} = 97.66a_0$$

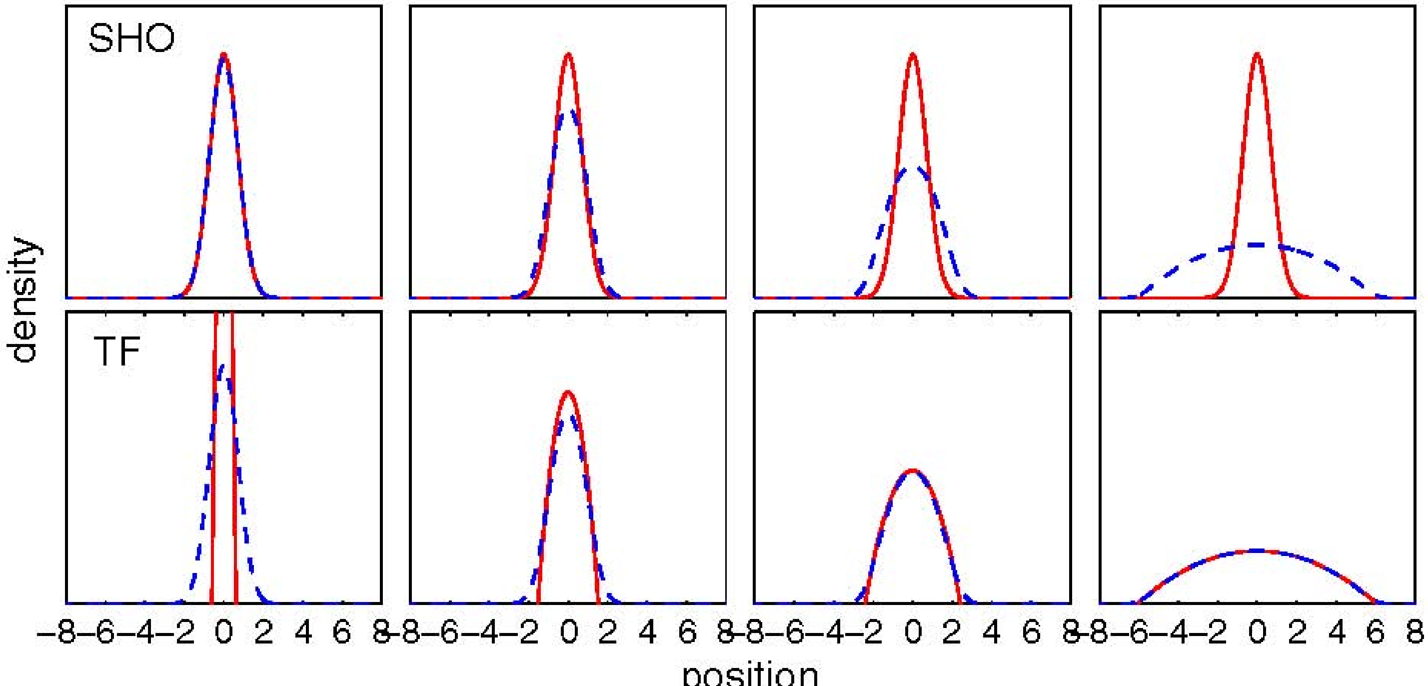
$$\tfrac{1}{2}(a_{11} - a_{22}) = 2.70a_0\ , \quad \tfrac{1}{2}(a_{11} + a_{22}) - a_{12} = 0.04a_0$$

Nearly pure NJ_z coupling to measure $\gamma = \frac{1}{2}(g_{11} - g_{22})$

Two-component BECs

Isotropic, harmonic trap with bare ground-state width r_0

$$\begin{pmatrix}\text{critical atom}\\ \text{number}\end{pmatrix} = N_c \simeq 1 + \frac{r_0}{6a}$$



$N = 10$ $N = 10^3$ $N = 10^4$ $N = 10^6$

J. E. Williams, PhD dissertation, University of Colorado, 1999.

Two-component BECs

Isotropic, harmonic trap with bare ground-state width r_0

$$\begin{pmatrix}\text{critical atom}\\ \text{number}\end{pmatrix} = N_c \simeq 1 + \frac{r_0}{6a}$$

Renormalization of scattering strength

$$\frac{r_N}{r_0} \sim \left(\frac{N-1}{N_L-1}\right)^{1/5} \qquad \frac{g}{r_N^3} \sim \frac{g}{r_0^3}\left(\frac{N_L-1}{N-1}\right)^{3/5}$$

$$\Delta\gamma \sim 1/N^{9/10}$$

Let’s start over.

Two-component BECs

Anisotropic, nonharmonic trap: d dimensions loosely confined by a power-law potential $V = \frac{1}{2}kr^q$, with bare ground-state width $r_0 \simeq (\hbar^2/mk)^{1/(q+2)}$; $D = 3 - d$ dimensions tightly confined in a harmonic potential with bare ground-state width $\rho_0 \ll r_0$.

$$\begin{pmatrix}\text{critical atom}\\ \text{number}\end{pmatrix} = N_L \simeq 1+\beta_d \frac{r_0}{a}\left(\frac{\rho_0}{r_0}\right)^D , \qquad \beta_d = \begin{cases} 1\,, & d=1,\\ \sqrt{\pi}/4\,, & d=2,\\ 1/6\,, & d=3.\end{cases}$$

Renormalization of scattering strength

$$\frac{r_N}{r_0} \sim \left(\frac{N-1}{N_L-1}\right)^{1/(d+q)} \qquad \frac{g}{\rho_0^D r_N^d} \sim \frac{g}{\rho_0^D r_0^d}\left(\frac{N_L-1}{N-1}\right)^{d/(d+q)}$$

$$\Delta\gamma \sim 1/N^{(d+3q)/2(d+q)}$$

Integrated vs. position-dependent phase

$$\frac{\tau_{\text{pd}}}{\tau_{\text{int}}} = \sqrt{\frac{2(d+3q)}{d}}$$

Two-component BECs for quantum metrology

? Perhaps ?
With hard, low-dimensional trap

Losses ?
Counting errors ?

Experiment in
H. Rubinsztein-Dunlop's group at University of Queensland

Measuring a metrologically relevant parameter ?

S. Boixo, A. Datta, M. J. Davis, A. Shaji, A. B. Tacla, and C. M. Caves, "Quantum-limited metrology and Bose-Einstein condensates," PRA **80**, 032103 (2009).

San Juan River canyons
Southern Utah

One metrology story

Range of $\Gamma\tau$	T	N	ν	n	$\Delta\omega$
$\frac{\nu_{\min}\Gamma}{R} \le \Gamma\tau < \frac{2\nu_{\min}\Gamma}{R}$	T_s	1	$\nu_{\min}$	$\nu_{\min}$	$\frac{e^{\Gamma T_s}}{T_s\sqrt{\nu_{\min}}}$
$\frac{2\nu_{\min}\Gamma}{R} \le \Gamma\tau < \sqrt{\frac{2\nu_{\min}\Gamma}{R}}$	$\frac{\tau}{2}$	$\frac{R\tau}{2\nu_{\min}}$	$\nu_{\min}$	$\frac{R\tau}{2}$	$\frac{4\sqrt{\nu_{\min}}}{R\tau^2}e^{\Gamma R\tau^2/4\nu_{\min}}$
$\sqrt{\frac{2\nu_{\min}\Gamma}{R}} \le \Gamma\tau < 1$	$\frac{\tau}{2}$	$\frac{1}{\Gamma\tau}$	$\frac{\Gamma R\tau^2}{2}$	$\frac{R\tau}{2}$	$\frac{2\sqrt{2e}}{\tau\sqrt{R/\Gamma}}$
$\Gamma\tau \ge 1$	T_p	1	$R(\tau - T_p)$	$R(\tau - T_p)$	$\frac{e^{\Gamma T_p}}{T_p\sqrt{R(\tau - T_p)}}$

$$T_s = \tau - \nu_{\min}/R$$

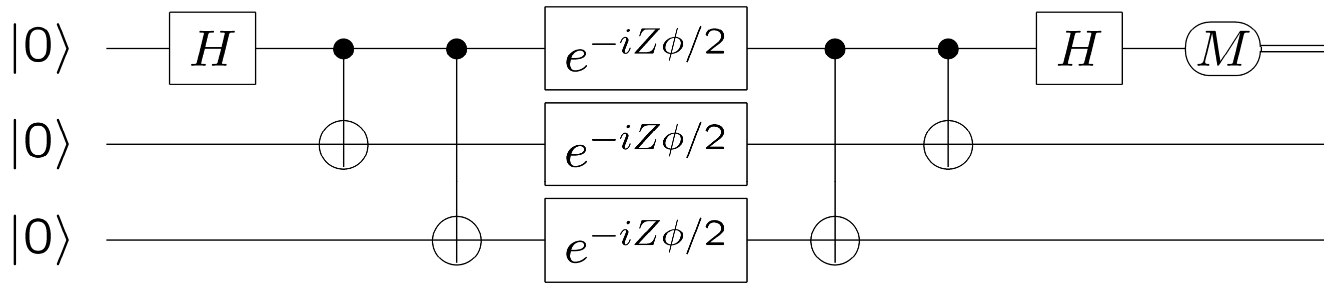
$$T_p = \frac{3/2 + \Gamma\tau - \sqrt{(3/2 + \Gamma\tau)^2 - 4\Gamma\tau}}{2\Gamma} \longrightarrow 1/\Gamma \text{ when } \Gamma\tau \gg 1$$

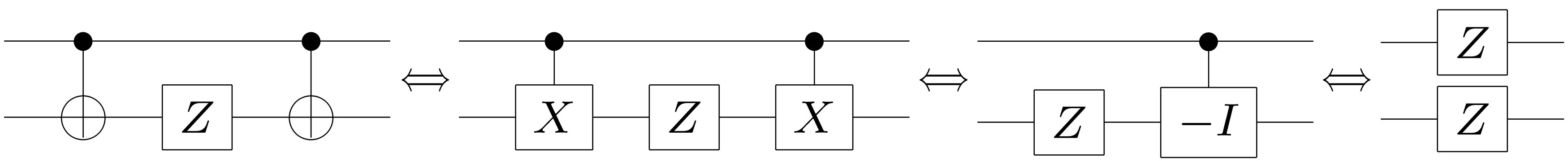
One metrology story

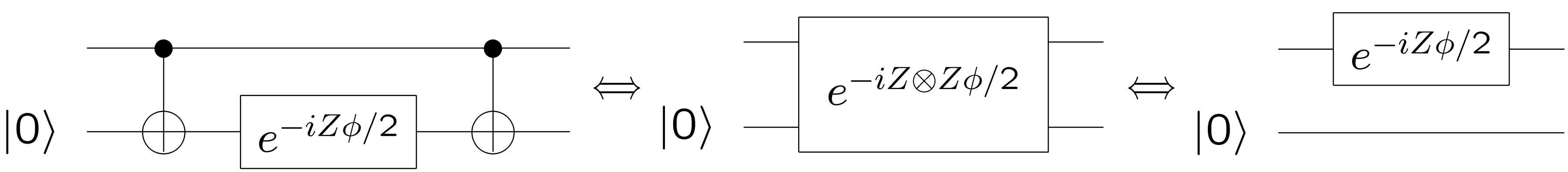
	Range of $R\tau/\nu_{\text{min}}$	T	N	ν	n	$\Delta\omega$
Qubit starvation	$1 \leq \frac{R\tau}{\nu_{\text{min}}} < 2$	$\sim \tau$	1	ν_{min}	ν_{min}	$\sim \frac{1}{\tau}$
Cat-state regime	$2 \leq \frac{R\tau}{\nu_{\text{min}}} < \sqrt{\frac{2R}{\nu_{\text{min}}\Gamma}}$	$\frac{\tau}{2}$	$\frac{R\tau}{2\nu_{\text{min}}}$	ν_{min}	$\frac{R\tau}{2}$	$\sim \frac{1}{\tau^2}$
Cat-state transition	$\sqrt{\frac{2R}{\nu_{\text{min}}\Gamma}} \leq \frac{R\tau}{\nu_{\text{min}}} < \frac{R}{\nu_{\text{min}}\Gamma}$	$\frac{\tau}{2}$	$\frac{1}{\Gamma\tau}$	$\frac{\Gamma R\tau^2}{2}$	$\frac{R\tau}{2}$	$\sim \frac{1}{\tau}$
Decoherence dominance	$\frac{R\tau}{\nu_{\text{min}}} \geq \frac{R}{\nu_{\text{min}}\Gamma}$	$\sim \frac{1}{\Gamma}$	1	$\sim R\tau$	$\sim R\tau$	$\sim \frac{1}{\sqrt{\tau}}$

Using quantum circuit diagrams

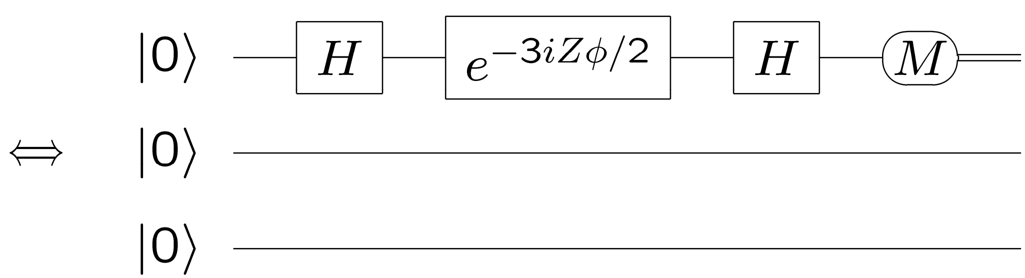
Cat-state interferometer







Cat-state interferometer



C. M. Caves and A. Shaji, "Quantum-circuit guide to optical and atomic interferometry," Opt. Comm., to be published, arXiv:0909.0803 [quant-ph].