Quantum metrology: An information-theoretic perspective

in ~~three~~ two lectures

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Quantum metrology: An information-theoretic perspective

Lecture 1

I. Introduction. What's the problem?
II. Squeezed states and optical interferometry
III. Ramsey interferometry, cat states, and spin squeezing

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I. Introduction. What's the problem?

View from Cape Hauy
Tasman Peninsula
Tasmania

Quantum information science

A new way of thinking

Computer science

Computational complexity depends on physical law.

New physics

Quantum mechanics as liberator.

What can be accomplished with quantum systems that can't be done in a classical world?

Explore what can be done with quantum systems, instead of being satisfied with what Nature hands us.

Quantum engineering

~~**Old physics**~~

~~*Quantum mechanics as nag.*~~

~~*The uncertainty principle restricts what can be done.*~~

Metrology

Taking the measure of things

The heart of physics

New physics

Quantum mechanics as liberator.

Explore what can be done with quantum systems, instead of being satisfied with what Nature hands us.

Quantum engineering

Old physics

Quantum mechanics as nag.

The uncertainty principle restricts what can be done.

Old conflict in new guise

Measuring a classical parameter

Phase shift in an (optical) interferometer

- Readout of anything that changes optical path lengths
- Michelson-Morley experiment
- Gravitational-wave detection
- Planck-scale, holographic uncertainties in positions

Torque on or free precession of a collection of spins

- Magnetometer
- Atomic clock

Lectures 1 and 2

Force on a linear system

- Gravitational-wave detection
- Accelerometer
- Gravity gradiometer
- Electrometer
- Strain meter

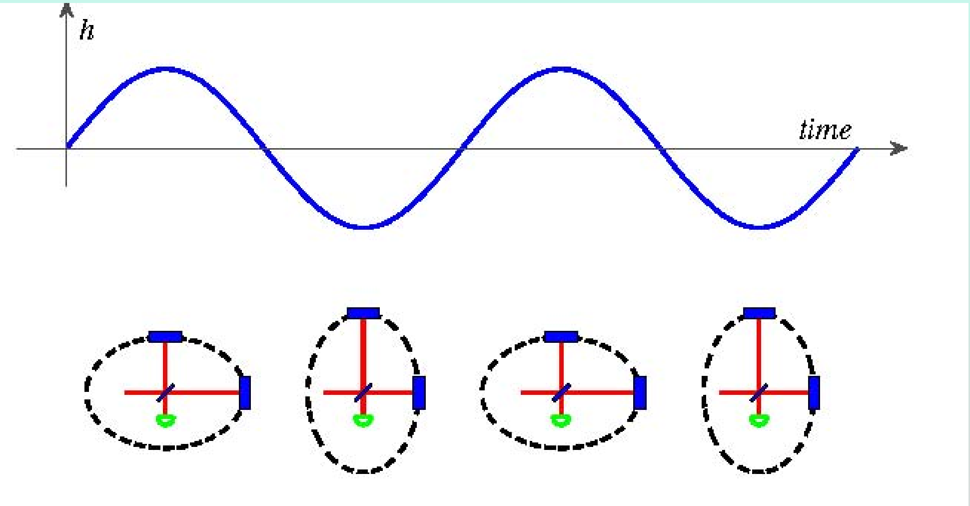
~~Lecture 3~~

II. Squeezed states and optical interferometry

Oljeto Wash
Southern Utah

(Absurdly) high-precision interferometry

Hanford, Washington



The LIGO Collaboration, Rep. Prog. Phys. 72, 076901 (2009).

Laser Interferometer Gravitational Observatory (LIGO)

Livingston, Louisiana

(Absurdly) high-precision interferometry

Initial LIGO

Hanford, Washington

$$\begin{pmatrix}\text{differential}\\ \text{strain}\\ \text{sensitivity}\end{pmatrix} \simeq 10^{-21}$$

$$\begin{pmatrix}\text{differential}\\ \text{displacement}\\ \text{sensitivity}\end{pmatrix} \simeq 4 \times 10^{-18}\,\text{m}$$

from 40 Hz to 7,000 Hz.

Laser Interferometer Gravitational Observatory (LIGO)

High-power, Fabry-Perot-cavity (multipass), power-recycled interferometers

Livingston, Louisiana

(Absurdly) high-precision interferometry

Advanced LIGO

Hanford, Washington

$$\begin{pmatrix}\text{differential}\\ \text{strain}\\ \text{sensitivity}\end{pmatrix} \simeq 3 \times 10^{-23}$$

Currently a factor of 3 short of this design goal.

$$\begin{pmatrix}\text{differential}\\ \text{displacement}\\ \text{sensitivity}\end{pmatrix} \simeq 10^{-19}\,\text{m}$$

from 10 Hz to 7,000 Hz.

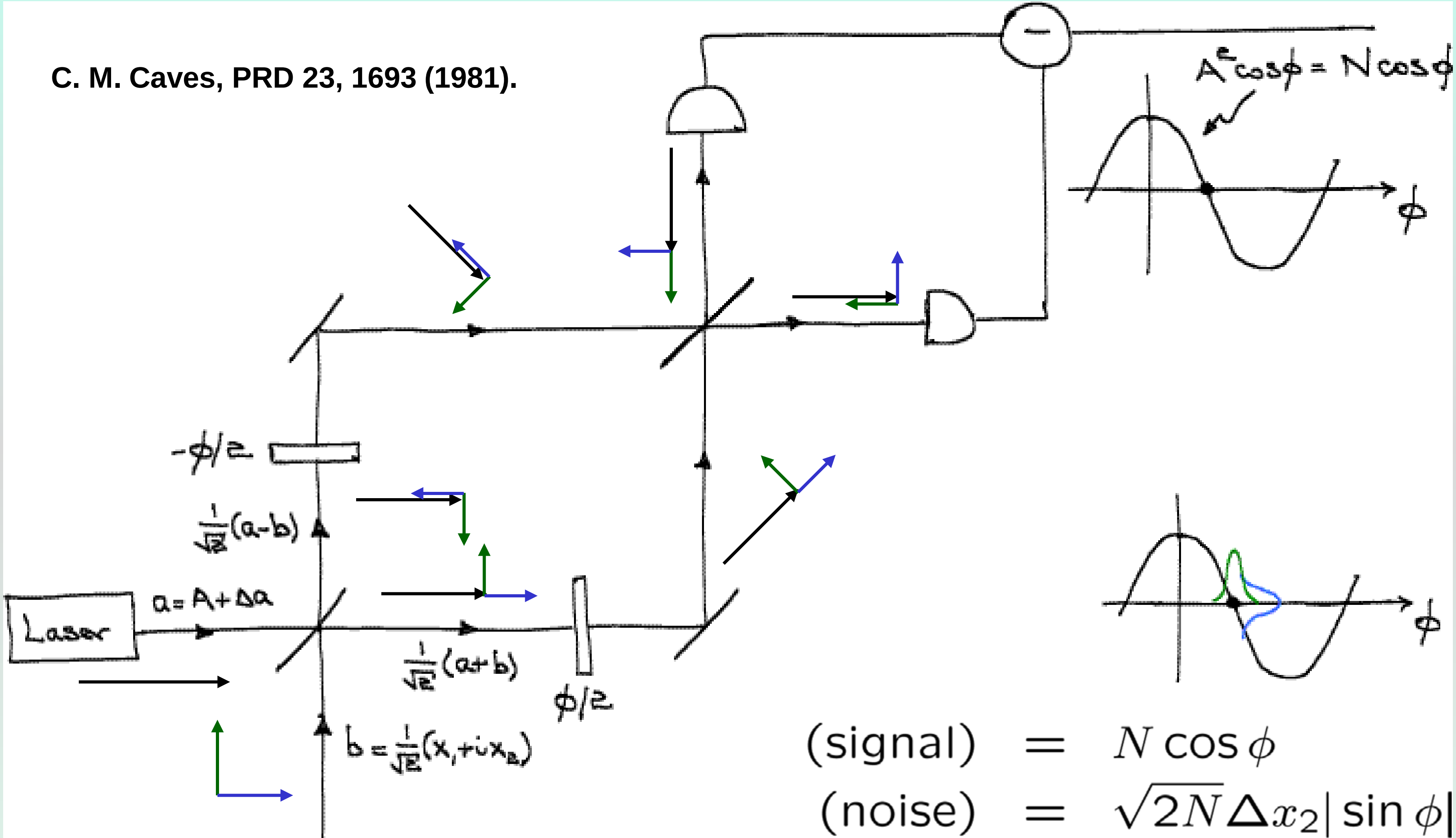
Laser Interferometer Gravitational Observatory (LIGO)

Livingston, Louisiana

High-power, Fabry-Perot-cavity (multipass), power- and signal-recycled, squeezed-light interferometers

Mach-Zender interferometer

C. M. Caves, PRD 23, 1693 (1981).



$$\text{(signal)} = N\cos\phi$$

$$\text{(noise)} = \sqrt{2N}\,\Delta x_2 |\sin\phi|$$

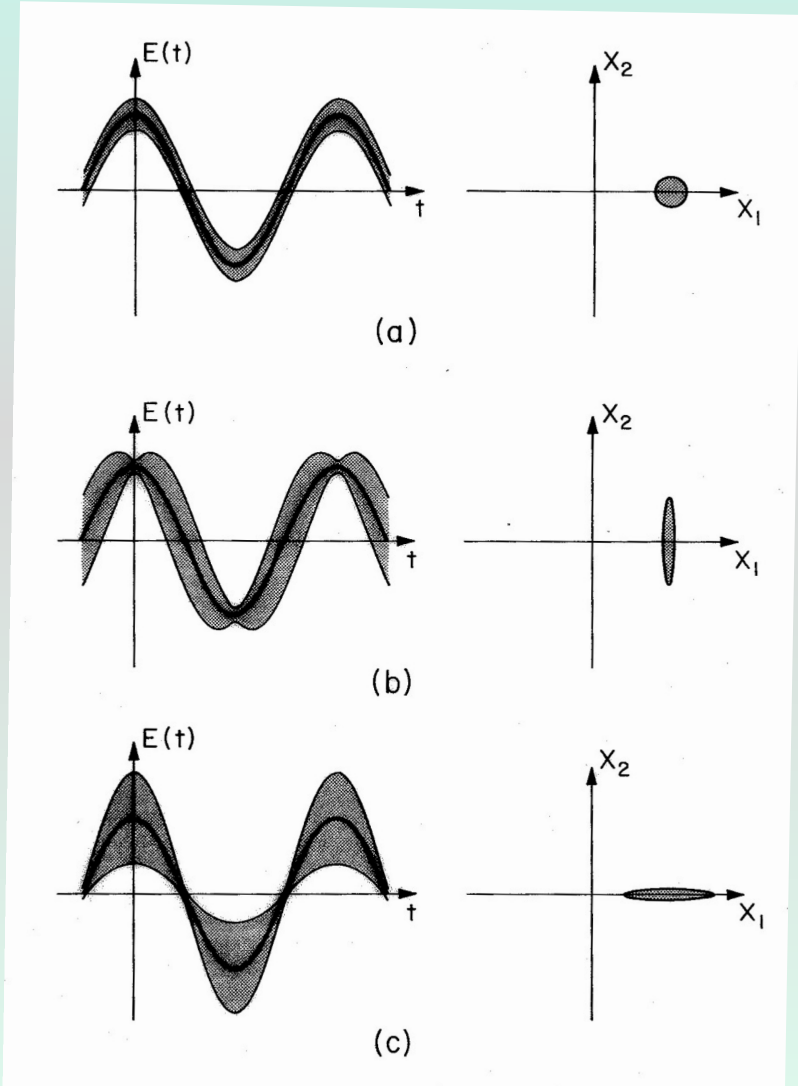
$$[x_1, x_2] = i$$

$$\Delta x_1 \Delta x_2 \geq 1/2$$

$$\Delta\phi = \frac{\text{(noise)}}{|d(\text{signal})/d\phi|} = \frac{\sqrt{2}\Delta x_2}{\sqrt{N}}$$

Squeezed states of light

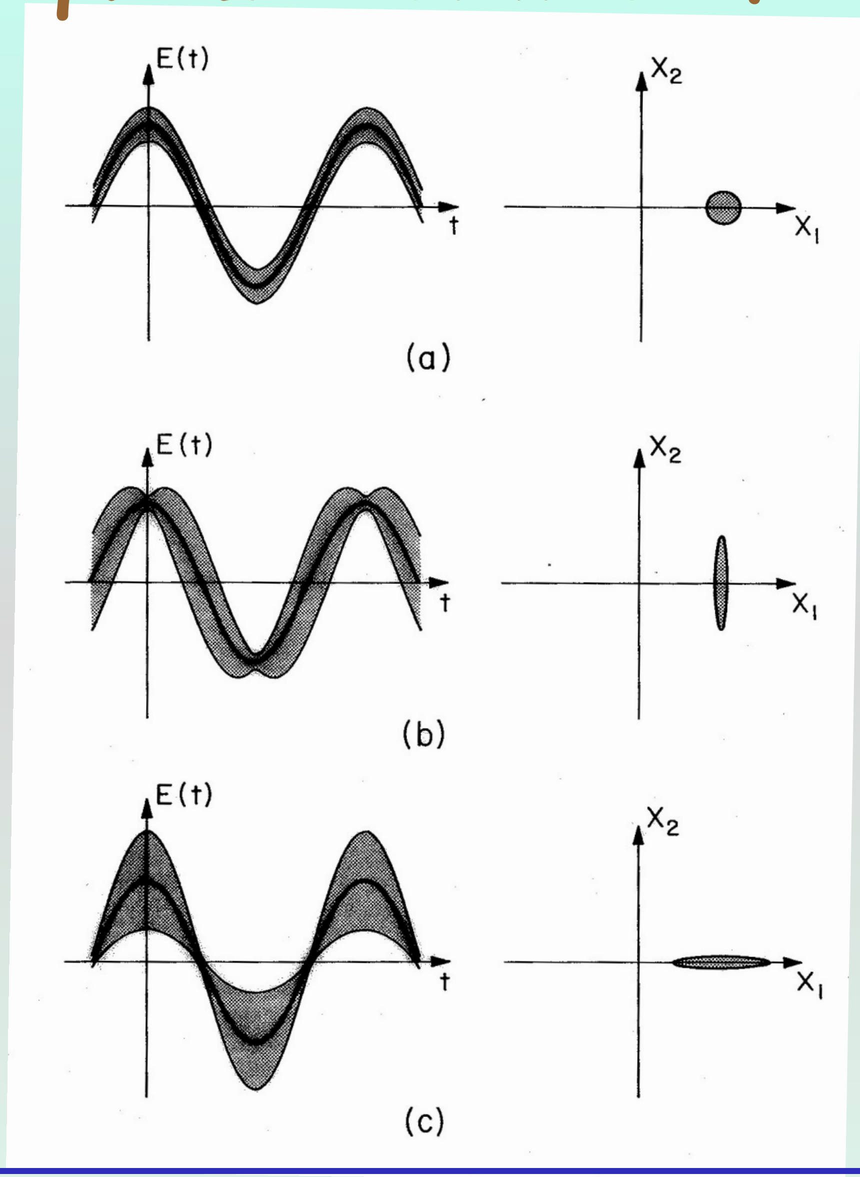
$$\Delta\phi \sim \frac{\Delta x_2}{A} = \frac{\Delta x_2}{\sqrt{N}}$$

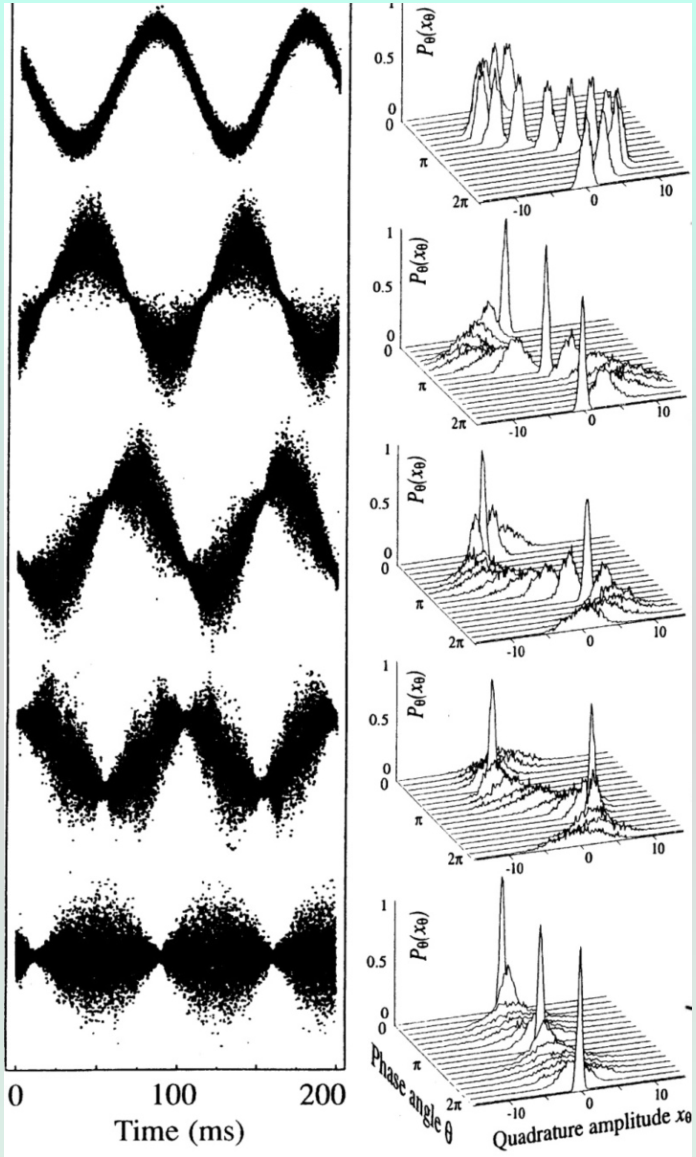


$$\Delta x_1 = \Delta x_2 = \frac{1}{\sqrt{2}}, \qquad \Delta\phi \sim \frac{1}{\sqrt{2N}}$$

$$\Delta x_1 = e^{r}/\sqrt{2}$$
$$\Delta x_2 = e^{-r}/\sqrt{2}, \qquad \Delta\phi \sim \frac{e^{-r}}{\sqrt{2N}}$$

Squeezed states of light



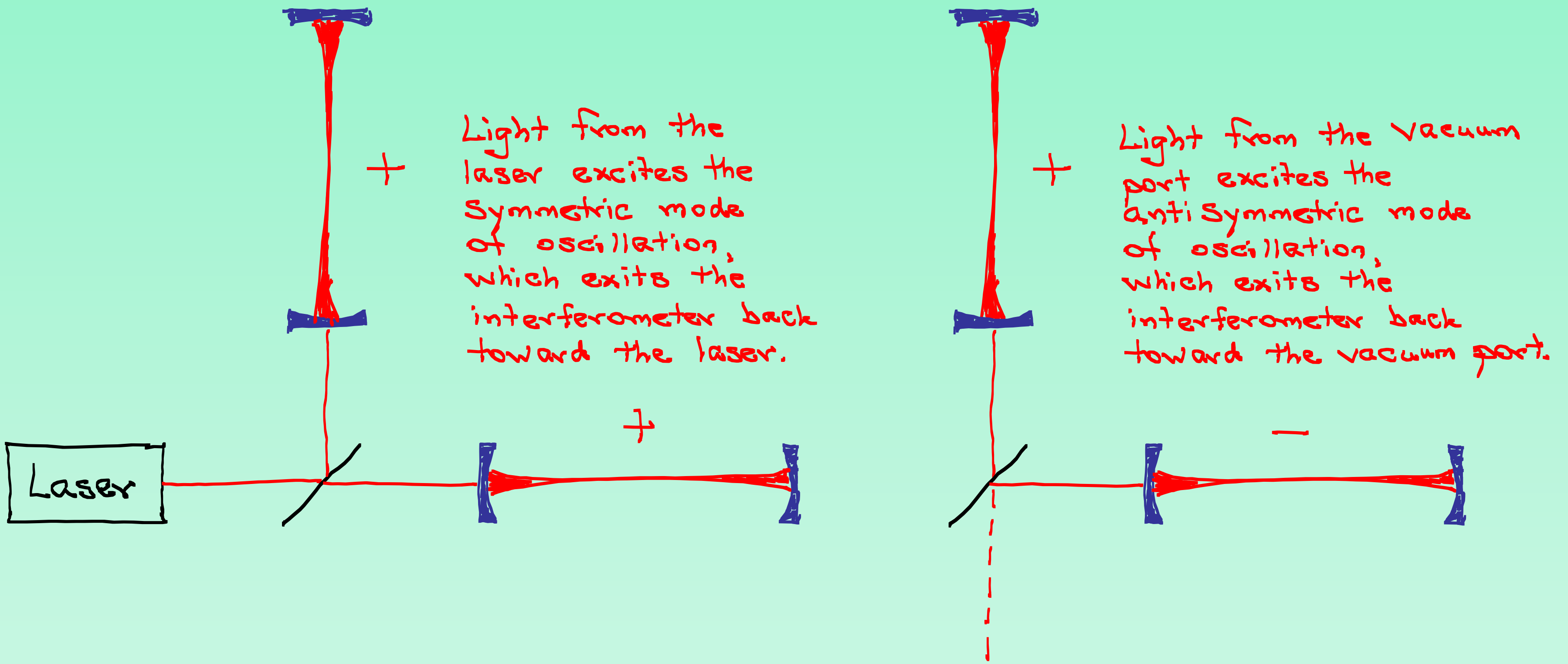


Groups at Australian National University, Hannover, and Tokyo have achieved up to 15 dB of squeezing at audio frequencies for use in Advanced LIGO, VIRGO, and GEO.

Squeezing by a factor of about 3.5

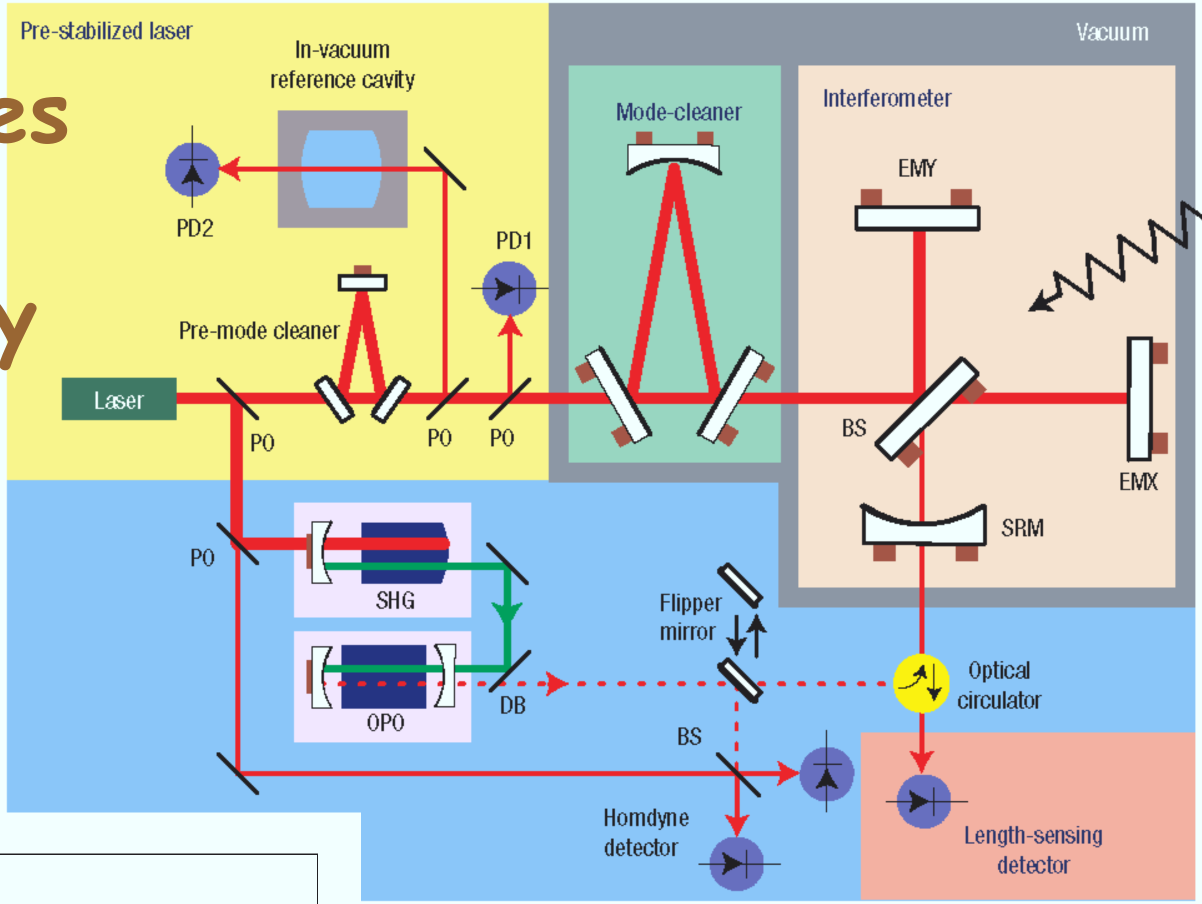
G. Breitenbach, S. Schiller, and J. Mlynek, Nature 387, 471 (1997).

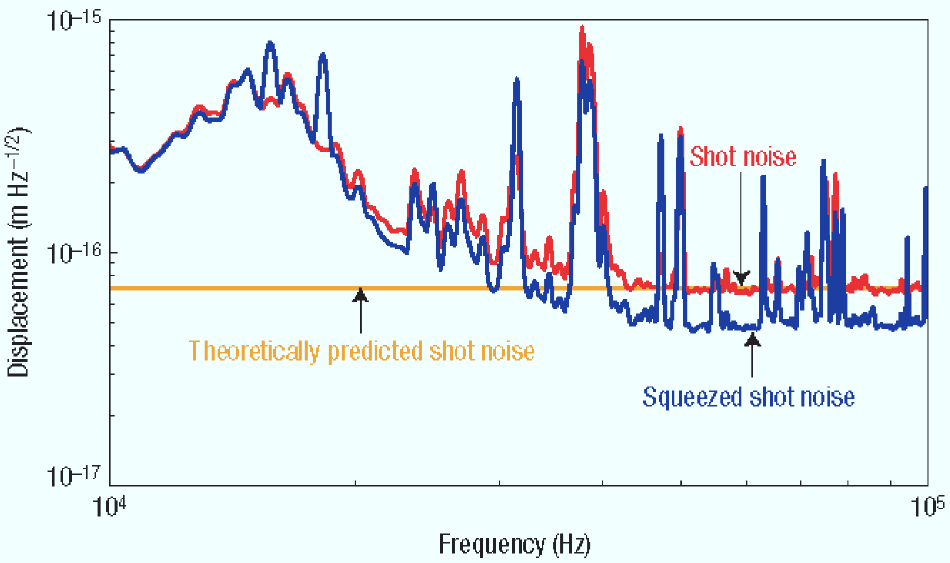
Fabry-Perot Michelson interferometer



Motion of the mirrors produced by a gravitational wave induces a transition from the symmetric mode to the antisymmetric mode; the resulting tiny signal at the vacuum port is contaminated by quantum noise that entered the vacuum port.

Squeezed states and optical interferometry

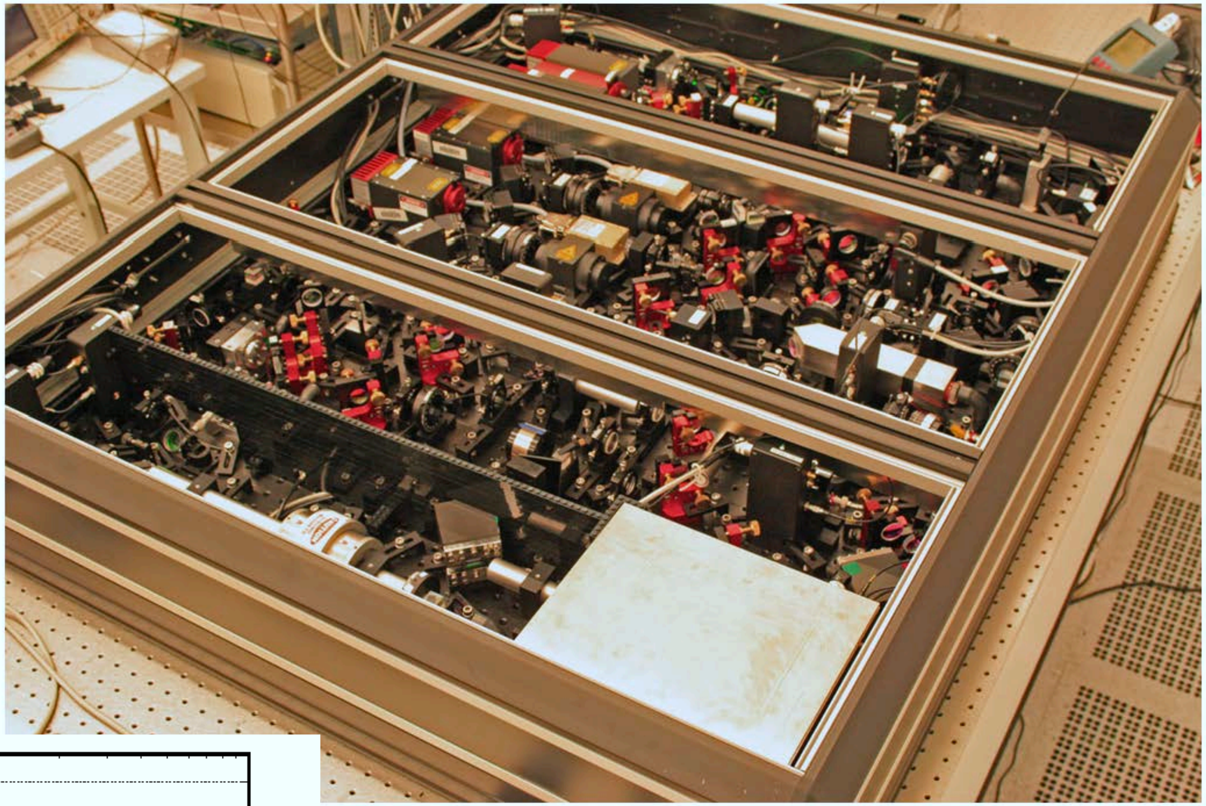


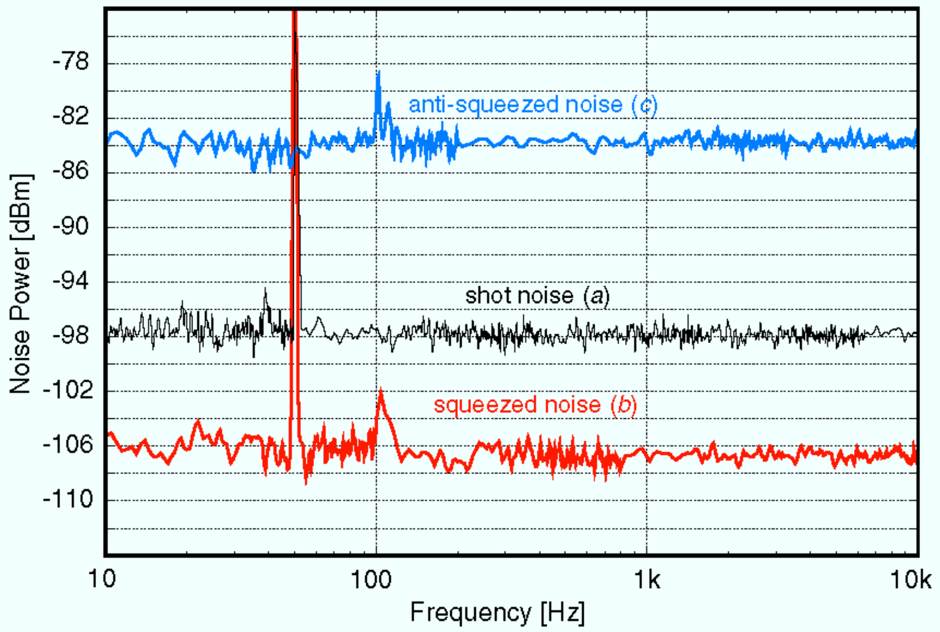


K. Goda, O. Miyakawa, E. E. Mikhailov, S. Saraf. R. Adhikari, K. McKenzie, R. Ward, S. Vass, A. J. Weinstein, and N. Mavalvala, Nature Physics 4, 472 (2008).

44% improvement in displacement sensitivity

Squeezed states for optical interferometry



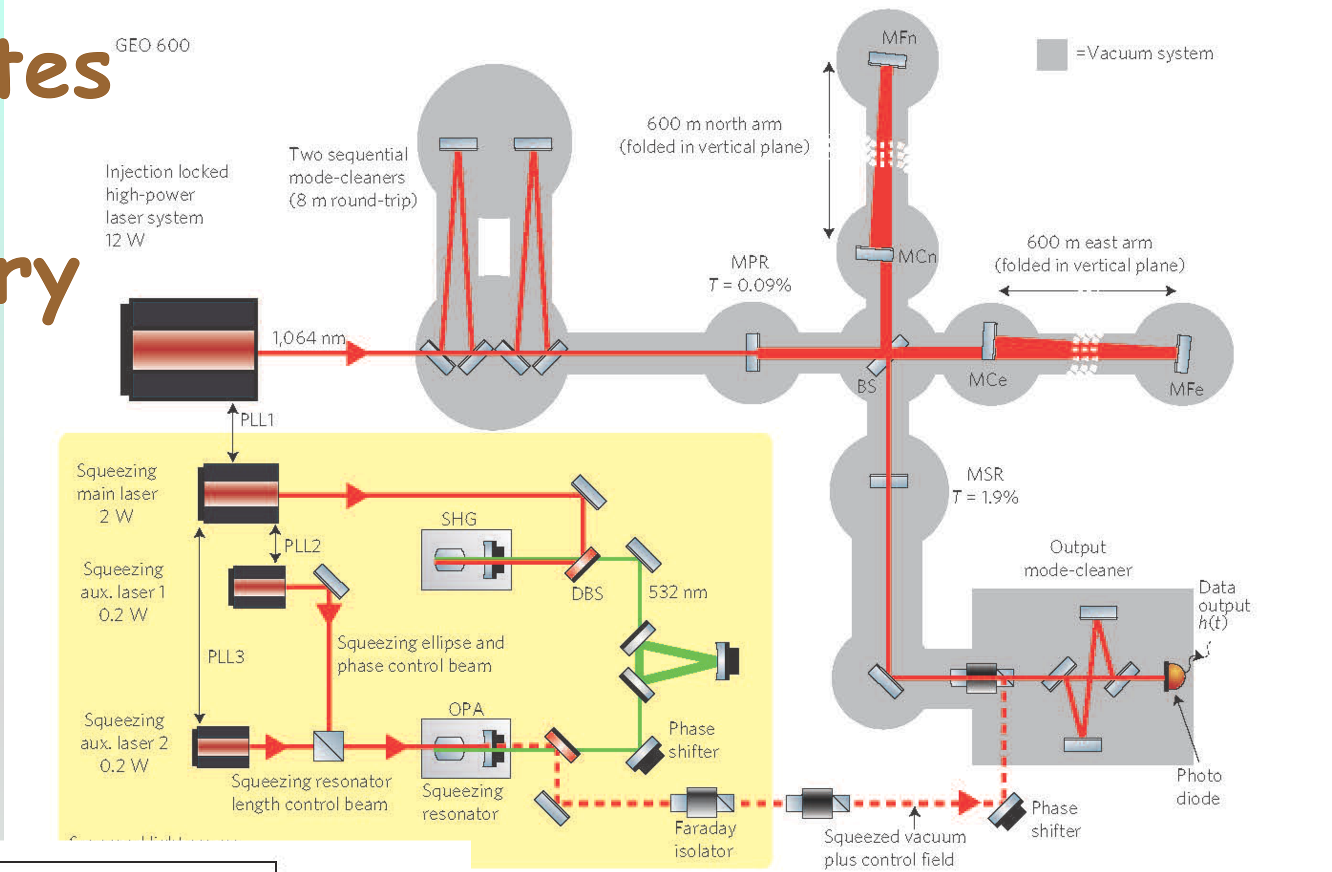


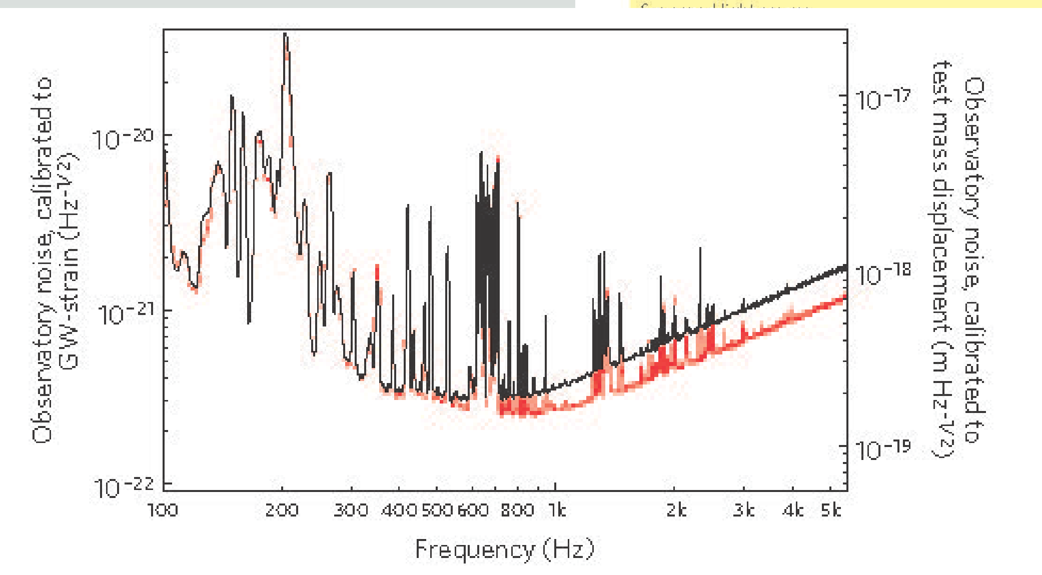
H. Vahlbruch, A. Khalaidovski, N. Lastzka, C. Graef, K. Danzmann, and R. Schnabel, Classical and Quantum Gravity 27, 084027 (2010).

9dB below shot noise from 10 Hz to 10 kHz

Squeezed states and optical interferometry

GEO 600 laser interferometer



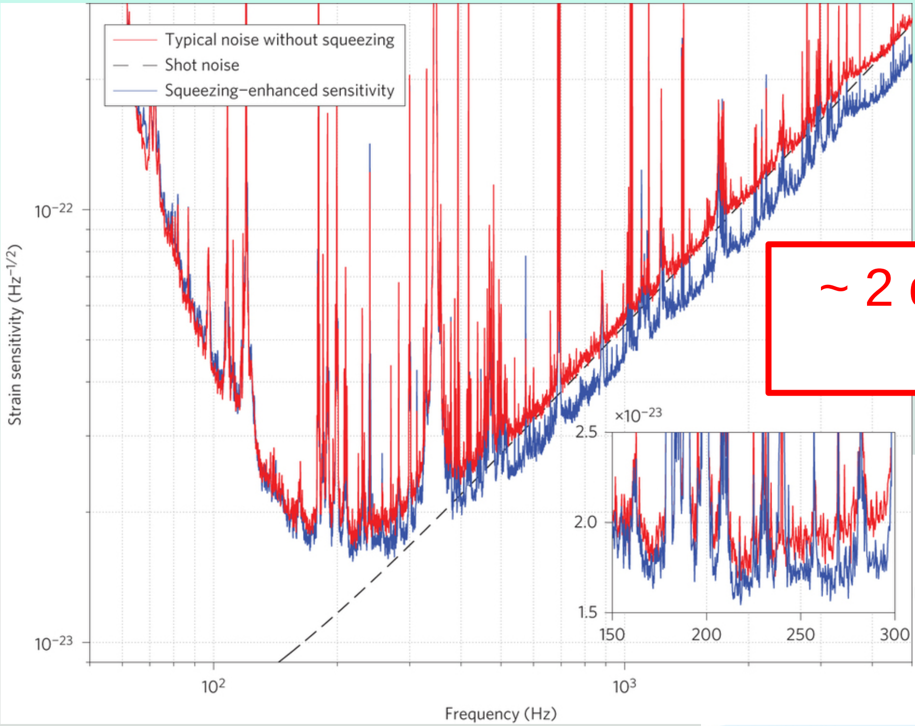


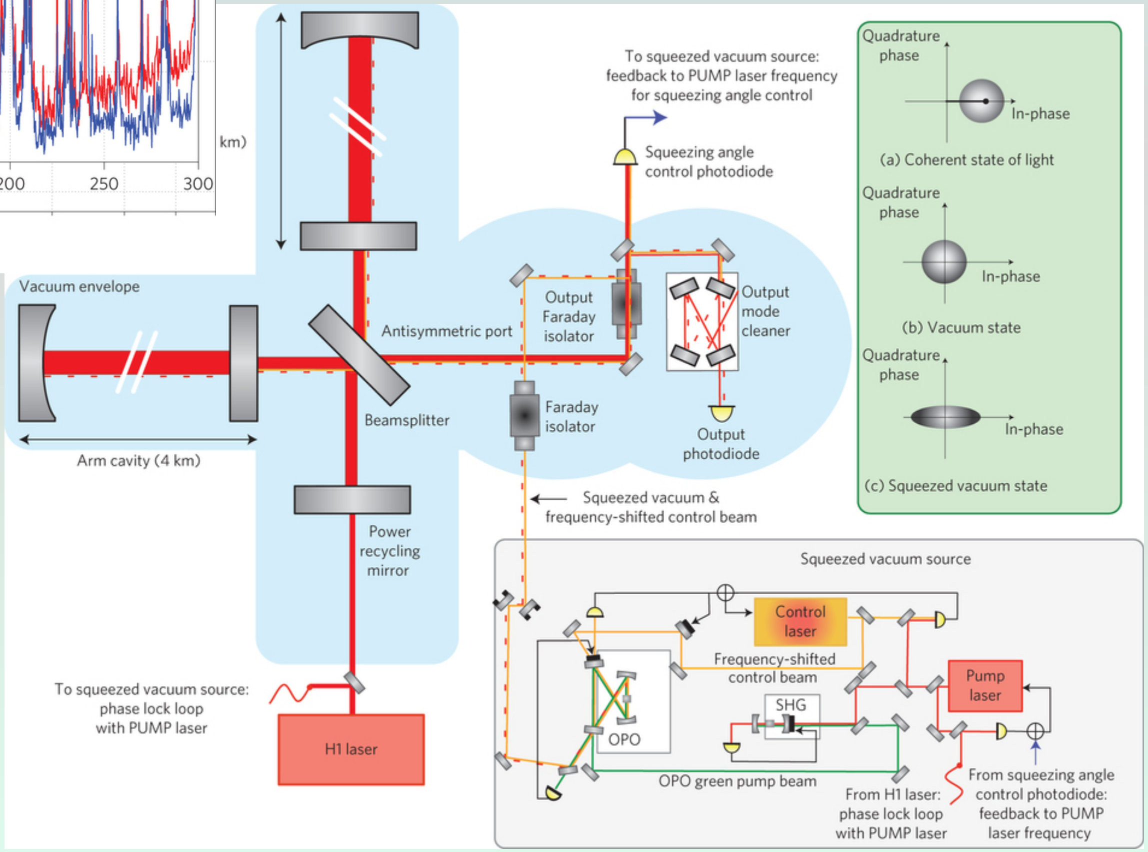
The LIGO Scientific Collaboration, Nature Physics 7, 962 (2011).

Up to 3.5dB improvement in sensitivity in the shot-noise-limited frequency band

Squeezed states and optical interferometry

~ 2 dB of shot-noise reduction





Squeezed light in the LIGO Hanford detector

The LIGO Scientific Collaboration, Nat. Phot. 7, 613 (2013).

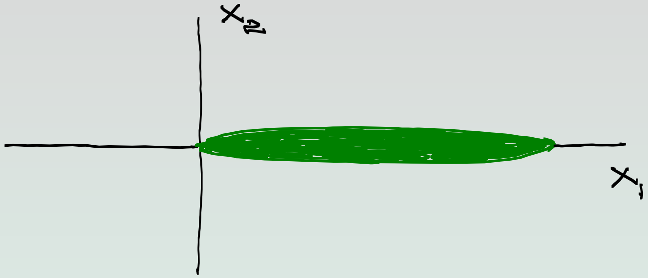
Quantum limits on optical interferometry

$$\Delta\phi = \frac{\sqrt{2}\Delta x_2}{\sqrt{N}}$$

Quantum Noise Limit (Shot-Noise Limit)

$$\Delta x_1 = \Delta x_2 = \frac{1}{\sqrt{2}}\,, \qquad \boxed{\Delta\phi = \frac{1}{\sqrt{N}}}$$

Heisenberg Limit



As much power in the squeezed light as in the main beam

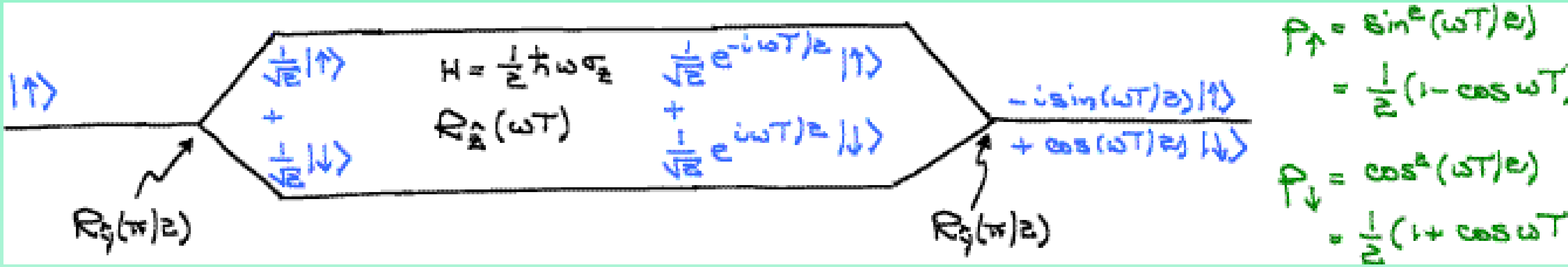
$$\frac{1}{2}(\Delta x_1)^2 \sim N$$

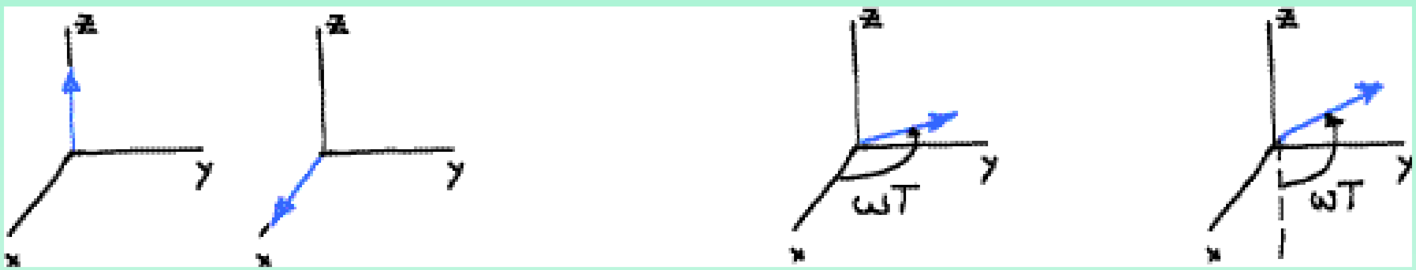
$$\Delta x_2 = \frac{1}{2\Delta x_1} \sim \frac{1}{2\sqrt{2N}}\,, \qquad \boxed{\Delta\phi \sim \frac{1}{2N}}$$

III. Ramsey interferometry, cat states, and spin squeezing

Truchas from East Pecos Baldy
Sangre de Cristo Range
Northern New Mexico

Ramsey interferometry





***N* independent "atoms"**

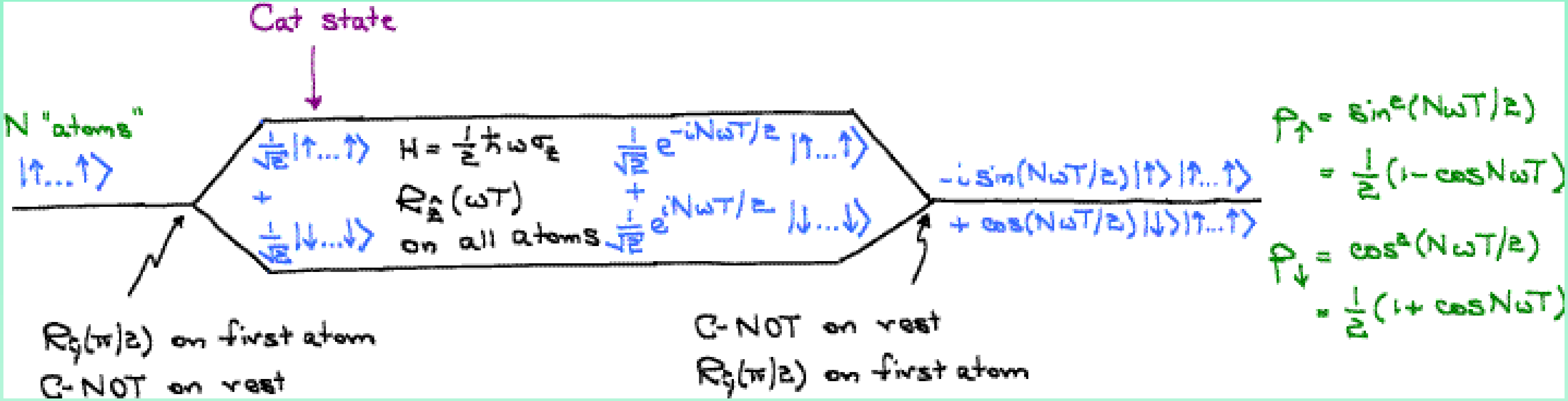
$$
\begin{aligned}
\text{(signal)} &= \langle \sigma_z \rangle = -\cos \omega T \\
\text{(noise)} &= \Delta\sigma_z = \sqrt{1-\cos^2 \omega T} = |\sin \omega T|
\end{aligned}
$$

$$
\Delta(\omega T) = \frac{1}{\sqrt{N}} \frac{\text{(noise)}}{|d(\text{signal})/d(\omega T)|} = \frac{1}{\sqrt{N}}
$$

Frequency measurement
Time measurement
Clock synchronization

Cat-state Ramsey interferometry

J. J. Bollinger, W. M. Itano, D. J. Wineland, and D. J. Heinzen, Phys. Rev. A 54, R4649 (1996).



Fringe pattern with period 2π/*N*

$$(\text{signal}) = \langle\sigma_z\rangle = -\cos N\omega T$$

$$(\text{noise}) = \Delta\sigma_z = \sqrt{1-\cos^2 N\omega T} = |\sin N\omega T|$$

$$\Delta(\omega T) = \frac{1}{\sqrt{\nu}}\frac{(\text{noise})}{|d(\text{signal})/d(\omega T)|} = \frac{1}{\sqrt{\nu}}\frac{1}{N}$$

$\nu = (\text{number of trials})$

***N* cat-state atoms**

Optical interferometry / Ramsey interferometry

$$\Delta\phi \sim \frac{1}{\sqrt{N}}$$

Quantum Noise Limit (Shot-Noise Limit)

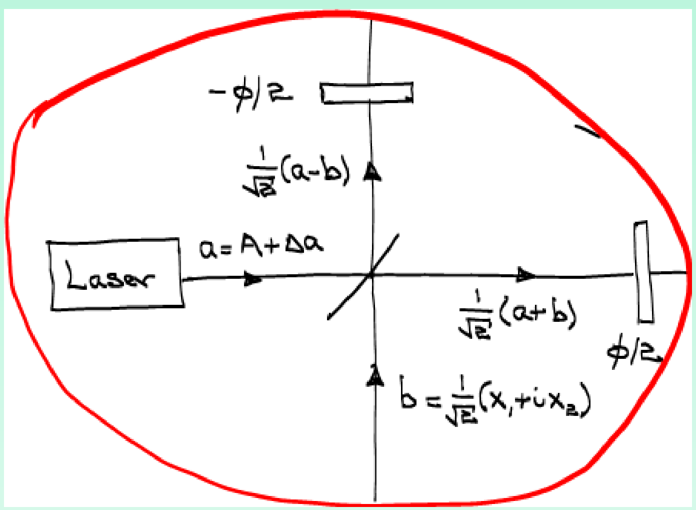
$$\Delta\phi \sim \frac{1}{\sqrt{N}}$$

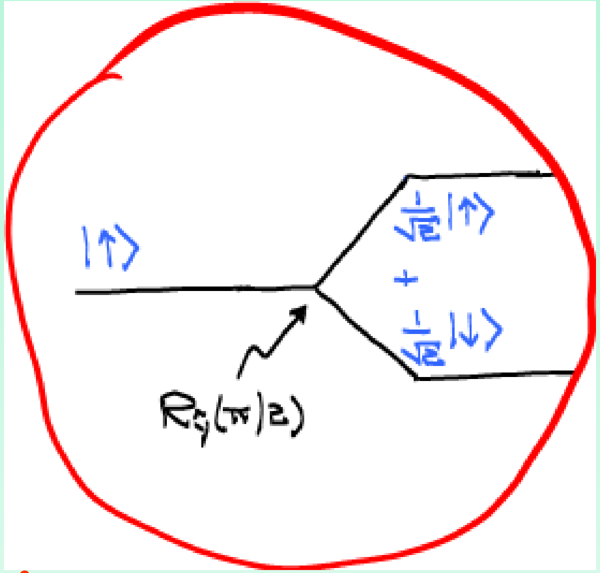
$$\phi = \omega T$$

$$\Delta\phi \sim \frac{1}{N}$$

Heisenberg Limit

$$\Delta\phi \sim \frac{1}{N}$$





Something's going on here.

Squeezed-state optical interferometry

Cat-state Ramsey interferometry

Entanglement before "beamsplitter"

Between arms
(wave or modal entanglement)

0 e-bit

Between photons
(particle entanglement)

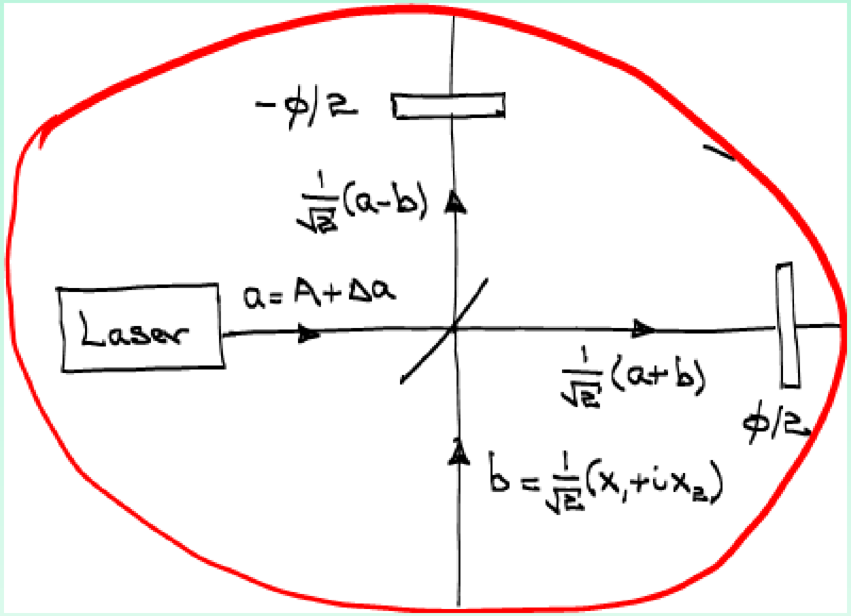
0 e-bit

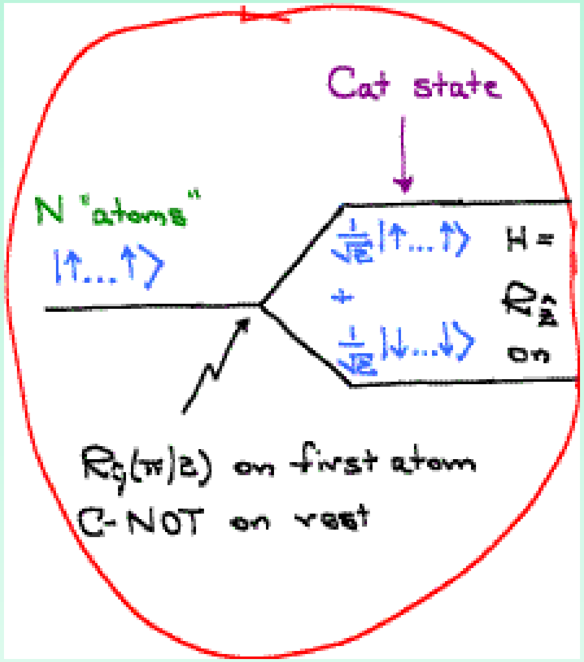
Between atoms
(particle entanglement)

0 e-bit

Between arms
(modal entanglement)

0 e-bit





Squeezed-state optical interferometry

Cat-state Ramsey interferometry

Entanglement after “beamsplitter”

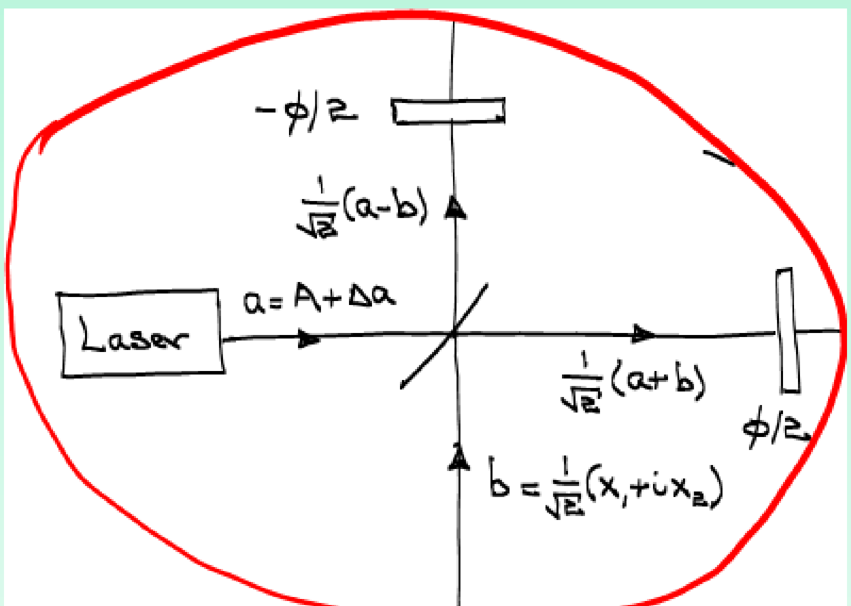
Squeezed-state optical interferometry

Between arms
(wave or modal entanglement)

$\sim r/\ln 2$ e-bits $\rightarrow \log N$ e-bits

Between photons
(particle entanglement)

?



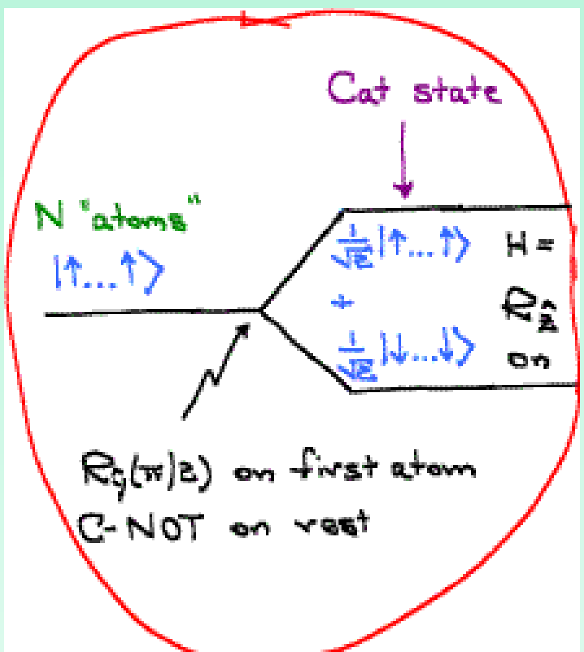
Cat-state Ramsey interferometry

Between atoms
(particle entanglement)

1 e-bit

Between arms
(modal entanglement)

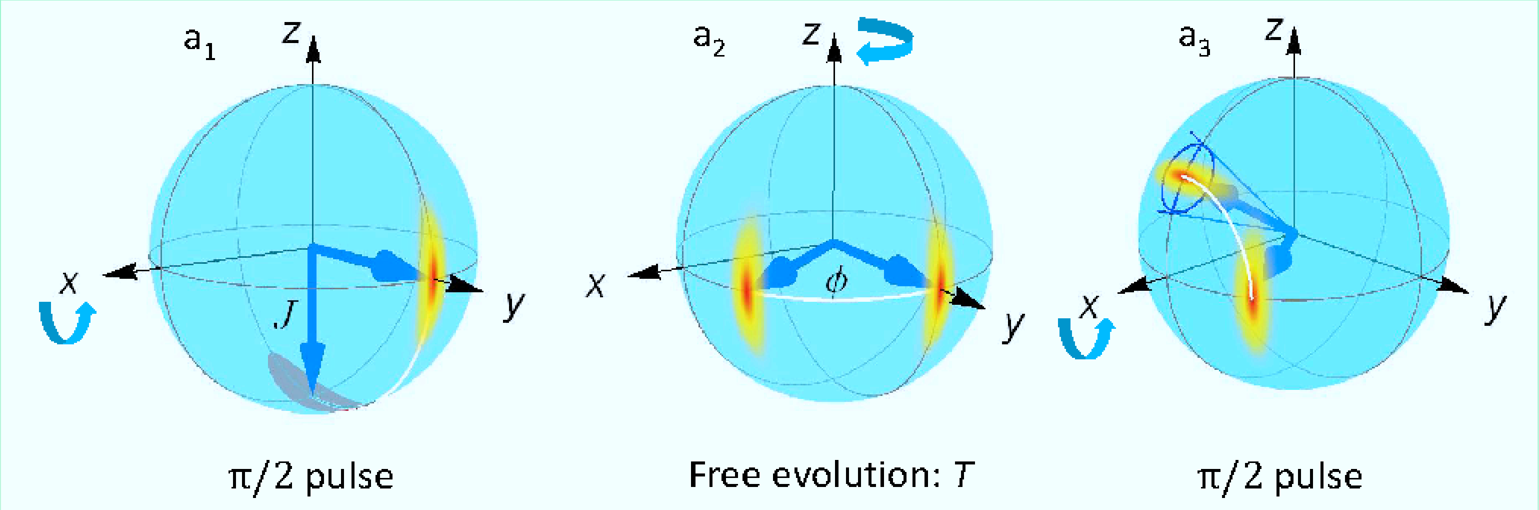
1 e-bit



Spin-squeezing Ramsey interferometry

Collective spin $J = N/2$

J. Ma, X. Wang, C. P. Sun, and F. Nori, Phys. Rep. 509, 89–165 (2011).



$$[J_z, J_x] = i\hbar J_y \quad \Longrightarrow \quad \Delta J_z \Delta J_x \geq \frac{\hbar}{2}|\langle J_y \rangle| = \frac{\hbar^2}{4}N$$

After 1st $\pi/2$ pulse

Spin coherent state $\quad \Delta J_x = \Delta J_z = \frac{1}{2}\hbar\sqrt{N}$

Spin-squeezed state $\quad \Delta J_x = \frac{1}{2}\hbar\sqrt{N}e^{-r}$

$$\Delta J_z = \frac{1}{2}\hbar\sqrt{N}e^{r}$$

Phase sensitivity $\quad \Delta\phi \sim \frac{\Delta J_x}{\hbar J} = \frac{e^{-r}}{\sqrt{N}}$

Heisenberg Limit

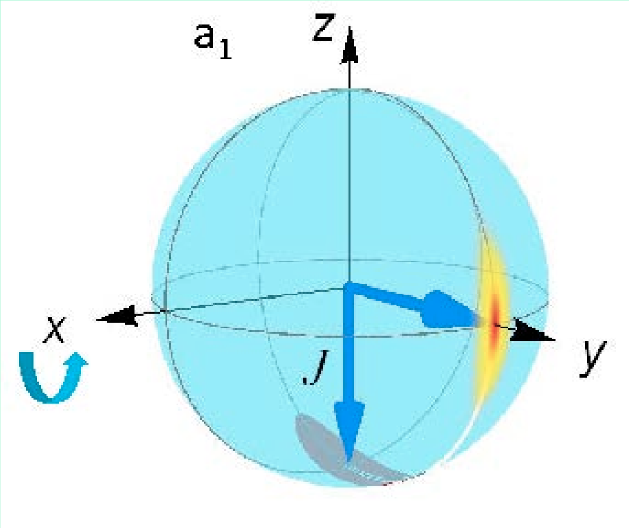
$$\Delta J_x \sim \hbar/2$$

$$\Delta J_z \sim \hbar N/2$$

$$\Delta\phi \sim \frac{1}{N}$$

This is really a cat state.

Spin-squeezing Ramsey interferometry



Collective spin $J = N/2$

What's squeezed?

The +*y* spin state has *N* particles; the –*y* spin state has single-mode squeezing. This is like the state of the two arms prior to the beamsplitter in an optical interferometer. The up and down spin states have correlated squeezing like that in the arms of a squeezed-state optical interferometer.

What's entangled?

No entanglement of +y and –y spin states.
Modal entanglement of up and down spin states.
Particle entanglement.

Squeezed-state optical interferometry

Spin-squeezing Ramsey interferometry

Entanglement

Between arms
(wave or modal entanglement)

$$\sim \frac{r}{\ln 2} \text{ e-bits} \rightarrow \log N \text{ e-bits}$$

Between atoms
(particle entanglement)

0 e-bit to 1 e-bit

Between photons
(particle entanglement)

?

Between arms
(modal entanglement)

$$\frac{r}{\ln 2} \text{ e-bits} \rightarrow \frac{\log N}{2} \text{ e-bit} \rightarrow 1 \text{ e-bit}$$

Role of entanglement

Entanglement is a resource …

for getting my paper into *Nature*.

Don't accept facile explanations.
Ask questions.

Transition

Telling stories is what physics is about.

Lecture 1 has been about understanding fringe patterns and sources of noise and designing devices to improve phase sensitivity based on this understanding. This is telling stories.

Lecture 2 is about proving that the stories aren't fooling us.

Which is better, stories or proofs? You need them both, but stories are going to get you farther.

Quantum metrology: An information-theoretic perspective

Lecture 2

I. Quantum Cramér-Rao Bound (QCRB)

II. Making quantum limits relevant. Loss and decoherence

III. Beyond the Heisenberg limit. Nonlinear interferometry

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Transition

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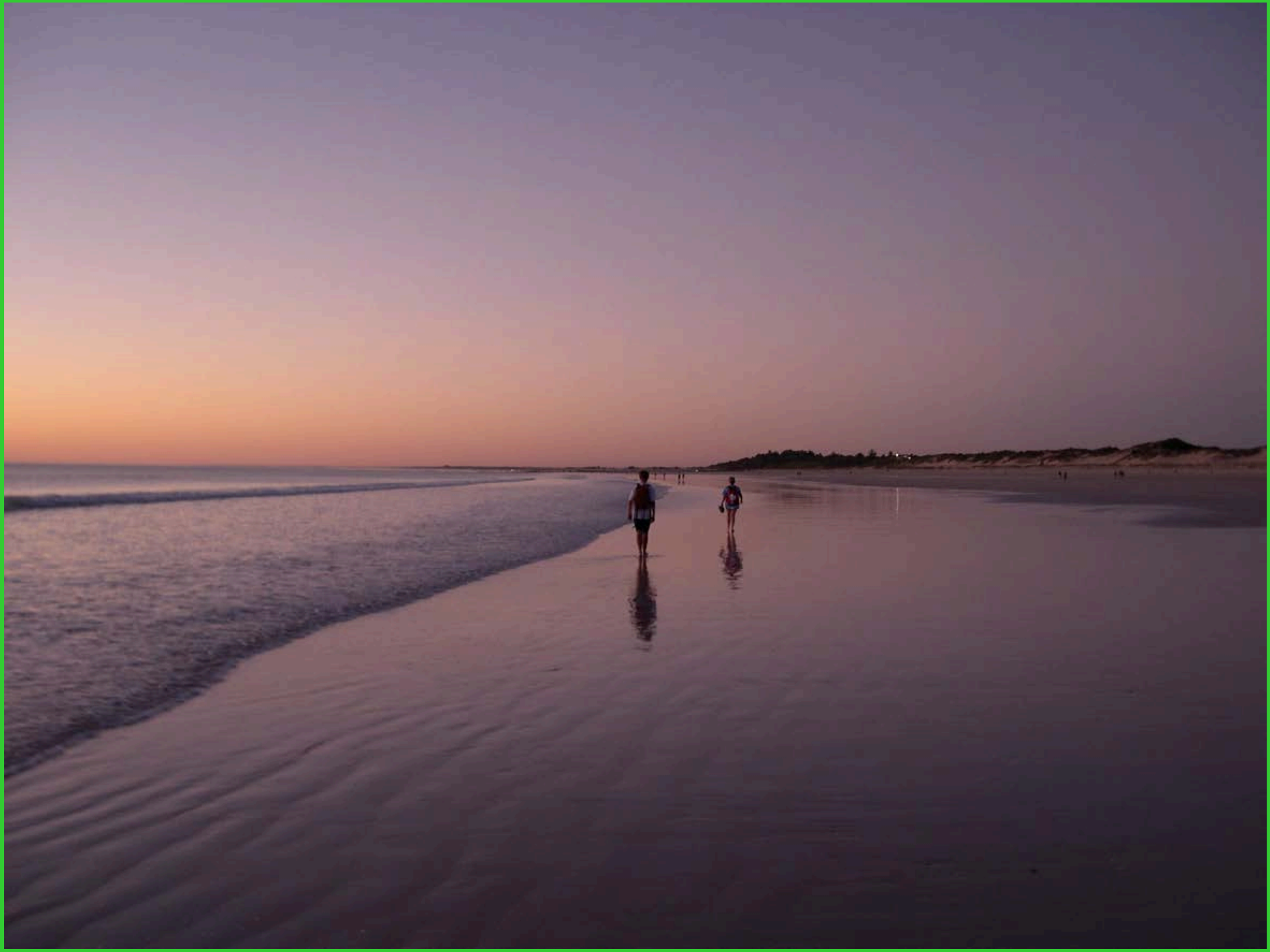
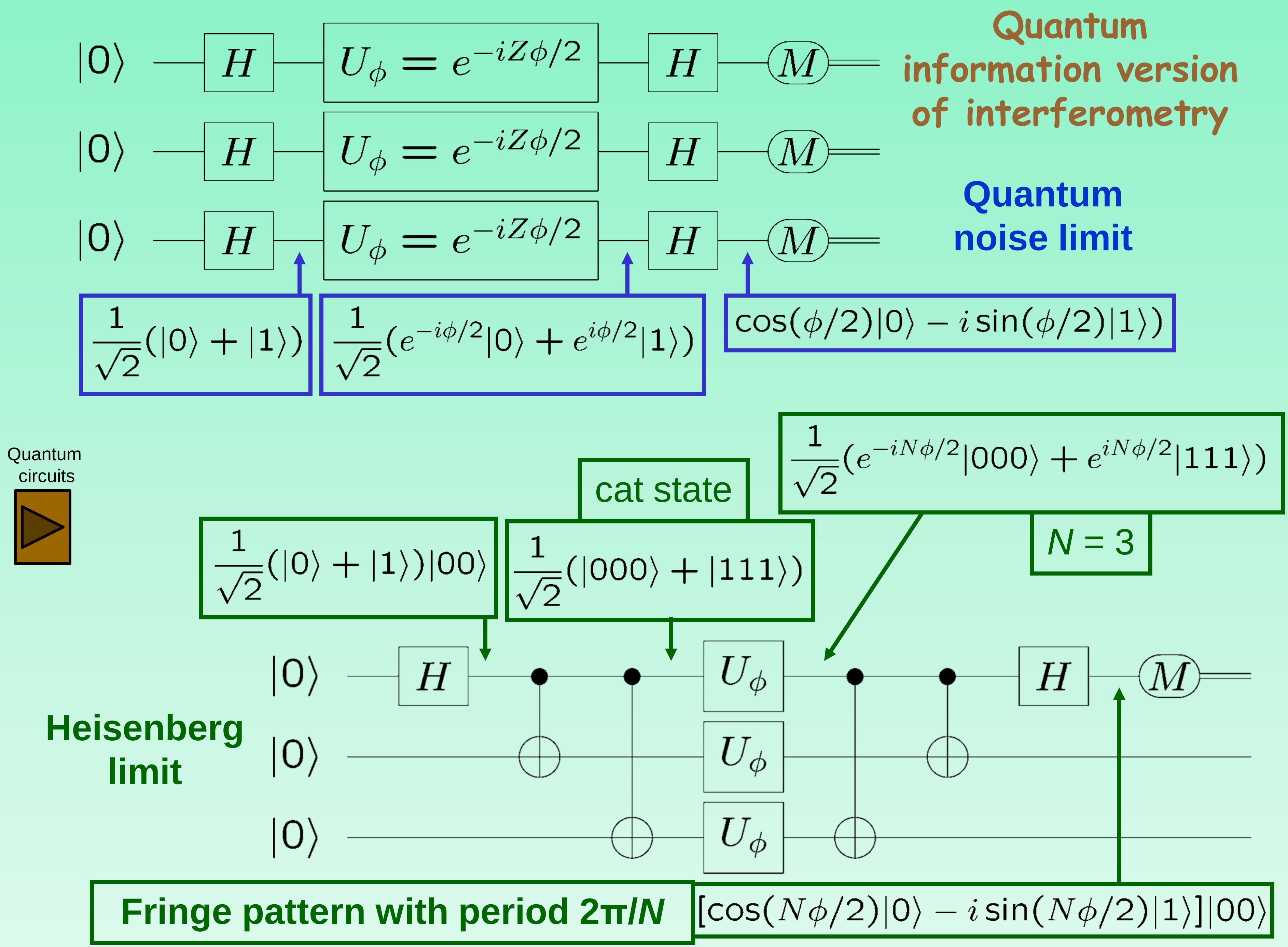
I. Quantum Cramér-Rao Bound (QCRB)

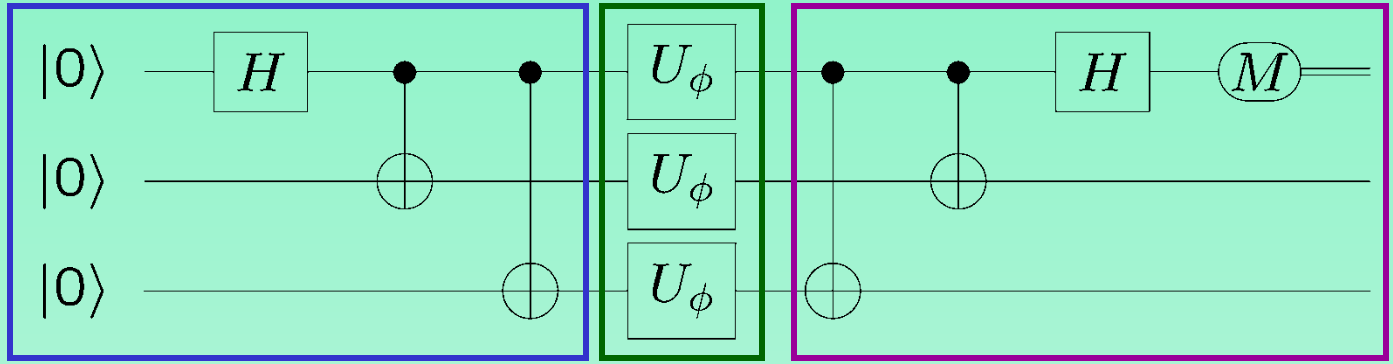


Cable Beach
Western Australia



Quantum information version of interferometry
Quantum noise limit
$|0\rangle$
H
$U_\phi = e^{-iZ\phi/2}$
H
M
$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
$\frac{1}{\sqrt{2}}(e^{-i\phi/2}|0\rangle + e^{i\phi/2}|1\rangle)$
$\cos(\phi/2)|0\rangle - i\sin(\phi/2)|1\rangle$
Quantum circuits
cat state
$\frac{1}{\sqrt{2}}(e^{-iN\phi/2}|000\rangle + e^{iN\phi/2}|111\rangle)$
$N = 3$
$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|00\rangle$
$\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$
Heisenberg limit
U_ϕ
Fringe pattern with period 2π/N
$[\cos(N\phi/2)|0\rangle - i\sin(N\phi/2)|1\rangle]|00\rangle$

Cat-state interferometer



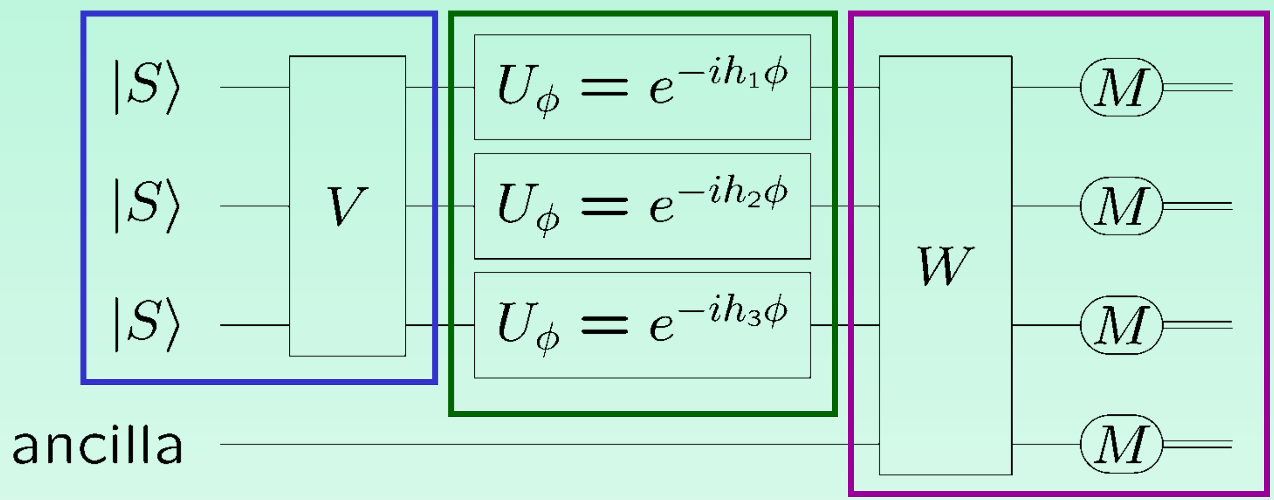
State preparation

$$U = e^{-ih\phi}$$

$$h = \sum_{j=1}^{N} h_j$$

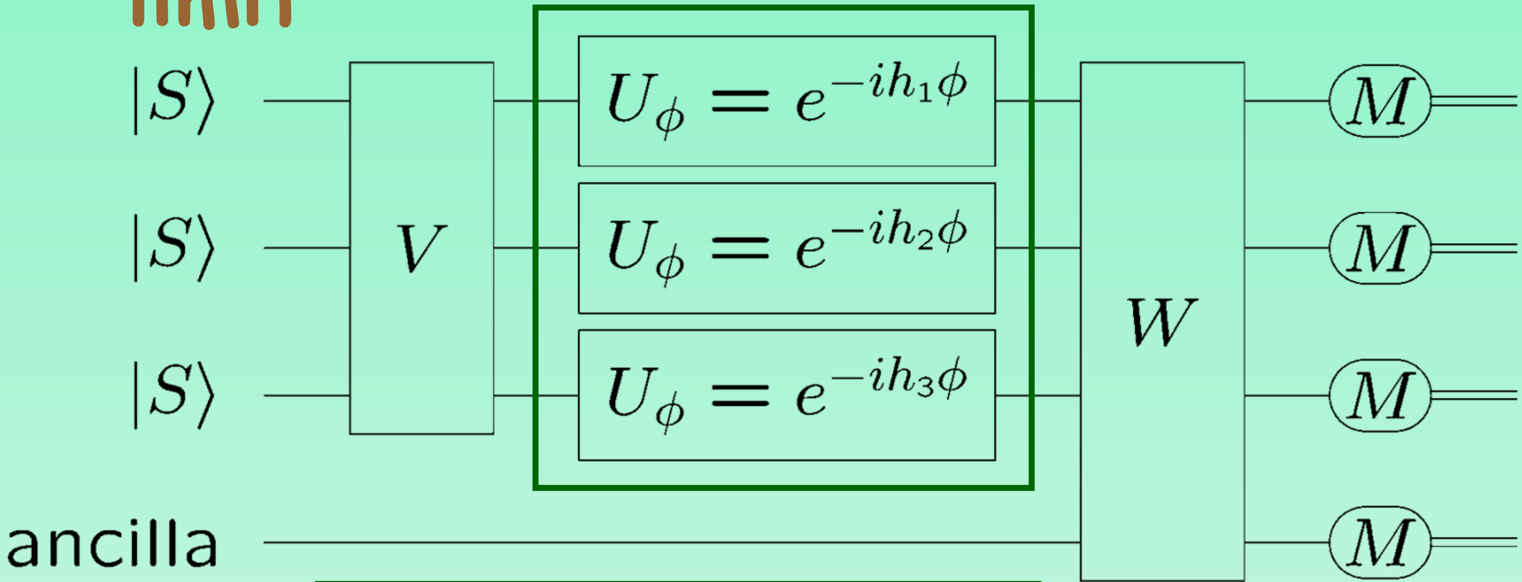
Measurement

Single-parameter estimation



Heisenberg limit

S. L. Braunstein, C. M. Caves, and G. J. Milburn, Ann. Phys. 247, 135 (1996).
V. Giovannetti, S. Lloyd, and L. Maccone, PRL 96, 041401 (2006).
S. Boixo, S. T. Flammia, C. M. Caves, and JM Geremia, PRL 98, 090401 (2007).



W, U_ϕ, and ancilla measurements can be interleaved.

$$U = e^{-ih\phi}\,, \qquad h = \sum_{j=1}^{N} h_j$$

$$\Delta\phi \ge \frac{1}{2\Delta h} \ge \frac{1}{N(\Lambda - \lambda)}$$

Generalized uncertainty principle

Quantum Cramér-Rao bound

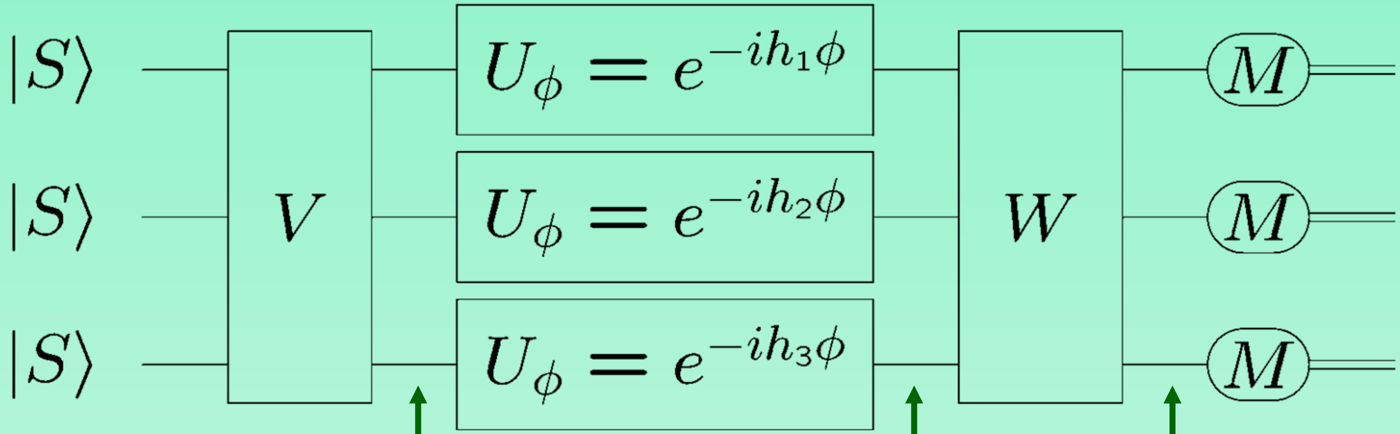
$$\Delta h \le \frac{1}{2}||h|| = \frac{1}{2}N(\Lambda - \lambda)$$

Separable inputs

$$\Delta h \le \frac{1}{2}\sqrt{N}(\Lambda - \lambda)$$

$$\Delta\phi \ge \frac{1}{\sqrt{N}(\Lambda - \lambda)}$$

Achieving the Heisenberg limit



cat state

$$\frac{1}{\sqrt{2}}(|\Lambda,\ldots,\Lambda\rangle + |\lambda,\ldots,\lambda\rangle)$$

$$\frac{1}{\sqrt{2}}\left(e^{-iN\Lambda\phi}|\Lambda,\ldots,\Lambda\rangle + e^{-iN\lambda\phi}|\lambda,\ldots,\lambda\rangle\right)$$

$$e^{-iN(\Lambda+\lambda)\phi/2}\Big(\cos[N(\Lambda-\lambda)\phi/2]|\Lambda,\ldots,\Lambda\rangle - i\sin[N(\Lambda-\lambda)\phi/2]|\lambda,\ldots,\lambda\rangle\Big)$$

Proof of QCRB

Fringe pattern with period $2\pi/N(\Lambda-\lambda)$

$$\Delta\phi = \frac{1}{N(\Lambda-\lambda)}$$

Is it entanglement?

It's the entanglement, stupid.

But what about?

- There are states with far more bipartite particle entanglement than the cat state—up to about $N/2$ e-bits for equal bipartite splits—yet they are typically useless—and definitely not as good as cat states—for metrology.

- Measurement sensitivity and optimal initial state depend on local Hamiltonians h_j, but entanglement measures are usually constructed to be independent of such mundane details.

For metrology, entanglement is part of the story, but only part. We need a generalized notion of entanglement/resources that includes information about the physical situation, particularly the relevant Hamiltonian.

II. Making quantum limits relevant. Loss and decoherence

Bungle Bungle Range
Western Australia

Making quantum limits relevant

Optimal (Heisenberg) sensitivity

$$\Delta\omega \sim \frac{1}{TN}$$

Quantum circuits

The serial resource, *T*, and the parallel resource, *N*, are equivalent and interchangeable, *mathematically.*

The serial resource, *T*, and the parallel resource, *N*, are not equivalent and not interchangeable, *physically.*

Information science perspective

Platform independence

Physics perspective

Distinctions between different physical systems

Making quantum limits relevant.
One metrology story

Resources

Let τ be the overall measurement time (τ^{-1} is the bandwidth), and let γ characterize the rate of decoherence or loss.

- The number of systems, n, within each nonclassical (entangled) probe is the *quantum parallel resource*.
- The coherent interaction time T of each probe is the *quantum serial resource*.
- The rate at which systems can be deployed, R, is the *classical resource*; alternatively, the number of probes, $\nu = R(\tau - T)/n$, can be regarded as the classical resource [$N = \nu n = R(\tau - T)$ is the total number of systems].

Problem

Given τ and γ, what is the best strategy for using n, T, and R to estimate a frequency $\omega = \phi/T$?

An answer has been worked out for squeezed-state optical interferometry and for Ramsey interferometry with phase decoherence: the quantum resources—extended coherent evolution and entanglement—are useful only if $\gamma\tau \lesssim 1$ and $R\tau \gg 1$.

A. Shaji and C. M. Caves, PRA 76, 032111 (2007).

Making quantum limits relevant

Rule of thumb for photon losses for large *N*

S. Knysh, V. N. Smelyanskiy and G. A. Durkin, PRA 83, 021804(R) (2011).

Given fractional photon loss $1-\eta$,

$$\Delta\omega \sim \frac{1}{T}\sqrt{\frac{e^{-2r}\eta + (1-\eta)}{\eta N}} \qquad e^{2r} \lesssim N$$

$$= \frac{1}{T}\sqrt{\frac{e^{-2r}}{N} + \frac{1-\eta}{\eta N}}$$

$$\simeq \begin{cases} \dfrac{1}{TN}, & 1-\eta \lesssim 1/N \simeq e^{-2r} \\ \dfrac{1}{T}\sqrt{\dfrac{1-\eta}{\eta N}}, & e^{-2r} \simeq 1-\eta \gtrsim 1/N \end{cases}$$

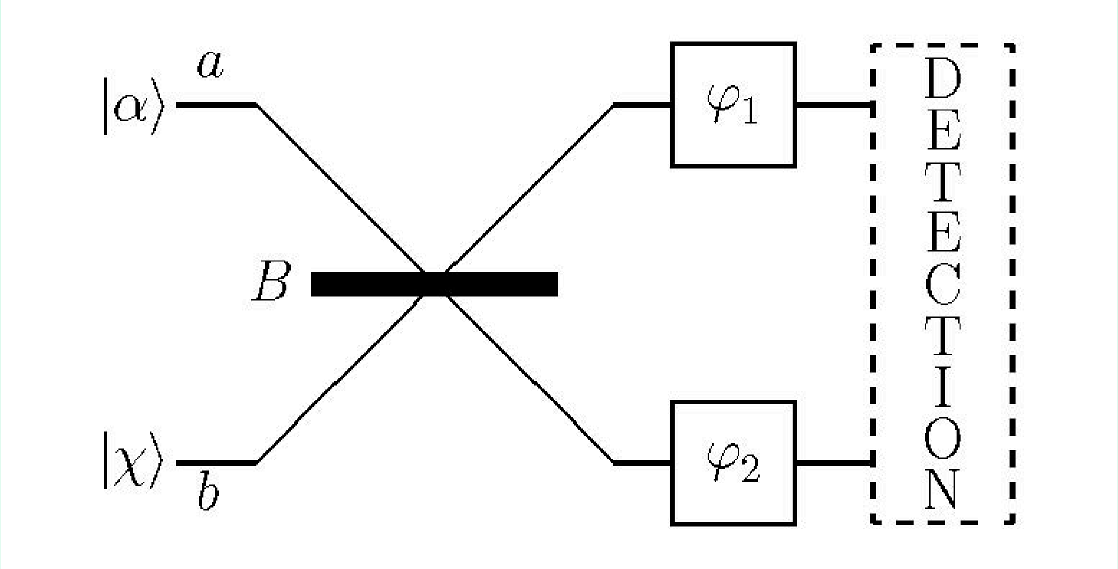
Heisenberg limit: less than one photon lost.

Typically, beat shot noise by square of root of fractional loss.

Quantum limit on practical optical interferometry

1. Cheap photons from a laser (coherent state)
2. Low, but nonzero losses on the detection timescale
3. Beamsplitter to make differential phase detection insensitive to laser fluctuations

Freedom: state input to the second input port; optimize relative to a mean-number constraint.
Entanglement: mixing this state with coherent state at the beamsplitter.



Generalized uncertainty principle QCRB

$$\Delta\phi^2_{d,\mathrm{est}} \geq \frac{1}{\Delta N_d^2} \equiv \frac{1}{\mathcal{F}}$$

$$\mathcal{F} = 2|\alpha|^2\langle(\Delta p)\rangle^2 + \bar{N}_b \leq |\alpha|^2\left(2\bar{N}_b + \sqrt{\bar{N}_b(\bar{N}_b+1)} + 1\right) + \bar{N}_b$$

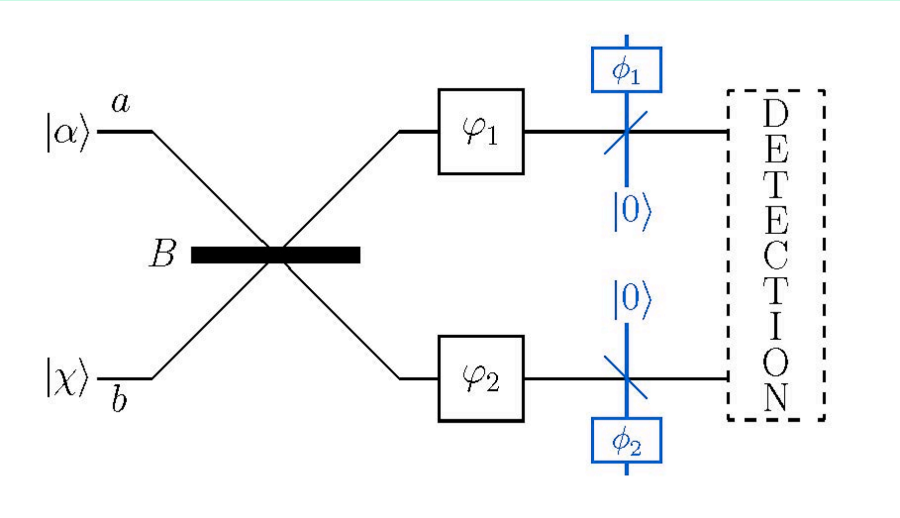
$$= |\alpha|^2 e^{2r} + \sinh^2 r$$

M. D. Lang and C. M. Caves, PRL 111, 17360 (2013).

Optimum achieved by differenced photodetection in a Mach-Zehnder configuration.

Achieved by squeezed vacuum into the second input port

Practical optical interferometry: Photon losses



M. D. Lang , UNM PhD dissertation, 2015.

$$1-\eta = \begin{pmatrix}\text{fractional loss}\\ \text{in each arm}\end{pmatrix}$$

$$\Delta\phi^2_{d,\text{est}} \ge \boxed{\frac{1}{\mathcal{F}_Q} \ge \frac{1}{\mathcal{C}_Q} \ge \frac{1}{I_Q}}$$

B. M. Escher, R. L. de Matos Filho, and L. Davidovich, Nat. Phys. 7, 406–411 (2011).

$$\mathcal{C}_Q = \begin{pmatrix}\text{Upper bound on quantum Fisher information}\\ \text{maximized over fake phase shifts } \phi_1 \text{ and } \phi_2\\ \text{and over all states input to second input port}\end{pmatrix}$$

Z. Jiang, PRA 89, 032128 (2014).

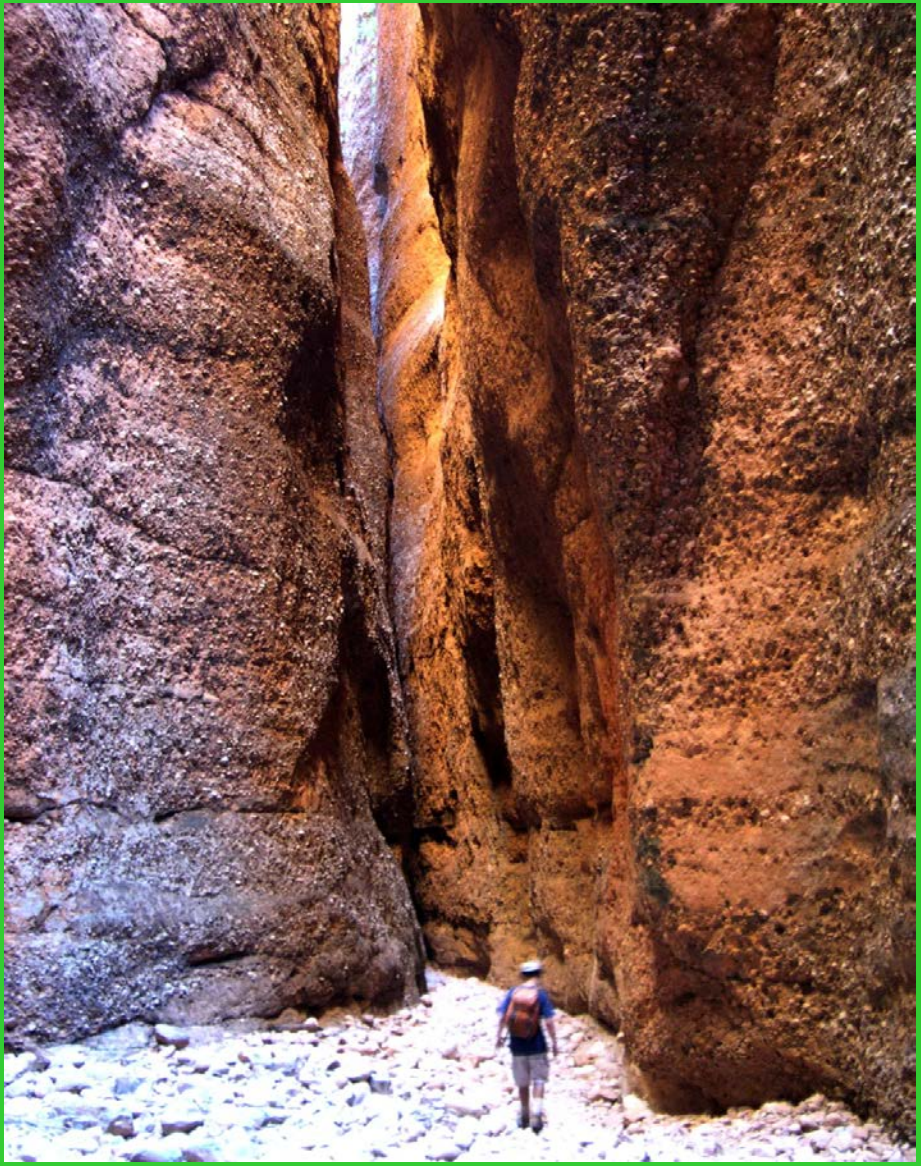
$$\mathcal{F}_Q = \begin{pmatrix}\text{Quantum Fisher information}\\ \text{for squeezed vacuum}\\ \text{input to second input port}\end{pmatrix}$$

$$I_Q = \boxed{\frac{|\alpha|^2+\bar{N}_b}{\dfrac{1-\eta}{\eta}+\dfrac{1}{2\langle(\Delta p)^2\rangle}} \simeq \frac{\eta}{1-\eta}|\alpha|^2}$$

When $|\alpha|^2 \gg \bar{N}_b$, all agree to within corrections of order $\bar{N}_b/|\alpha|^2$.

Optimum achieved by differenced photodetection in a Mach-Zehnder configuration.

III. Beyond the Heisenberg limit. Nonlinear interferometry



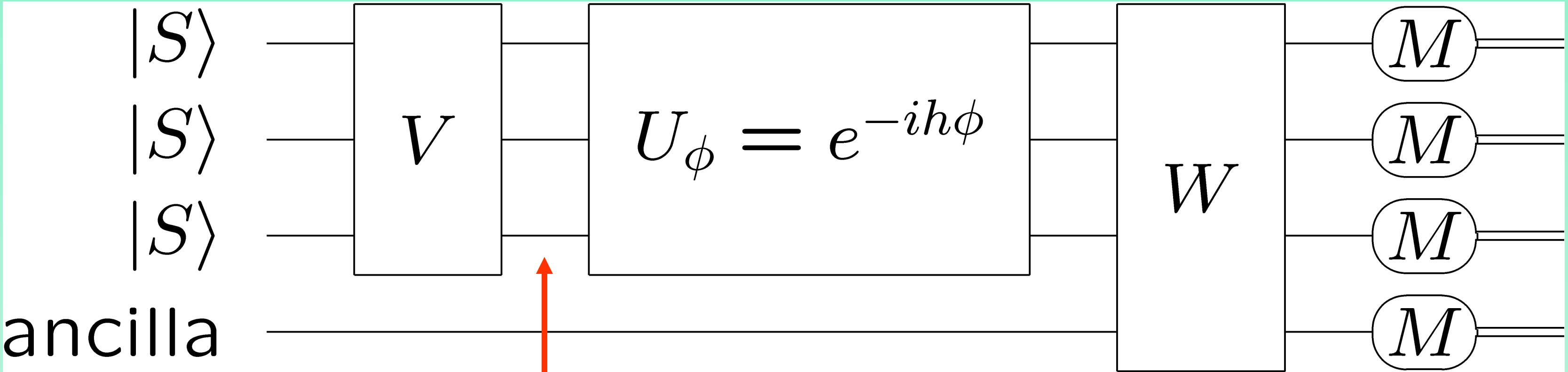
Echidna Gorge
Bungle Bungle Range
Western Australia

Beyond the Heisenberg limit

The purpose of theorems in physics is to lay out the assumptions clearly so one can discover which assumptions have to be violated.

Improving the scaling with N

S. Boixo, S. T. Flammia, C. M. Caves, and JM Geremia, PRL **98**, 090401 (2007).



Cat state does the job.

Nonlinear Ramsey interferometry

$$\Delta\phi \ge \frac{1}{2\Delta h} \ge \frac{1}{||h||} = \frac{1}{N^k(\Lambda^k - \lambda^k)}$$

Metrologically relevant *k*-body coupling

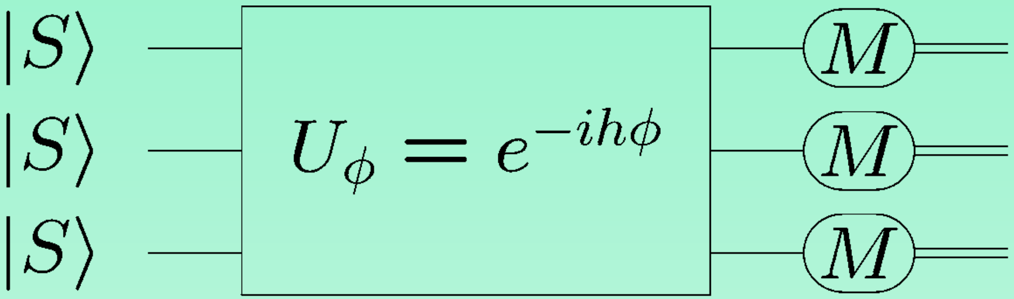
$$h = \left(\sum_{j=1}^{N} h_j\right)^k = \underbrace{\sum_{j_1,\ldots,j_k} h_{j_1} h_{j_2} \cdots h_{j_k}}_{N^k \text{ terms in sum}}$$

$$||h|| = N^k(\Lambda^k - \lambda^k)$$

Improving the scaling with *N* without entanglement

S. Boixo, A. Datta, S. T. Flammia, A. Shaji, E. Bagan, and C. M. Caves, PRA 77, 012317 (2008).

Product input

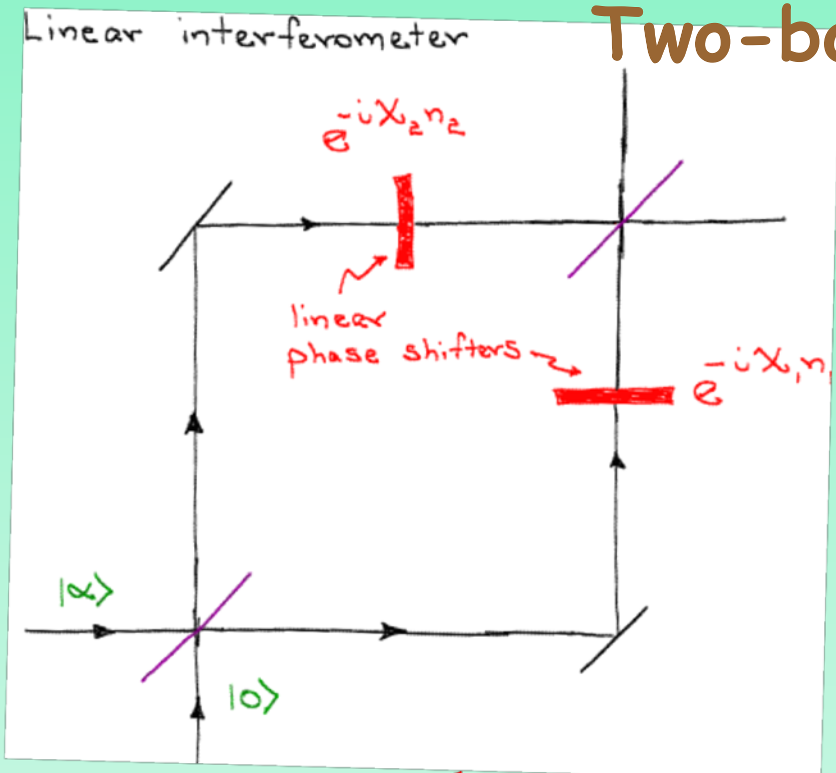


Product measurement

$$h = \left(\sum_{j=1}^{N} Z_j/2 \right)^k = J_z^k$$

$$\Delta\phi \sim \frac{1}{N^{k-1/2}}$$

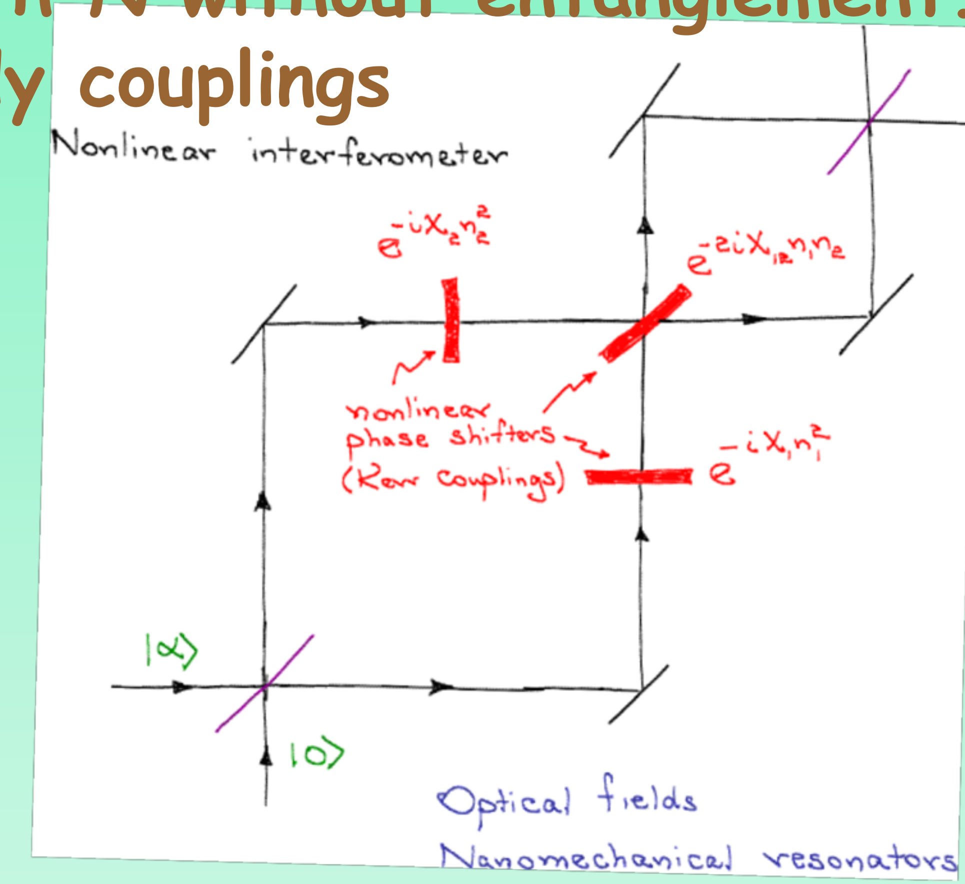
Improving the scaling with N without entanglement. Two-body couplings



$$\chi_1 n_1 + \chi_2 n_2 = \tfrac{1}{2}(\chi_1+\chi_2)N + \underbrace{(\chi_1-\chi_2)}_{\equiv\,\phi}\delta n$$

$$N = n_1 + n_2\,, \quad \delta n = \tfrac{1}{2}(n_1 - n_2)$$

$$\Delta\phi = 1/\sqrt{N}$$



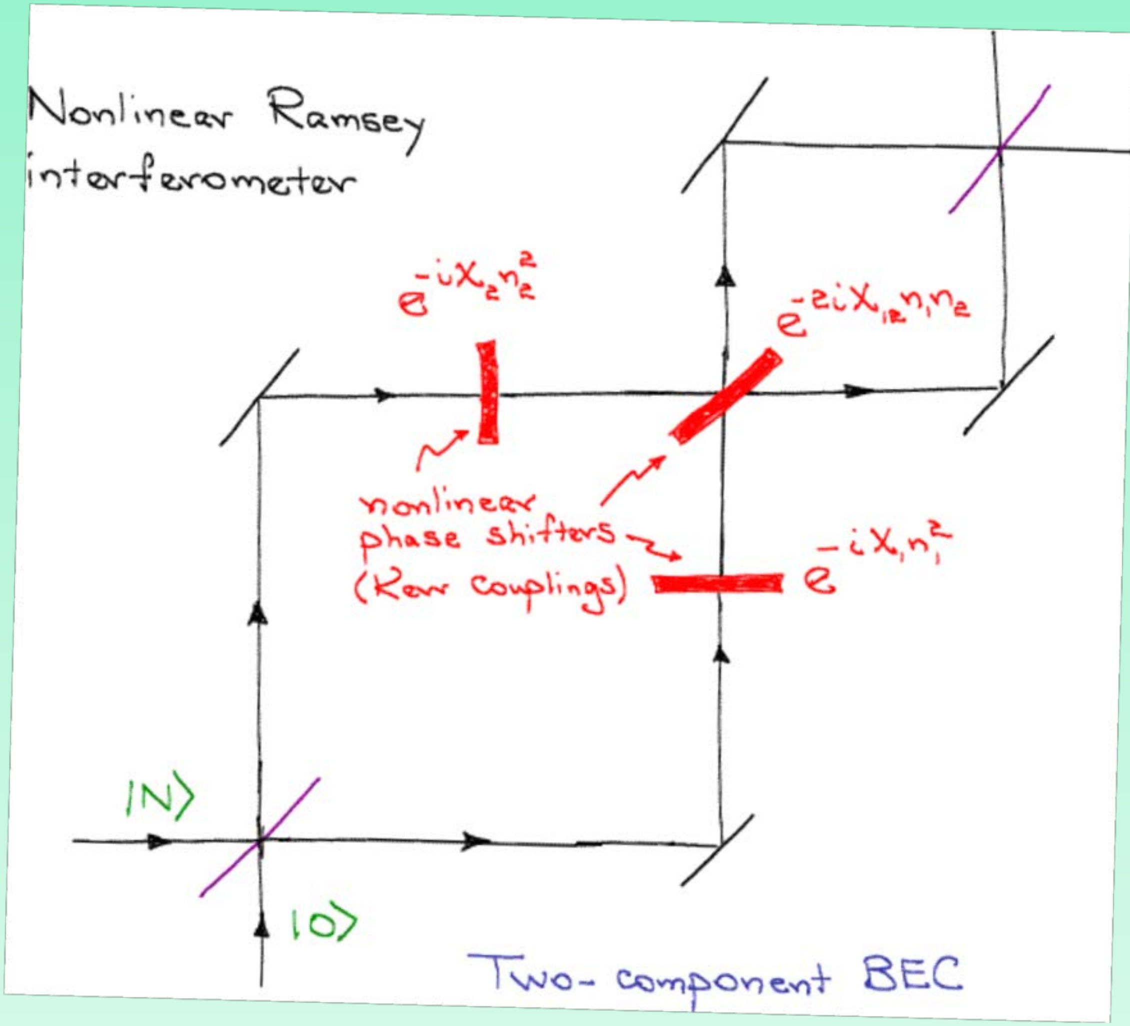
$$\chi_1 n_1^2 + \chi_2 n_2^2 + 2\chi_{12} n_1 n_2 = \tfrac{1}{4}(\chi_1+\chi_2+2\chi_{12})N^2 + (\chi_1-\chi_2)N\delta n + (\chi_1+\chi_2-2\chi_{12})\delta n^2$$

$$\Delta\phi = 1/N^{3/2}$$

Loss and decoherence?

S. Boixo, A. Datta, S. T. Flammia, A. Shaji, E. Bagan, and C. M. Caves, PRA 77, 012317 (2008); M. J. Woolley, G. J. Milburn, and C. M. Caves, NJP 10, 125018 (2008).

Improving the scaling with *N* without entanglement. Two-body couplings



$$\chi_1 n_1^2 + \chi_2 n_2^2 + 2\chi_{12} n_1 n_2$$
$$= \tfrac{1}{4}(\chi_1 + \chi_2 + 2\chi_{12})N^2$$
$$+\underbrace{(\chi_1 - \chi_2)}_{\equiv \phi} N\delta n$$
$$+\underbrace{(\chi_1 + \chi_2 - 2\chi_{12})}_{\simeq 0}\delta n^2$$

$$\Delta\phi = 1/N^{3/2}$$

Super-Heisenberg scaling from nonlinear dynamics (*N*-enhanced rotation of a spin coherent state), without any particle entanglement

S. Boixo, A. Datta, M. J. Davis, S. T. Flammia, A. Shaji, and C. M. Caves, PRL 101, 040403 (2008); A. B. Tacla, S. Boixo, A. Datta, A. Shaji, and C. M. Caves, PRA 82, 053636 (2010).

Loss and decoherence?

Improving the scaling with N without entanglement. Optical experiment

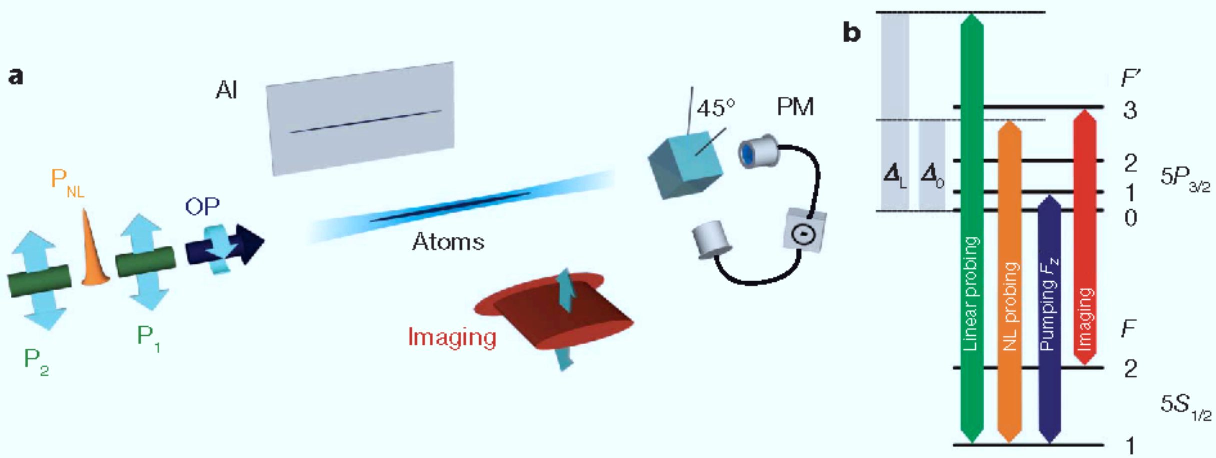
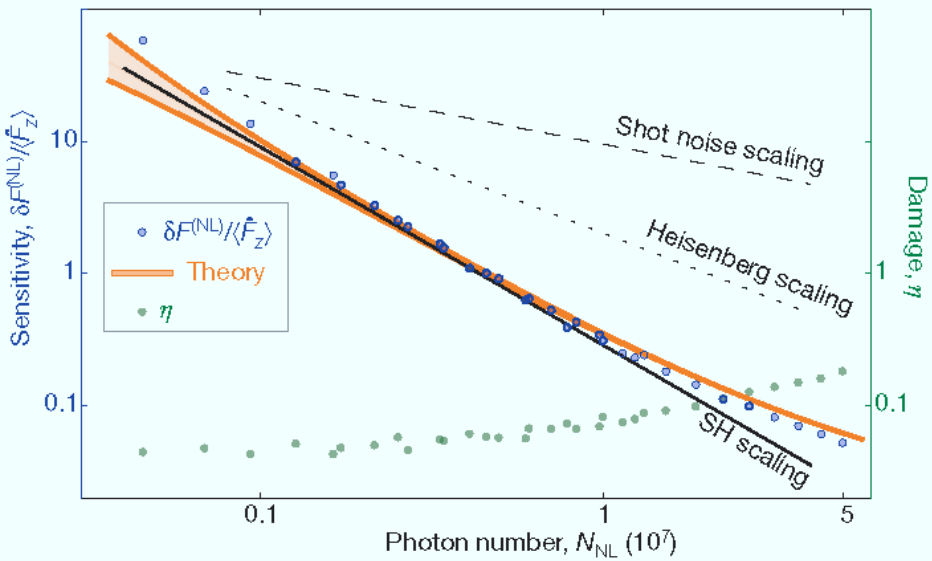


Figure 1 | Atom–light interface. a, Experimental schematic: an ensemble of 7×10^5 ^{87}Rb atoms, held in an optical dipole trap, is prepared in the state $|F = 1, m_F = 1\rangle$ by optical pumping (OP). Linear (P_1, P_2) and nonlinear (P_{NL}) Faraday rotation probe pulses (in the order P_1, P_{NL}, P_2) measure the atomic magnetization, detected by a shot-noise-limited polarimeter (PM). The atom number is measured by quantitative absorption imaging (AI). **b**, Spectral positions of the pump, probe and imaging light on the D_2 transition.



M. Napolitano, M. Koschorreck, B. Dubost, N. Behbood, R. J. Sewell, and M. W. Mitchell, Nature 471, 486 (2011).

Quantum metrology: An information-theoretic perspective

Lecture 3

I. Introduction. What's the problem?

II. Standard quantum limit (SQL) for force detection. The right wrong story

III. Beating the SQL. Three strategies

Carlton M. Caves

Center for Quantum Information and Control, University of New Mexico

Centre for Engineered Quantum Systems, University of Queensland

http://info.phys.unm.edu/~caves

CQuIC Center for Quantum Information and Control

I. Introduction. What's the problem?

Pecos Wilderness
Sangre de Cristo Range
Northern New Mexico

Measuring a classical parameter

Phase shift in an (optical) interferometer

- Readout of anything that changes optical path lengths
- Michelson-Morley experiment
- Gravitational-wave detection
- Planck-scale, holographic uncertainties in positions

Torque on or free precession of a collection of spins

- Magnetometer
- Atomic clock

Lectures 1 and 2

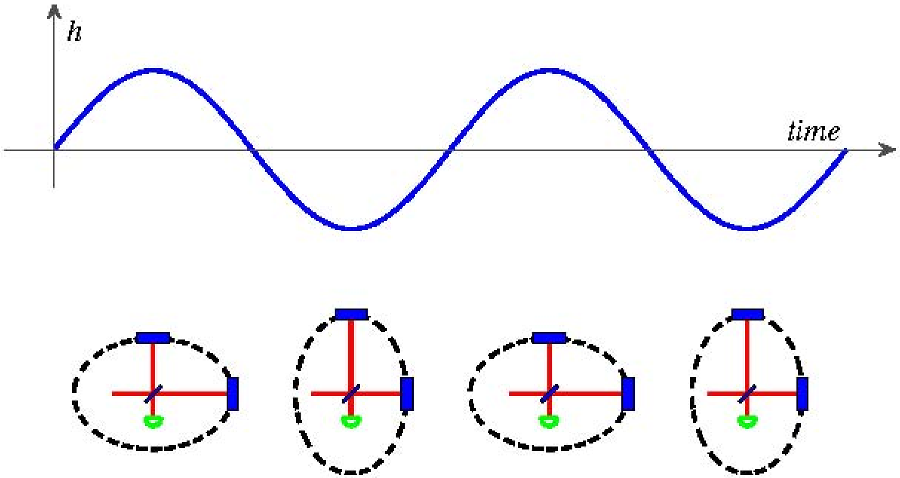
Force on a linear system

- Gravitational-wave detection
- Accelerometer
- Gravity gradiometer
- Electrometer
- Strain meter

Lecture 3

(Absurdly) high-precision interferometry for force sensing

Hanford, Washington



The LIGO Collaboration, Rep. Prog. Phys. 72, 076901 (2009).

Laser Interferometer Gravitational Observatory (LIGO)

Livingston, Louisiana

(Absurdly) high-precision interferometry for force sensing

Hanford, Washington

Initial LIGO

$$\begin{pmatrix}\text{differential}\\ \text{strain}\\ \text{sensitivity}\end{pmatrix} \simeq 10^{-21}$$

$$\begin{pmatrix}\text{differential}\\ \text{displacement}\\ \text{sensitivity}\end{pmatrix} \simeq 4\times 10^{-18}\,\text{m}$$

from 40 Hz to 7,000 Hz.

Laser Interferometer Gravitational Observatory (LIGO)

High-power, Fabry-Perot-cavity (multipass), power-recycled interferometers

Livingston, Louisiana

(Absurdly) high-precision interferometry for force sensing

Hanford, Washington

Advanced LIGO

$$\begin{pmatrix}\text{differential}\\ \text{strain}\\ \text{sensitivity}\end{pmatrix} \simeq 3\times 10^{-23}$$

$$\begin{pmatrix}\text{differential}\\ \text{displacement}\\ \text{sensitivity}\end{pmatrix} \simeq 10^{-19}\,\text{m}$$

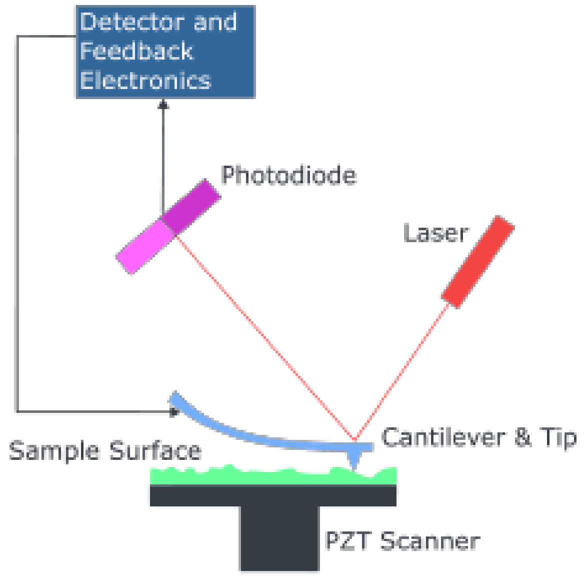
from 10 Hz to 7,000 Hz.

Laser Interferometer Gravitational Observatory (LIGO)

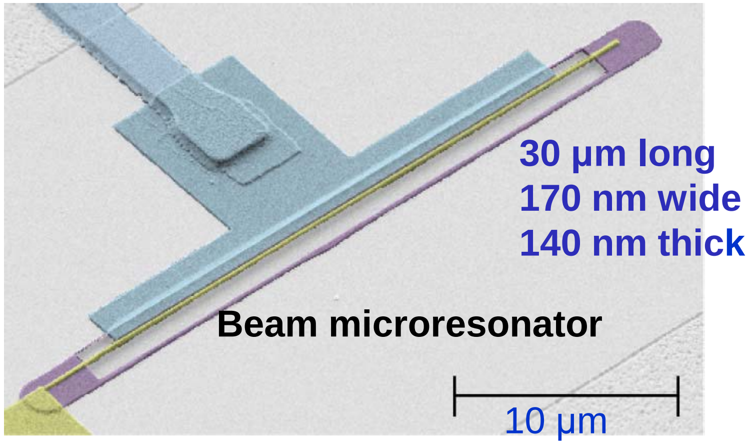
Livingston, Louisiana

High-power, Fabry-Perot-cavity (multipass), power- and signal-recycled, squeezed-light interferometers

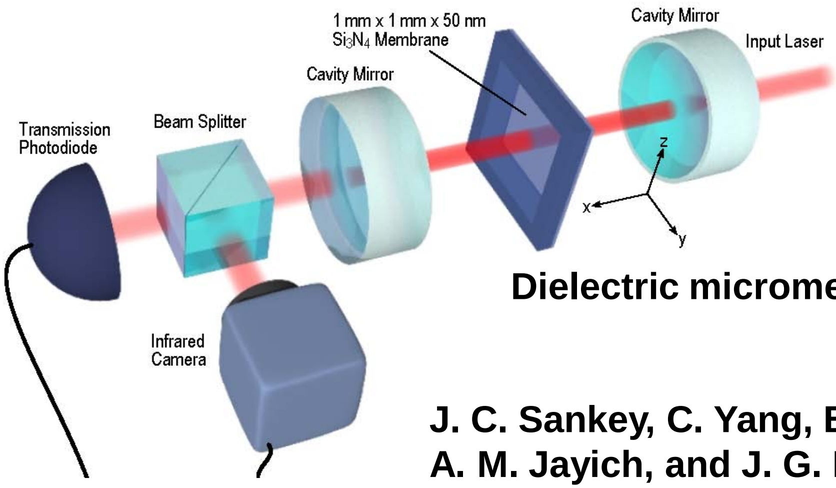
Opto, atomic, electro micromechanics



Atomic force microscope



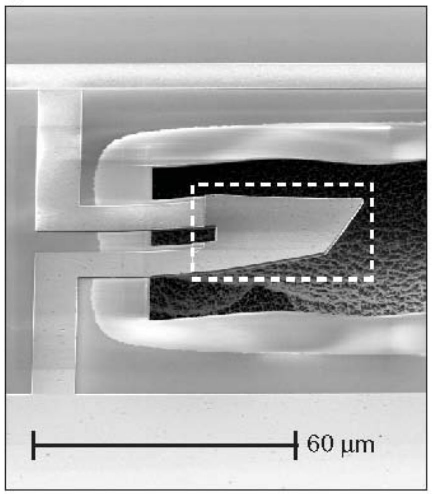
T. Rocheleau, T. Ndukum, C. Macklin, J. B. Hertzberg, A. A. Clerk, and K. C. Schwab, Nature 463, 72 (2010).

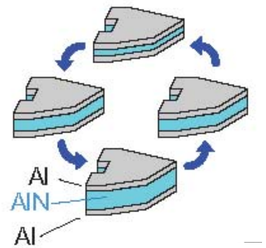


Dielectric micromembrane

J. C. Sankey, C. Yang, B. M. Zwickl, A. M. Jayich, and J. G. E. Harris, Nature Physics 6, 707 (2010).

Opto,atomic, electro micromechanics

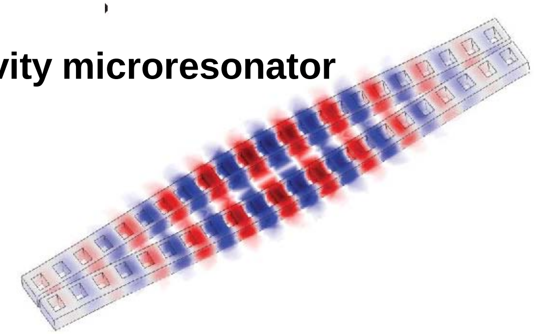


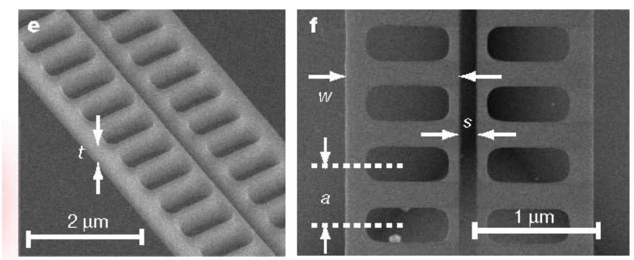


Drum microresonator

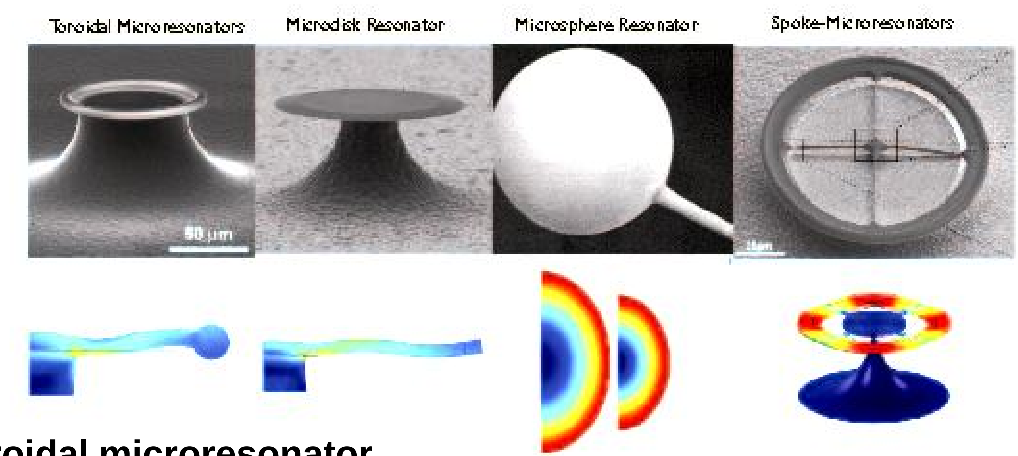
A. D. O'Connell *et al.*, Nature 464, 697 (2010).

Zipper-cavity microresonator





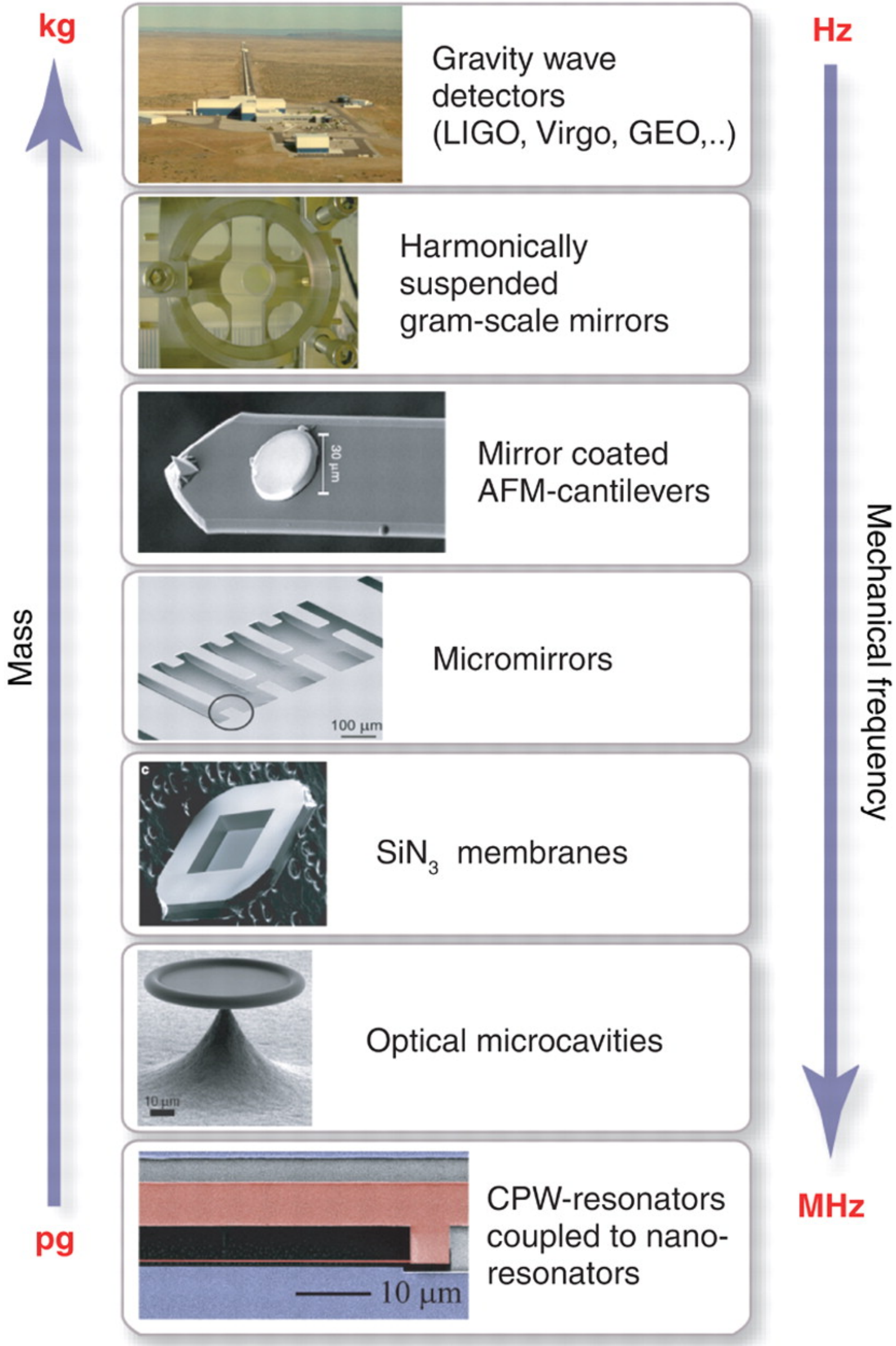
M. Eichenfield, R. Camacho, J. Chan, K. J. Vahala, and O. Painter, Nature 459, 550 (2009).



Toroidal microresonator

A. Schliesser and T. J. Kippenberg, Advances in Atomic, Molecular, and Optical Physics, Vol. 58, (Academic Press, San Diego, 2010), p. 207.

Mechanics for force sensing



T. J. Kippenberg and K. J. Vahala, Science 321, 172 (2008).

Science AAAS

Standard quantum limit (SQL)

Wideband detection of force *f* on free mass *m*

LIGO interferometer

$$\Delta q \simeq \sqrt{\Delta q_0^2 + \boxed{\frac{\Delta p_0^2 \tau^2}{m^2}}} \geq \sqrt{\frac{2\tau \Delta q_0 \Delta p_0}{m}} \geq \sqrt{\frac{\hbar \tau}{m}} \equiv \Delta q_{\mathsf{SQL}}$$

Back action

$$\delta q \simeq \frac{f\tau^2}{2m} \quad \Longrightarrow \quad f_{\mathsf{SQL}} \equiv \frac{2m}{\tau^2}\Delta q_{\mathsf{SQL}} = \sqrt{\frac{4\hbar m}{\tau^3}}$$

$$m \simeq 50\,\mathsf{kg}, \qquad \Delta\nu = 1/\tau \simeq 100\,\mathsf{Hz}$$

$$\Longrightarrow \quad \Delta q_{\mathsf{SQL}} \simeq 10^{-19}\,\mathsf{m}, \qquad f_{\mathsf{SQL}} \simeq 100\,\mathsf{fN}$$

Standard quantum limit (SQL)

Narrowband, on-resonance detection of force *f* on oscillator of mass *m* and resonant frequency ω_0

Nanoresonator

$$\Delta q_{\mathrm{SQL}} \equiv \sqrt{\frac{\hbar}{2m\omega_0}}$$

Back action?

$$\delta q \simeq \frac{f\tau}{2m\omega_0} \quad \Longrightarrow \quad f_{\mathrm{SQL}} \equiv \frac{2m\omega_0}{\tau}\Delta q_{\mathrm{SQL}} = \sqrt{\frac{2\hbar m\omega_0}{\tau^2}}$$

$$m \simeq 10\,\mathrm{pg}, \quad 1/\tau_0 = \omega_0/2\pi \simeq 10\,\mathrm{MHz}, \quad Q \simeq 10^4\text{–}10^6$$

$$\Longrightarrow \quad \Delta q_{\mathrm{SQL}} \simeq 10\,\mathrm{fm}, \quad f_{\mathrm{SQL}} \simeq 100\,\mathrm{fN} \times \frac{\tau_0}{\tau}$$

$$\left(\begin{array}{c}\text{force between two}\\ \text{Bohr magnetons}\\ \text{separated by } r = 1\,\mathrm{nm}\end{array}\right) = \frac{\mu_0}{4\pi} \times \frac{\mu_B^2}{r^4} \simeq 10\,\mathrm{aN}$$

$$\mu_B = e\hbar/2m_e c = e\lambda_c/4\pi \simeq e \times 0.2\,\mathrm{pm}$$

SQL

Wideband force *f* on free mass *m*

$$\Delta q_{\mathrm{SQL}} = \sqrt{\frac{\hbar\tau}{m}} \qquad f_{\mathrm{SQL}} = \sqrt{\frac{4\hbar m}{\tau^3}} = \Delta\nu\sqrt{4\hbar m(\Delta\nu)}$$

On-resonance force *f* on oscillator of mass *m* and resonant frequency ω_0

$$\Delta q_{\mathrm{SQL}} = \sqrt{\frac{\hbar}{2m\omega_0}} \qquad f_{\mathrm{SQL}} = \sqrt{\frac{2\hbar m\omega_0}{\tau^2}} = \Delta\nu\sqrt{2\hbar m\omega_0}$$

It's wrong.

It's not even the right wrong story.

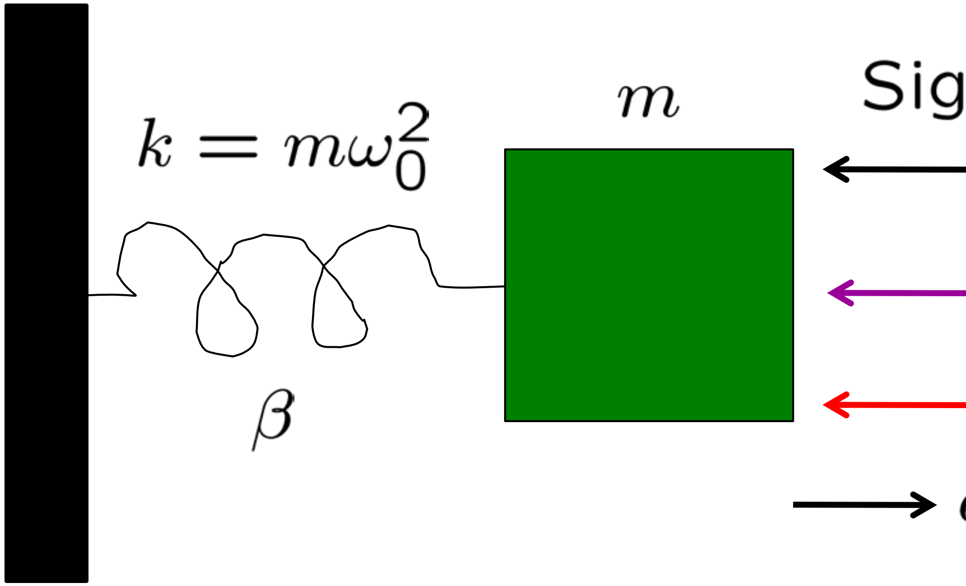
The right wrong story. Waveform estimation.

$$S_{\mathrm{SQL}}(\omega) = \frac{\hbar}{|G(\omega)|} = \hbar m\sqrt{(\omega^2-\omega_0^2)^2+4\beta^2\omega^2}$$

II. Standard quantum limit (SQL) for force detection. The right wrong story

San Juan River canyons
Southern Utah

SQL for force detection



Signal force $f(t)$ (waveform)

coupling $-qf(t)$

Langevin force $L(t)$

Back-action force $\xi(t)$

$q(t)$

Monitor position

Readout $y(t) = q(t) + \eta(t)$

Measurement noise

Back-action force

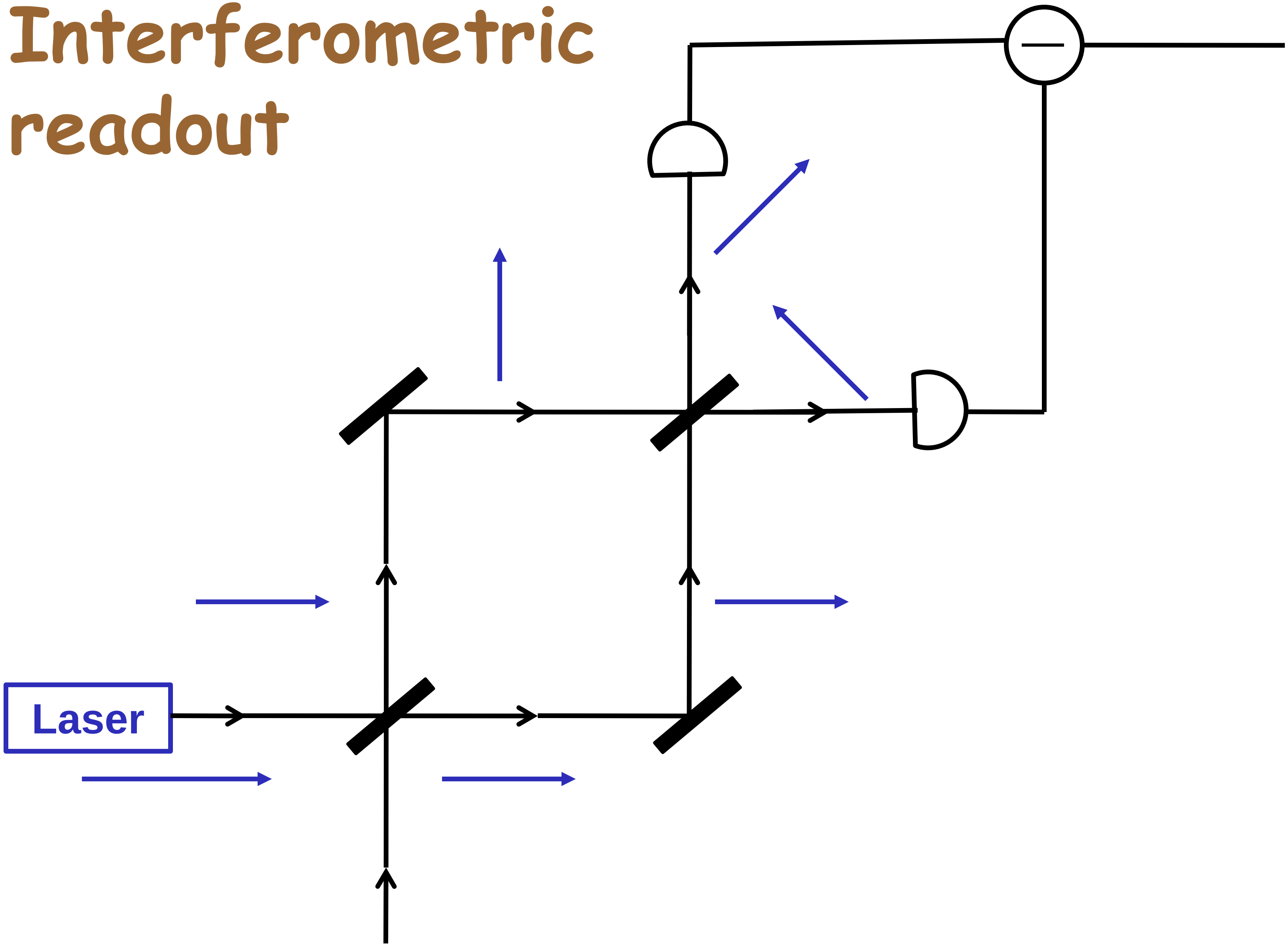
Langevin force

$$\ddot{q} + 2\beta\dot{q} + \omega_0^2 q = \frac{f(t)}{m} + \frac{\xi(t)}{m} + \frac{L(t)}{m}$$

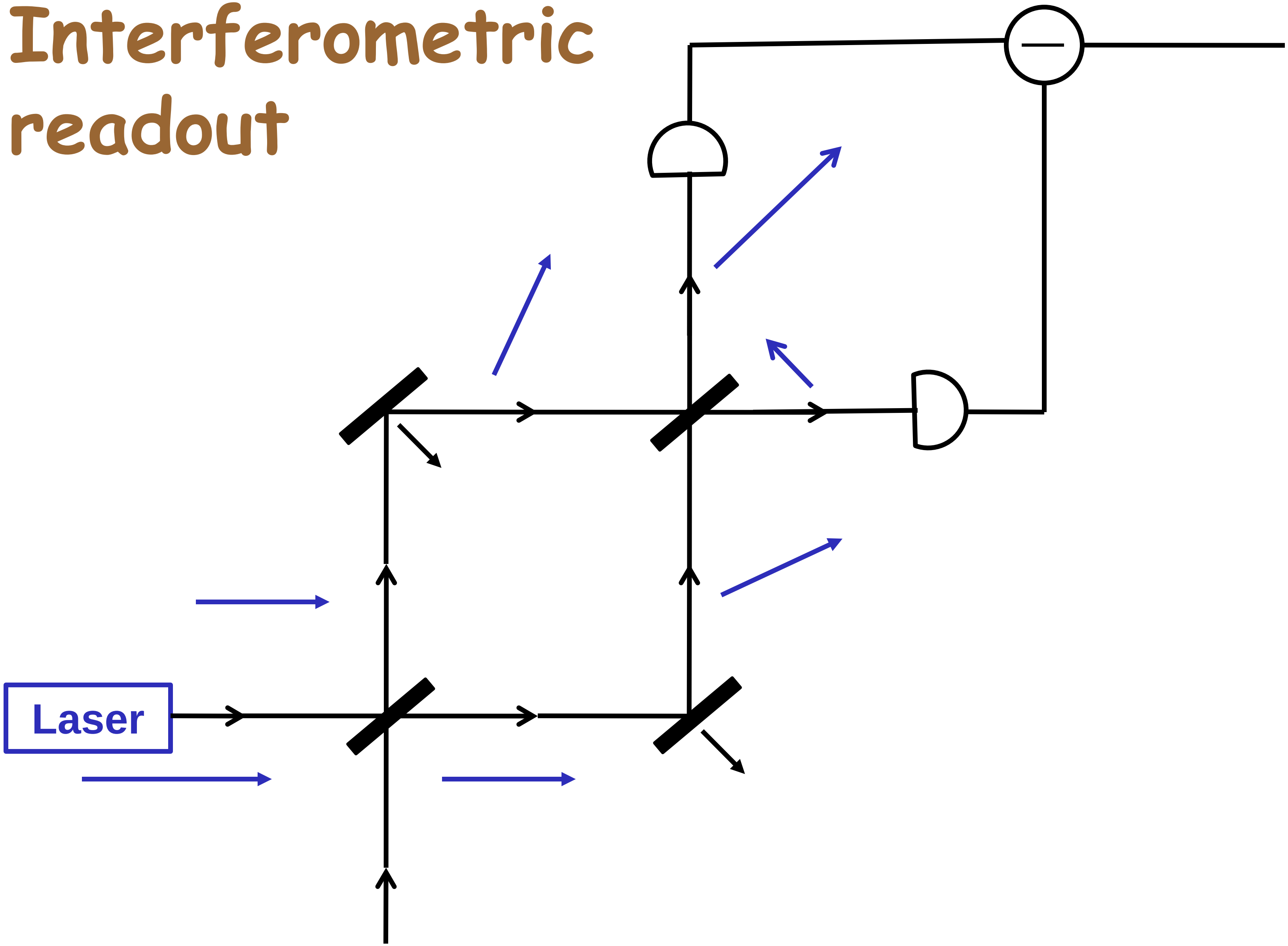
$$y(t) = q(t) + \eta(t)$$

measurement (shot) noise

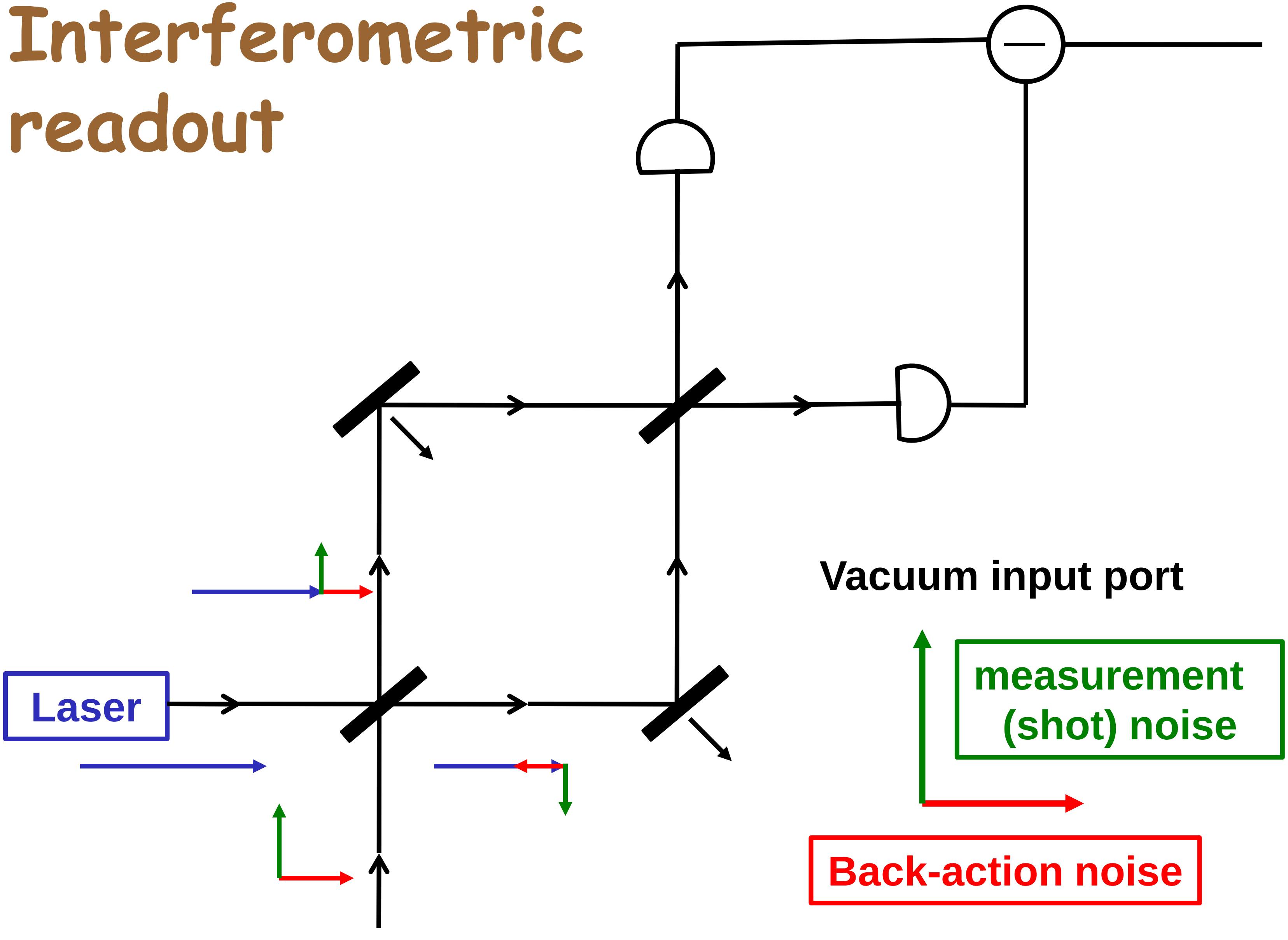
Interferometric readout



Interferometric readout

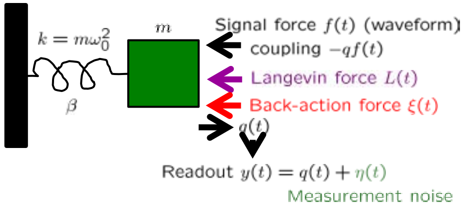


Interferometric readout



If shot noise dominates, squeeze the phase quadrature.

SQL for force detection



Time domain

Back-action force

Langevin force

$$\ddot{q} + 2\beta\dot{q} + \omega_0^2 q = \frac{f(t)}{m} + \frac{\xi(t)}{m} + \frac{L(t)}{m}$$

$$y(t) = q(t) + \eta(t)$$

measurement noise

$$q(\omega) = G(\omega)[f(\omega) + \xi(\omega) + L(\omega)]$$

Frequency domain

$$\begin{pmatrix} \text{response or} \\ \text{transfer function} \end{pmatrix} = G(\omega) \equiv \frac{1}{m(\omega_0^2 - \omega^2 - 2i\beta\omega)}$$

Back-action force

$$z(\omega) = \frac{1}{G(\omega)} y(\omega) = f(\omega) + \frac{\eta(\omega)}{G(\omega)} + \xi(\omega) + L(\omega)$$

Langevin force

measurement noise

Noise-power spectral densities

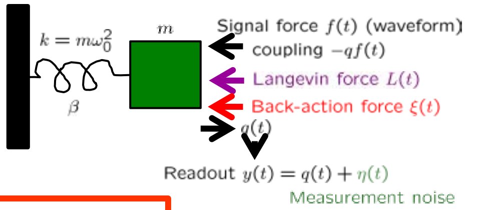
Zero-mean, time-stationary random process *u(t)*

$$u(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, u(\omega) e^{-i\omega t} \qquad u(\omega) = \int_{-\infty}^{\infty} dt\, u(t) e^{i\omega t}$$

Noise-power spectral density of *u*

$$\langle u^2 \rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, S_u(\omega)$$

SQL for force detection



Back-action force

$$z(\omega) = \frac{1}{G(\omega)} y(\omega) = f(\omega) + \boxed{\frac{\eta(\omega)}{G(\omega)}} + \boxed{\xi(\omega)} + \boxed{L(\omega)}$$

Langevin force

measurement noise

$$S_{\Delta z}(\omega) = \boxed{\frac{S_\eta(\omega)}{|G(\omega)|^2}} + \boxed{S_\xi(\omega)} + \boxed{S_L(\omega)}$$

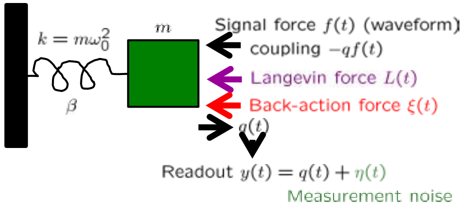
Fluctuation-dissipation theorem

$$S_L(\omega) = 4m\beta\hbar\omega \left(\frac{1}{2} + \frac{1}{e^{\hbar\omega/kT} - 1} \right)$$

Quantum mechanics of continuous measurement

$$S_\eta(\omega) S_\xi(\omega) \geq \hbar^2/4$$

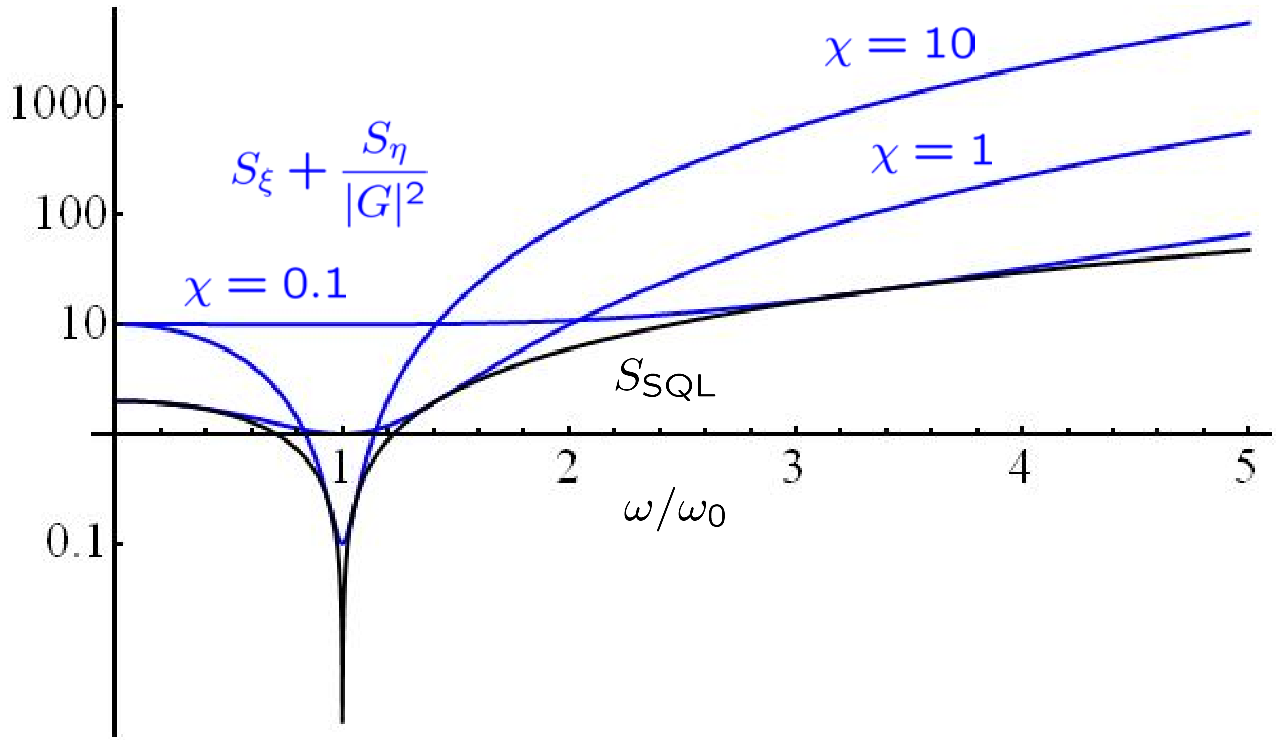
SQL for force detection



$$S_{\Delta z}(\omega) = \frac{S_\eta(\omega)}{|G(\omega)|^2} + S_\xi(\omega) \geq \frac{2\sqrt{S_\eta(\omega) S_\xi(\omega)}}{|G(\omega)|} \geq \frac{\hbar}{|G(\omega)|} \equiv S_{\mathsf{SQL}}(\omega)$$

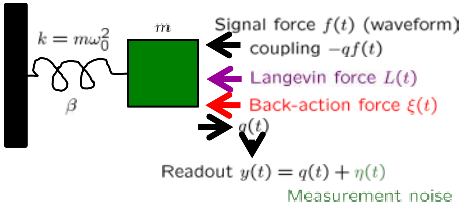
$$= \iff S_\eta(\omega) = S_\xi(\omega)|G(\omega)|^2$$

$$= \iff S_\eta(\omega) S_\xi(\omega) = \hbar^2/4$$



$$S_\eta S_\xi = \hbar^2/4 \qquad \frac{m^2\omega_0^4 S_\eta}{S_\xi} \equiv \chi \qquad Q = \frac{\omega_0}{2\beta} = 10^3$$

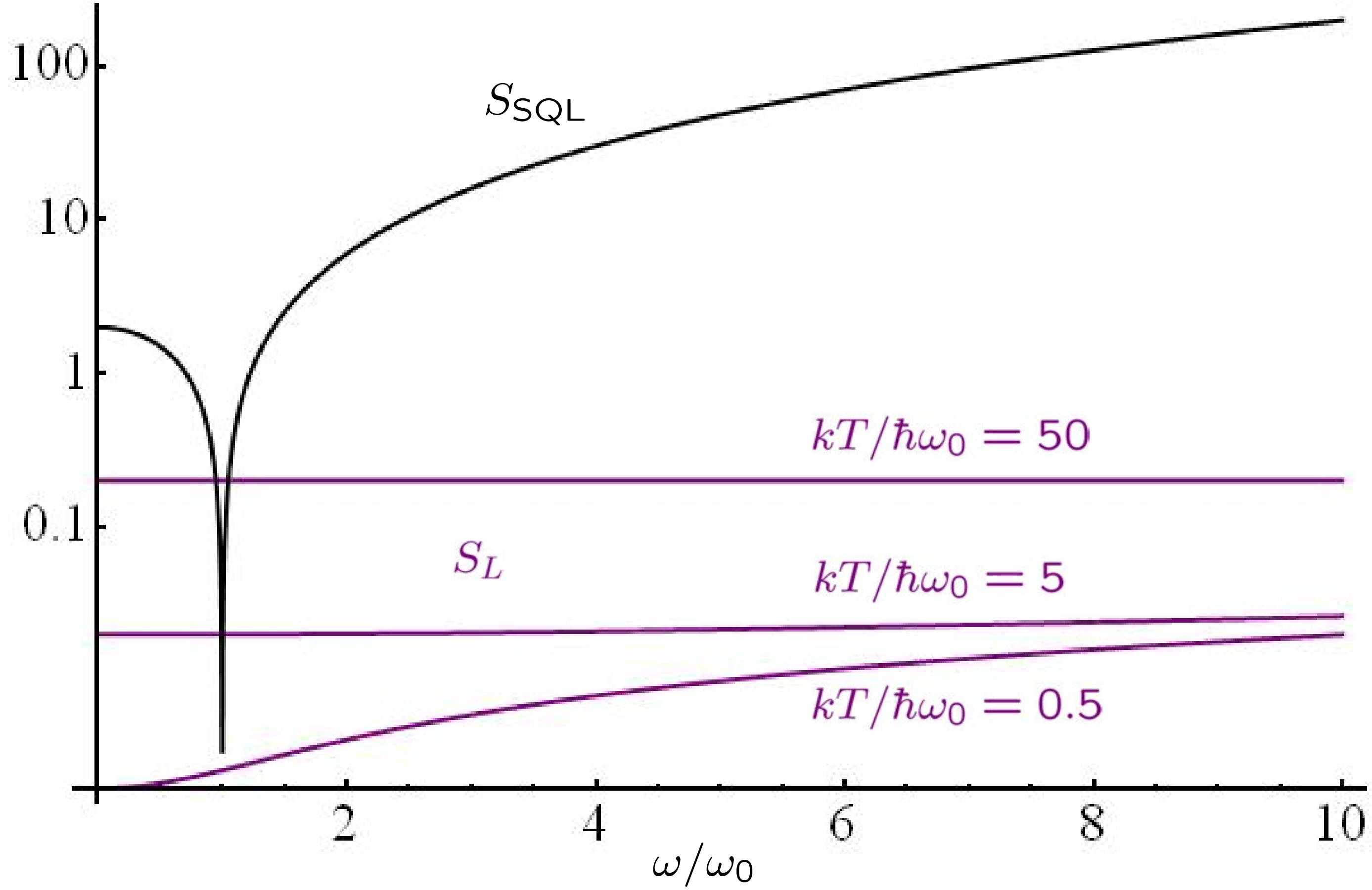
Langevin force



$$S_L(\omega) = 4m\beta\hbar\omega\left(\frac{1}{2} + \frac{1}{e^{\hbar\omega/kT} - 1}\right)$$

$$S_{\mathrm{SQL}}(\omega) = \frac{\hbar}{|G(\omega)|} = \hbar m\sqrt{(\omega^2 - \omega_0^2)^2 + 4\beta^2\omega^2}$$

$$Q = \frac{\omega_0}{2\beta} = 10^3$$



SQL for force detection

$$S_{\mathsf{SQL}}(\omega) = \frac{\hbar}{|G(\omega)|} = \hbar m \sqrt{(\omega^2 - \omega_0^2)^2 + 4\beta^2\omega^2}$$

The right wrong story.

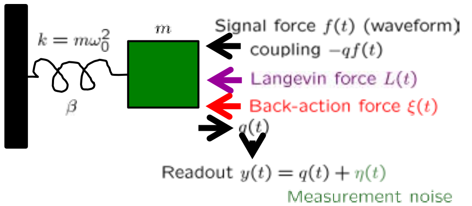
$$\frac{S_\eta(\omega)}{S_\xi(\omega)} = |G(\omega)|^2 \qquad S_\eta(\omega) S_\xi(\omega) = \hbar^2/4$$

In an opto-mechanical setting, achieving the SQL at a particular frequency requires squeezing at that frequency, and achieving the SQL over a wide bandwidth requires frequency-dependent squeezing.

III. Beating the SQL. Three strategies

Truchas from East Pecos Baldy
Sangre de Cristo Range
Northern New Mexico

Beating the SQL. Strategy 1



1. Couple parameter to observable *h*, and monitor observable *o* conjugate to *h.*
2. Arrange that *h* and *o* are *conserved* in the absence of the parameter interaction; *o* is the simplest sort of *quantum nondemolition* (QND) or *back-action-evading* (BAE) observable.
3. Give *o* as small an uncertainty as possible, thereby giving *h* as big an uncertainty as possible (back action).

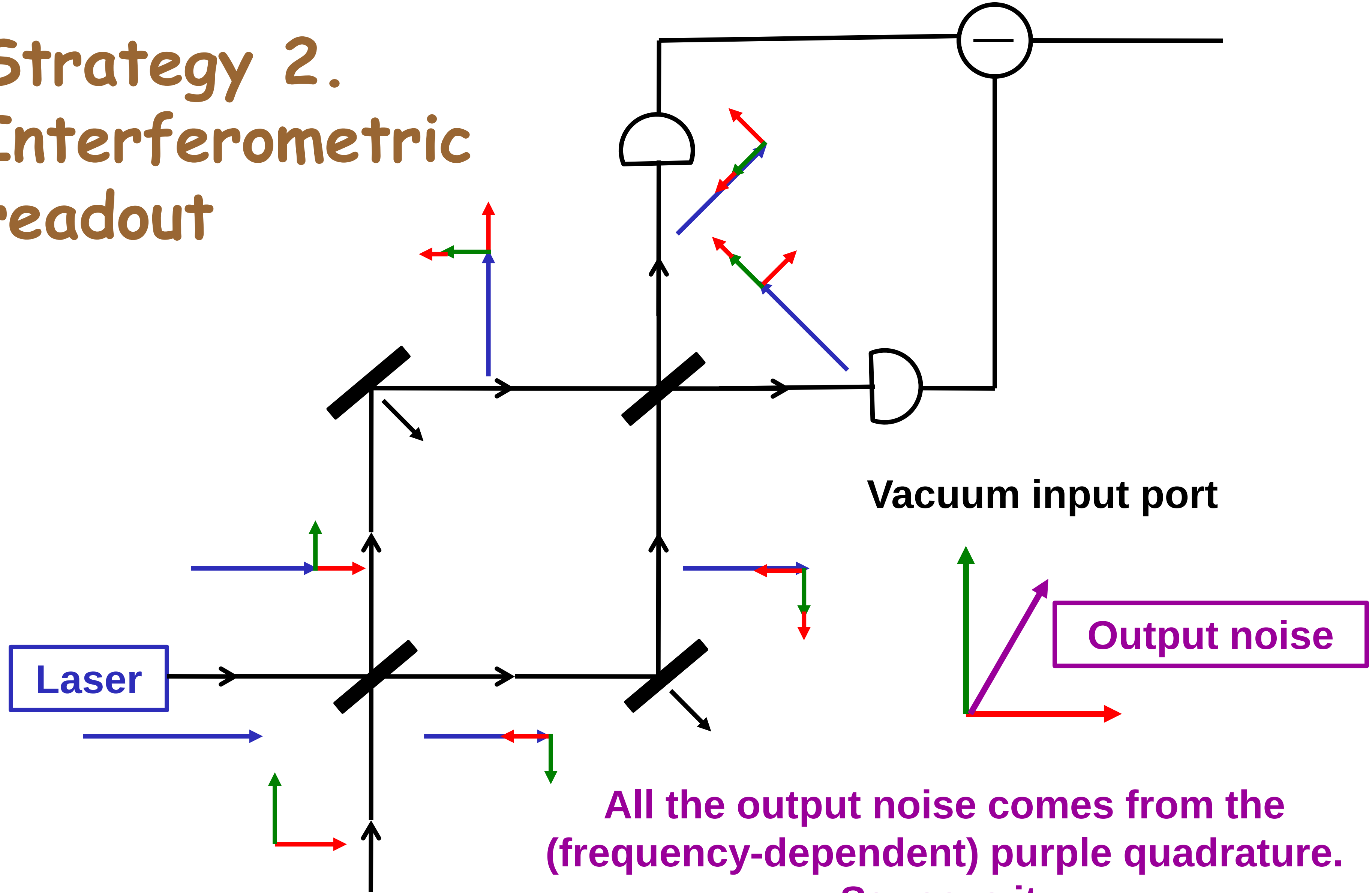
Strategy 1. Monitor a quadrature component.

$$q = \mathrm{Re}[(X_1 + iX_2)e^{-i\omega_0 t}] = X_1 \cos\omega_0 t + X_2 \sin\omega_0 t$$
$$p/m\omega_0 = \mathrm{Im}[(X_1 + iX_2)e^{-i\omega_0 t}] = -X_1 \sin\omega_0 t + X_2 \cos\omega_0 t$$

Downsides

1. **Detect only one quadrature of the force.**
2. **Mainly narrowband (no convenient free-mass version).**
3. **Need new kind of coupling to monitor oscillator.**

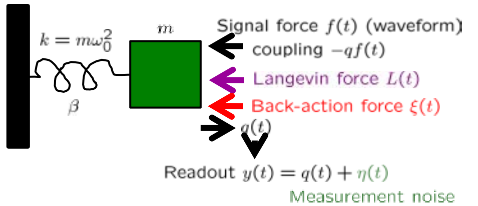
Strategy 2. Interferometric readout



All the output noise comes from the (frequency-dependent) purple quadrature. Squeeze it.

W. G. Unruh, in Quantum Optics, Experimental Gravitation, and Measurement Theory, edited by P. Meystre and M. O. Scully (Plenum, 1983), p. 647; F. Ya. Khalili, PRD 81, 122002 (2010).

Beating the SQL. Strategy 2



Strategy 2. Squeeze the entire output noise by correlating the measurement and back-action noise.

$$y(\omega) = \underbrace{G(\omega) f(\omega) + G(\omega)\xi(\omega)}_{= q(\omega)} + \eta(\omega)$$

Squeeze this output noise by correlating η and ξ. Quantum mechanics requires that an orthogonal linear combination of η and $G\xi$ become very noisy, thus making η, ξ, *and* q very noisy.

Quantum Cramér-Rao Bound (QCRB)

Single-parameter estimation: Bound on the error in estimating a classical parameter that is coupled to a quantum system in terms of the inverse of the quantum Fisher information.

Multi-parameter estimation: Bound on the covariance matrix in estimating a set of classical parameters that are coupled to a quantum system in terms of the inverse of a quantum Fisher-information matrix.

Waveform estimation: Bound on the continuous covariance matrix for estimating a continuous waveform that is coupled to a quantum system in terms of the inverse of a continuous, two-time quantum Fisher-information matrix.

Waveform QCRB. Spectral uncertainty principle

M. Tsang, H. M. Wiseman, and C. M. Caves,
PRL 106, 090401 (2011).

$$S_{\text{est}}(\omega)\left(S_{\Delta q}(\omega)+\frac{\hbar^2}{4S_{\Delta f}(\omega)}\right)\geq\frac{\hbar^2}{4}$$

$$S_{\Delta q}(\omega)=|G(\omega)|^2 S_\xi(\omega)$$

Prior-information term

At frequencies where there is little prior information,

$$S_{\text{est}}(\omega)\geq\frac{\hbar^2}{4S_{\Delta q}(\omega)}=\frac{1}{|G(\omega)|^2}\frac{\hbar^2}{4S_\xi(\omega)}=\frac{S_\eta(\omega)}{|G(\omega)|^2}$$

Minimum-uncertainty noise

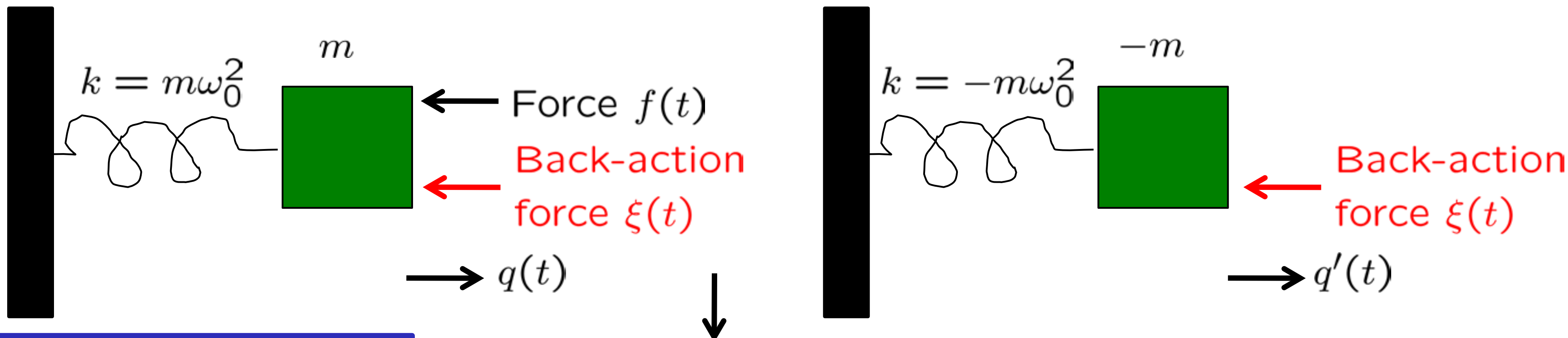
No hint of SQL—no back-action noise, only measurement noise—but can the bound be achieved?

Beating the SQL. Strategy 3

Strategy 3. Quantum noise cancellation (QNC) using oscillator and negative-mass oscillator.

Primary oscillator

Negative-mass oscillator



Monitor collective position Q

Readout $y(t) = Q(t) + \eta(t) = q(t) + q'(t) + \eta(t)$

Measurement noise

Conjugate pairs

$$Q = q + q' \qquad P = (p + p')/2$$

$$\delta q = (q - q')/2 \qquad \delta p = p - p'$$

Oscillator pairs

QCRB

$$S_{\Delta z}(\omega) = \frac{S_\eta(\omega)}{|G(\omega)|^2}$$

Quantum noise cancellation

M. Tsang and C. M. Caves, PRL 105,123601 (2010).

Oscillator (q,p) and negative-mass oscillator (q',p')

Conjugate pairs

$$Q = q + q' \longleftrightarrow P = (p + p')/2$$

$$\delta q = (q - q')/2 \longleftrightarrow \delta p = p - p'$$

Oscillator pairs

Back-action noise in q and q' cancels in $Q = q + q'$

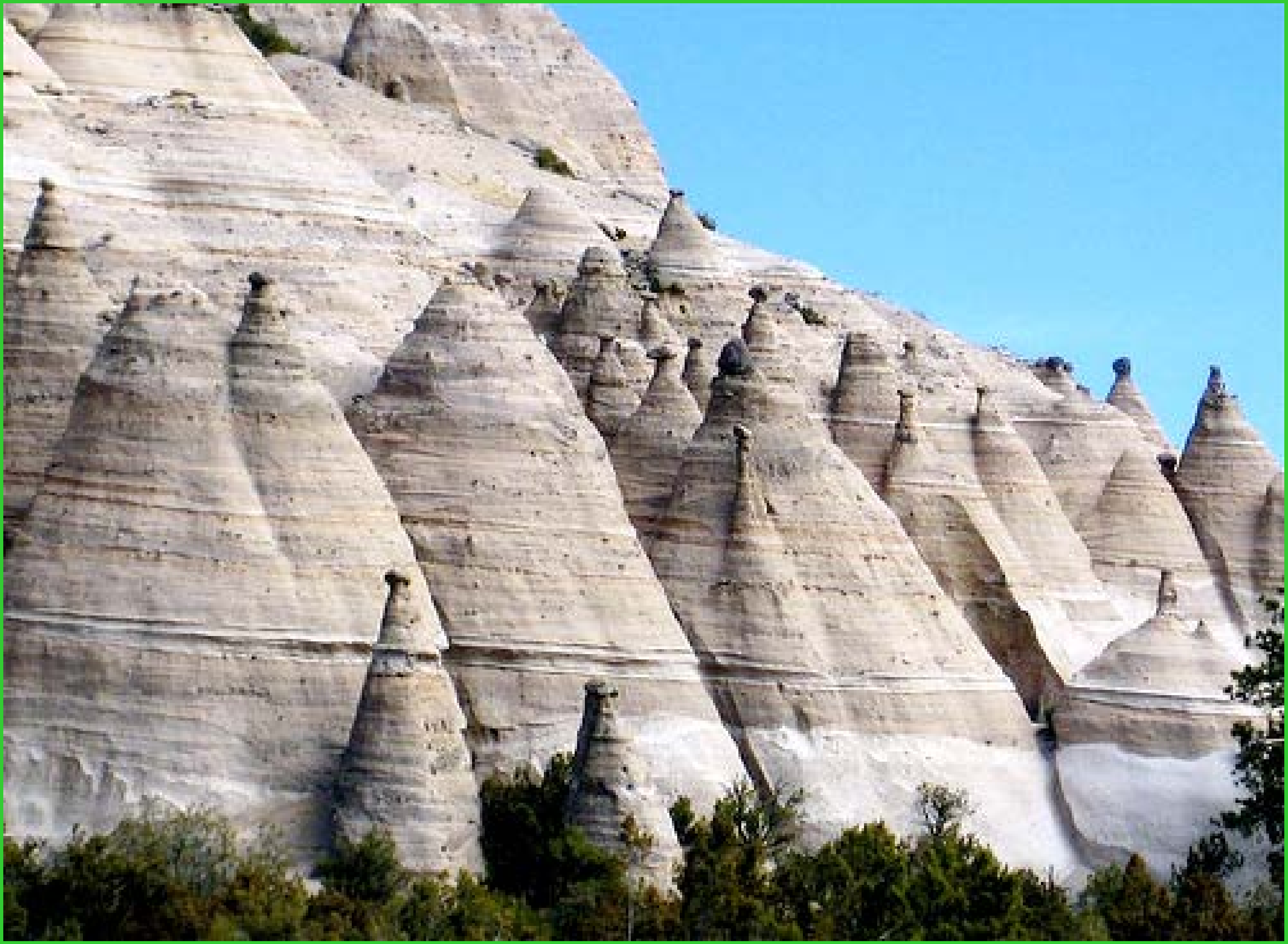
OR

$Q = q + q'$ is a new BAE observable, which, rather than being conserved, acts just like oscillator position, responding to a force in the same way.

Paired sidebands about a carrier frequency
Paired collective spins, polarized along opposite directions

W. Wasilewski , K. Jensen, H. Krauter, J. J. Renema, M. V. Balbas, and E. S. Polzik, PRL 104, 133601 (2010).

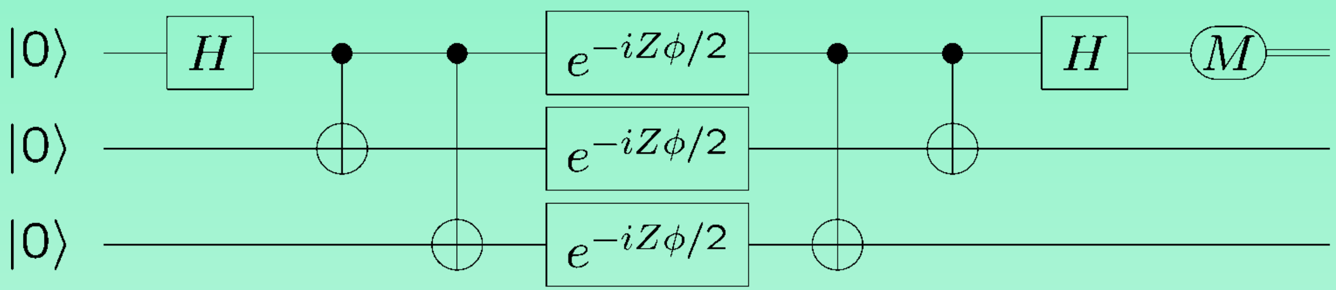
That's all. Thanks for your attention.

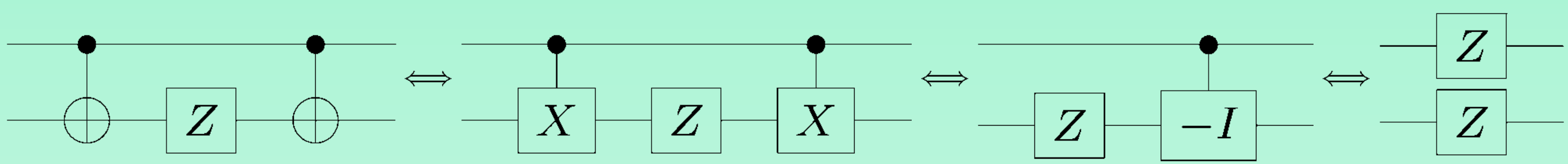


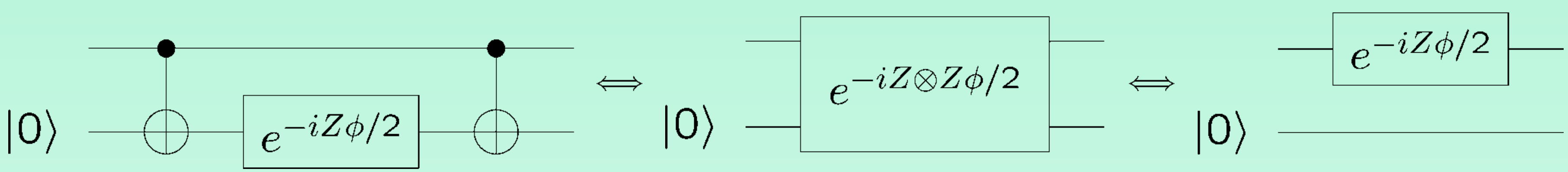
Tent Rocks
Kasha-Katuwe National Monument
Northern New Mexico

Using quantum circuit diagrams

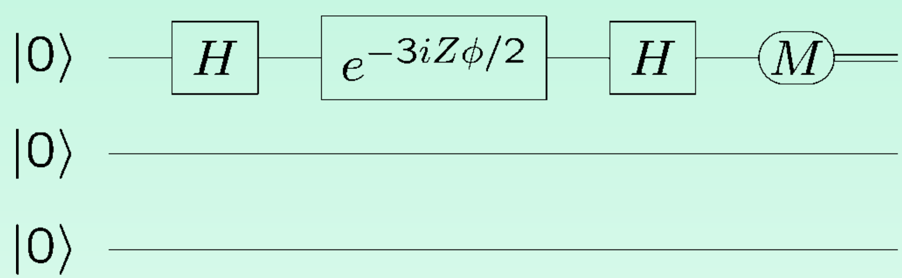
Cat-state interferometer







Cat-state interferometer ⟺



2.I

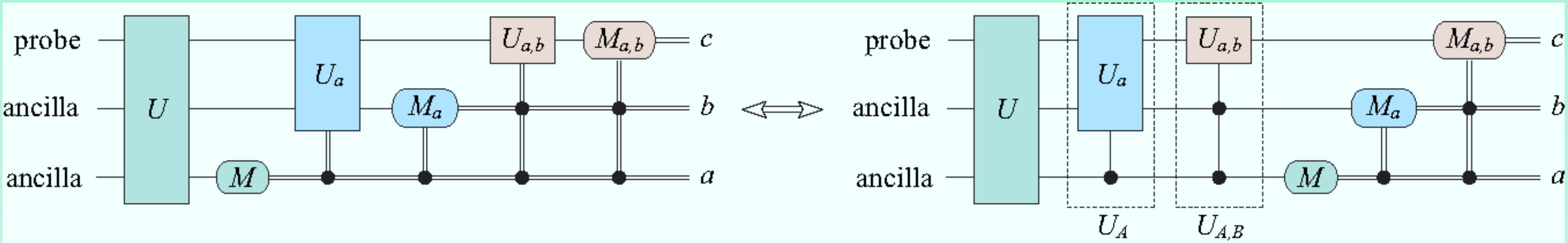
2.II

C. M. Caves and A. Shaji, Opt. Commun. 283, 695 (2010) .

Proof of QCRB. Setting

$$H_\gamma(t) = \hbar\gamma h + \tilde{H}(t)$$

Parameter γ
Generator of parameter displacements, h
Everything else, $\tilde{H}$



$$\rho_\gamma(t) = U_\gamma(t)\rho_0 U_\gamma^\dagger(t) \qquad i\hbar\frac{\partial U_\gamma(t)}{\partial t} = H_\gamma(t)U_\gamma(t)$$

$$p(\gamma_{\text{est}}|\gamma) = \text{tr}(E_{\gamma_{\text{est}}}\rho_\gamma(t)) \qquad \int d\gamma_{\text{est}}\, E_{\gamma_{\text{est}}} = I$$

Initial state ρ_0
Evolution operator $U_\gamma(t)$
Estimator POVM E_{est}

Proof of QCRB. Classical CRB

$$\Delta\gamma_{\text{est}} \equiv \gamma_{\text{est}} - \langle\gamma_{\text{est}}\rangle$$

$$\delta\gamma \equiv \frac{\gamma_{\text{est}}}{|d\langle\gamma_{\text{est}}\rangle/d\gamma|} - \gamma = \frac{\Delta\gamma_{\text{est}}}{|d\langle\gamma_{\text{est}}\rangle/d\gamma|} + \langle\delta\gamma\rangle$$

$$0 = \int d\gamma_{\text{est}}\, \Delta\gamma_{\text{est}}\, p(\gamma_{\text{est}}|\gamma)$$

Differentiate with respect to γ

$$\begin{aligned}
\frac{d\langle\gamma_{\text{est}}\rangle}{d\gamma} &= \int d\gamma_{\text{est}}\, \Delta\gamma_{\text{est}} \frac{\partial p(\gamma_{\text{est}}|\gamma)}{\partial\gamma} \\
&= \int d\gamma_{\text{est}}\, p(\gamma_{\text{est}}|\gamma) \Delta\gamma_{\text{est}} \frac{\partial \ln p(\gamma_{\text{est}}|\gamma)}{\partial\gamma} \\
&= \left\langle \Delta\gamma_{\text{est}} \frac{\partial \ln p(\gamma_{\text{est}}|\gamma)}{\partial\gamma} \right\rangle
\end{aligned}$$

Proof of QCRB. Classical CRB

$$\begin{aligned}\frac{d\langle\gamma_{\text{est}}\rangle}{d\gamma} &= \int d\gamma_{\text{est}}\,\Delta\gamma_{\text{est}}\frac{\partial p(\gamma_{\text{est}}|\gamma)}{\partial\gamma}\\ &= \int d\gamma_{\text{est}}\,p(\gamma_{\text{est}}|\gamma)\Delta\gamma_{\text{est}}\frac{\partial \ln p(\gamma_{\text{est}}|\gamma)}{\partial\gamma}\\ &= \left\langle\Delta\gamma_{\text{est}}\frac{\partial \ln p(\gamma_{\text{est}}|\gamma)}{\partial\gamma}\right\rangle\end{aligned}$$

$$\left(\frac{d\langle\gamma_{\text{est}}\rangle}{d\gamma}\right)^2 \le \langle(\Delta\gamma_{\text{est}})^2\rangle\left\langle\left(\frac{\partial \ln p(\gamma_{\text{est}}|\gamma)}{\partial\gamma}\right)^2\right\rangle = \langle(\Delta\gamma_{\text{est}})^2\rangle F(\gamma)$$

Schwarz inequality

Classical Fisher information $F(\gamma)$

$$\langle(\delta\gamma)^2\rangle = \frac{(\Delta\gamma_{\text{est}})^2}{|d\langle\gamma_{\text{est}}\rangle/d\gamma|^2} + \langle\delta\gamma\rangle^2 \ge \frac{1}{F(\gamma)} + \langle\delta\gamma\rangle^2 \ge \frac{1}{F(\gamma)}$$

$$\delta\gamma \equiv \frac{\gamma_{\text{est}}}{|d\langle\gamma_{\text{est}}\rangle/d\gamma|} - \gamma = \frac{\Delta\gamma_{\text{est}}}{|d\langle\gamma_{\text{est}}\rangle/d\gamma|} + \langle\delta\gamma\rangle$$

Proof of QCRB.
Classical Fisher information

$$\langle(\delta\gamma)^2\rangle = \frac{(\Delta\gamma_{\rm est})^2}{|d\langle\gamma_{\rm est}\rangle/d\gamma|^2} + \langle\delta\gamma\rangle^2 \geq \frac{1}{F(\gamma)} + \langle\delta\gamma\rangle^2 \geq \frac{1}{F(\gamma)}$$

$$\begin{aligned} F(\gamma) &\equiv \left\langle \left(\frac{\partial \ln p(\gamma_{\rm est}|\gamma)}{\partial\gamma} \right)^2 \right\rangle \\ &= \int d\gamma_{\rm est}\, p(\gamma_{\rm est}|\gamma) \left(\frac{\partial \ln p(\gamma_{\rm est}|\gamma)}{\partial\gamma} \right)^2 \\ &= \int d\gamma_{\rm est} \frac{1}{p(\gamma_{\rm est}|\gamma)} \left(\frac{\partial p(\gamma_{\rm est}|\gamma)}{\partial\gamma} \right)^2 \end{aligned}$$

Proof of QCRB. Quantum mechanics

$$F(\gamma) = \int d\gamma_{\text{est}} \frac{1}{p(\gamma_{\text{est}}|\gamma)} \left(\frac{\partial p(\gamma_{\text{est}}|\gamma)}{\partial \gamma} \right)^2$$

$$p(\gamma_{\text{est}}|\gamma)) = \text{tr}(E_{\gamma_{\text{est}}} \rho_\gamma(t))$$

$$\frac{\partial p(\gamma_{\text{est}}|\gamma)}{\partial \gamma} = \text{tr}\left(E_{\gamma_{\text{est}}} \frac{\partial \rho_\gamma(t)}{\partial \gamma} \right) = \text{Re}[\text{tr}(E_{\gamma_{\text{est}}} \mathcal{L}_\gamma \rho_\gamma(t))]$$

$$\frac{\partial \rho_\gamma(t)}{\partial \gamma} = \frac{1}{2}[\mathcal{L}_\gamma \rho_\gamma(t) + \rho_\gamma(t)\mathcal{L}_\gamma] = -iU_\gamma(t)[K_\gamma, \rho_0]U_\gamma^\dagger(t)$$

γ-generator K_γ referred to initial time

Symmetric logarithmic derivative $\mathcal{L}_\gamma$

$$K_\gamma = iU_\gamma^\dagger(t) \frac{\partial U_\gamma(t)}{\partial \gamma} = \int_0^t ds\, U_\gamma^\dagger(s) h U_\gamma(s) = t\overline{h}$$

$$(\overline{h} = h \text{ if } \tilde{H} = 0)$$

Proof of QCRB. Quantum mechanics

$$F(\gamma) = \int d\gamma_{\text{est}} \frac{1}{p(\gamma_{\text{est}}|\gamma)} \left(\frac{\partial p(\gamma_{\text{est}}|\gamma)}{\partial \gamma} \right)^2$$

$$\frac{\partial p(\gamma_{\text{est}}|\gamma)}{\partial \gamma} = \text{tr}\left(E_{\gamma_{\text{est}}} \frac{\partial \rho_\gamma(t)}{\partial \gamma} \right) = \text{Re}[\text{tr}(E_{\gamma_{\text{est}}} \mathcal{L}_\gamma \rho_\gamma(t))]$$

$$\begin{aligned}
\left(\frac{\partial p(\gamma_{\text{est}}|\gamma)}{\partial \gamma} \right)^2 &\leq |\text{tr}(E_{\gamma_{\text{est}}} \mathcal{L}_\gamma \rho_\gamma)|^2 \\
&= \left| \text{tr}\left(\sqrt{\rho_\gamma} \sqrt{E_{\gamma_{\text{est}}}} \sqrt{E_{\gamma_{\text{est}}}} \mathcal{L}_\gamma \sqrt{\rho_\gamma} \right) \right|^2 \\
\text{Schwarz inequality} &\leq \text{tr}\left(\sqrt{\rho_\gamma} E_{\gamma_{\text{est}}} \sqrt{\rho_\gamma} \right) \text{tr}\left(\sqrt{\rho_\gamma} \mathcal{L}_\gamma E_{\gamma_{\text{est}}} \mathcal{L}_\gamma \sqrt{\rho_\gamma} \right) \\
&= \text{tr}(E_{\gamma_{\text{est}}} \rho_\gamma) \, \text{tr}(E_{\gamma_{\text{est}}} \mathcal{L}_\gamma \rho_\gamma \mathcal{L}_\gamma) \\
&= p(\gamma_{\text{est}}|\gamma) \text{tr}(E_{\gamma_{\text{est}}} \mathcal{L}_\gamma \rho_\gamma \mathcal{L}_\gamma)
\end{aligned}$$

$$F(\gamma) \leq \text{tr}(\mathcal{L}_\gamma^2 \rho_\gamma(t)) \equiv \mathcal{F}(\gamma)$$

Quantum Fisher information $\mathcal{F}(\gamma)$

Proof of QCRB. Quantum mechanics

$$\mathcal{F}(\gamma) = \mathrm{tr}(\mathcal{L}_\gamma^2 \rho_\gamma(t))$$

$$\frac{\partial \rho_\gamma(t)}{\partial \gamma} = \frac{1}{2}[\mathcal{L}_\gamma \rho_\gamma(t) + \rho_\gamma(t)\mathcal{L}_\gamma] = -iU_\gamma(t)[K_\gamma, \rho_0]U_\gamma^\dagger(t)$$

Pure-state input: differentiating $\rho_\gamma = \rho_\gamma^2$ gives

$$\mathcal{L}_\gamma = 2\frac{\partial \rho_\gamma(t)}{\partial \gamma} = -2iU_\gamma(t)[K_\gamma, \rho_0]U_\gamma^\dagger(t)\ .$$

$$\begin{aligned} \mathcal{F}(\gamma) &= -4\mathrm{tr}\big([K_\gamma, \rho_0]^2 \rho_0\big) \\ &= 4\big(\mathrm{tr}(K_\gamma^2 \rho_0) - \mathrm{tr}(K_\gamma \rho_0)^2\big) \\ &= 4\langle(\Delta K_\gamma)^2\rangle \\ &= 4t^2\langle(\Delta \overline{h})^2\rangle \end{aligned}$$

The End

$||h||$ is difference between largest and smallest eigenvalues of h

$$\frac{1}{\langle(\delta\gamma)^2\rangle^{1/2}} \le \sqrt{F(\gamma)} \le \sqrt{\mathcal{F}(\gamma)}$$

$$= 2t\langle(\Delta\overline{h})^2\rangle^{1/2} \le 2t\overline{\langle(\Delta h)^2\rangle^{1/2}} \le t||h||$$