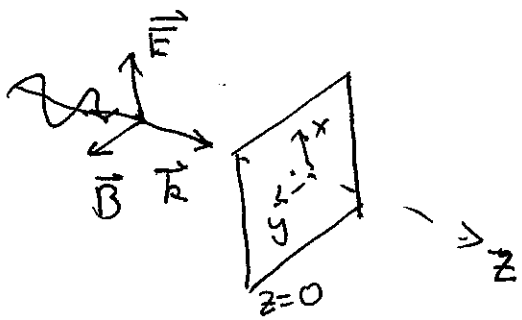
Physics 406

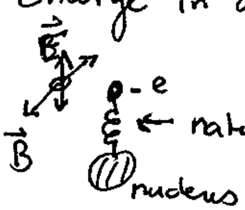
Lecture 11: Momentum and Angular Momentum in E&M

We have seen that electromagnetic waves carry energy. Thus also carry momentum. To see this, consider the momentum imparted to charges the absorb EM wave.



$$\vec{E} = E_0 \cos(kz-\omega t)\,\hat{x}$$
$$\vec{B} = \frac{E_0}{c} \cos(kz-\omega t)\,\hat{y}$$

Imagine a charge in a molecule (electron) bound. (Lorentz oscillator model)



Equation of motion of electron

$$m_e\left(\frac{d^2\vec{r}}{dt^2} + \Gamma\frac{d\vec{r}}{dt} + \omega_0^2\vec{r}\right) = \underbrace{-e\vec{E}(z=0,t)} - e\left(\frac{d\vec{r}}{dt}\right)\times\vec{B}(z=0,t)$$

$$= -eE_0\cos(\omega t)\,\hat{x} - eE_0\left(\frac{\dot{\vec{r}}}{c}\right)\times\hat{y}\,\cos\omega t$$

For non relativistic motion, $|\dot{\vec{r}}| \ll c$, effect of magnetic force $\ll$ effect of electric force.

Motion along x-direction

$$\ddot{x} + \Gamma \dot{x} + \omega_0^2 x = -\frac{e}{m_e} E_0 \cos\omega t$$

Steady state solution $\quad x(t) = \mathrm{Re}(\tilde{x}\, e^{-i\omega t})$

$$\Rightarrow \tilde{x} = \frac{-e/m_e}{\omega_0^2 - \omega^2 - i\omega\Gamma} E_0$$

$$\Rightarrow x(t) = \mathrm{Re}(\tilde{x} e^{-i\omega t}) = -e/m_e E_0 \; \mathrm{Re}\left(\frac{1}{\omega_0^2-\omega^2-i\omega\Gamma} e^{-i\omega t}\right)$$

$$= -\frac{e}{m_e} E_0 \left[\mathrm{Re}\left(\frac{1}{\omega_0^2-\omega^2-i\omega\Gamma}\right)\cos\omega t + \mathrm{Im}\left(\frac{1}{\omega_0^2-\omega^2-i\omega\Gamma}\right)\sin\omega t\right]$$

$$= \underbrace{-\frac{eE_0}{m_e}\left[\frac{\omega_0^2-\omega^2}{(\omega_0^2-\omega^2)^2+\omega^2\Gamma^2}\right]}_{\text{in phase: } x_0} \cos\omega t \; \underbrace{-\frac{eE_0}{m_e}\left[\frac{\omega\Gamma}{\omega_0^2-\omega^2+\omega^2\Gamma^2}\right]}_{\text{in quadrature } x_{\pi/2}} \sin\omega t$$

Now consider ~~motion~~ Force in the z-direction $\Rightarrow$ radiation pressure

$$F_z = -e v_x B_y = -e \dot{x} E_0 \cos\omega t$$

$$\Rightarrow \dot{x} = -\omega x_0 \sin\omega t + \omega x_{\frac{\pi}{2}} \cos\omega t$$

Time averaged force $\langle F_z \rangle = -e \langle v_x B_y \rangle$

$$\Rightarrow \langle F_z \rangle = -e\omega \frac{E_0}{c}\Big(x_0 \underbrace{\langle \sin\omega t \cos\omega t\rangle}_{0} - x_{\frac{\pi}{2}} \underbrace{\langle \cos^2\omega t\rangle}_{=\frac{1}{2}}\Big)$$

$$\Rightarrow \boxed{\langle F_z \rangle = \frac{e^2}{2mc} E_0 \left(\frac{\omega^2 \Gamma^2}{(\omega_0^2 - \omega^2)^2 + (\omega\Gamma)^2} \right)}$$

Using complex amplitudes

$$v_x = \mathrm{Re}(\tilde{v}_x e^{-i\omega t}) \quad \Rightarrow \quad \tilde{v}_x = -i\omega \tilde{x}$$

$$\langle F_z \rangle = -e \langle v_x B_y \rangle = -\frac{e}{2} \mathrm{Re}(\tilde{v}_x \tilde{B}_y)$$

$$= -\frac{e}{2} \mathrm{Re}\left(-i\omega \tilde{x} \frac{E_0}{c}\right) = -\frac{\omega e E_0}{2c} \underbrace{\mathrm{Re}(-i\tilde{x})}_{\mathrm{Im}(\tilde{x})}$$

Note: Radiation pressure, arising from the absorption of E+M wave depends on imaginary part of oscillation response function. Why?

Rate at which EM waves does work on charge

$$\left\langle \frac{d}{dt} W \right\rangle = \langle \vec{v} \cdot \vec{F} \rangle = \langle v_x F_x \rangle = -e \langle v_x E_x \rangle$$

$$= -\frac{e}{2} \mathrm{Re}(\tilde{v}_x \tilde{E}_x) = -\frac{e}{2} \mathrm{Re}(-i\omega \tilde{x} E_0)$$

$$= -\frac{e\omega E_0}{2} \mathrm{Re}(-i\tilde{x})$$

Time averaged work done on charge depends on $\vec{v}$ being in phase with $\vec{F}$ $\Rightarrow$ $\vec{x}$ in quadrature with $\vec{F}$

Momentum: Rate at which momentum is transfered

$$\langle F_z \rangle = \left\langle \frac{dp_z}{dt} \right\rangle = -e \langle v_x B_y \rangle = -e \frac{E_0}{c} \langle v_x \cos\omega t \rangle$$

Compared to rate at which energy is transfered

$$\left\langle \frac{dW}{dt} \right\rangle = -e \langle v_x E_x \rangle = -e E_0 \langle v_x \cos\omega t \rangle$$

$$\Rightarrow \boxed{\left\langle \frac{d}{dt} W \right\rangle = c \left\langle \frac{dp_z}{dt} \right\rangle}$$

$$\Rightarrow \boxed{\langle W \rangle = c \langle p_z \rangle}$$

Note: From relativity, given particle with mass m, momentum p

$$E = \sqrt{(cp)^2 + (mc^2)^2} \quad \text{: "Photon" } m = 0$$

$$\Rightarrow E = cp$$

Define momentum density in EM wave along $\hat{k}$

$$\vec{g} = \frac{u}{c} \hat{k} \quad \text{where } u = \frac{\epsilon_0}{2} |\vec{E}|^2 + \frac{1}{2\mu_0} |\vec{B}|^2$$

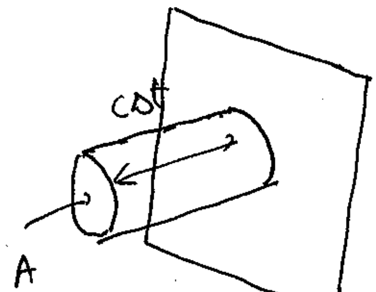
(energy density in field)

But $u \hat{k} = \frac{\vec{S}}{c}$ (Poynting vector)

$$\Rightarrow \boxed{\vec{g} = \frac{\vec{S}}{c^2} = \frac{\vec{E} \times \vec{B}}{\mu_0 c^2} = \frac{\vec{E} \times \vec{H}}{c^2}}$$

momentum density in field

Radiation Pressure



Momentum is volume

$$\Delta p = g\, A\, c\Delta t$$

$$\Rightarrow \text{Force} = \frac{\Delta p}{\Delta t} = g A c$$

$$\Rightarrow \text{Pressure} = g c = \frac{|\vec{S}|}{c^2} c = \frac{|\vec{S}|}{c} = u$$

Energy density = pressure

Example: Laser diode: $I = 300 \frac{mW}{cm^2} = 30 \frac{W}{m^2}$

$$\text{Pressure} = \frac{I}{c} = \frac{30 \frac{W}{m^2}}{3\times 10^8 \ m/s} = 10^{-7} \frac{\text{Newtons}}{m^2}$$: Pretty small

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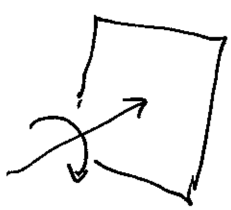
Angular Momentum: Consider circularly polarized light incident on the absorbing sheet.



$$\vec{E}(z=0,t) = \frac{E_0}{\sqrt{2}}\left(\hat{x}\cos\omega t \pm \hat{y}\sin\omega t\right)$$

$$= \mathrm{Re}\left(E_0 \frac{(\hat{x}\pm i\hat{y})}{\sqrt{2}} e^{-i\omega t}\right)$$

$$= \mathrm{Re}\left(E_0\, \vec{e}_\pm\, e^{-i\omega t}\right)$$

Aside: $\left(\vec{e}_+ = \frac{\hat{x}\pm i\hat{y}}{\sqrt{2}}\right)$

Motion in $\hat{x}$-$\hat{y}$ plane drive by $\vec{E}$-field

$$\frac{d^2\vec{r}}{dt^2} + \Gamma \frac{d\vec{r}}{dt} + \omega_0^2 \vec{r} = -\frac{e}{m}\vec{E}(t)$$

Steady State

$$\vec{r}(t) = \mathrm{Re}\left(\underset{\sim}{\vec{r}}\, e^{-i\omega t}\right) \qquad \vec{E}(t) = \mathrm{Re}\left(\underset{\sim}{\vec{E}}\, e^{-i\omega t}\right)$$

$$\Rightarrow \quad \left(\omega_0^2 - \omega^2 - i\omega\Gamma\right)\underset{\sim}{\vec{r}} = -\frac{e}{m}\underset{\sim}{\vec{E}}$$

$$\Rightarrow \quad \underset{\sim}{\vec{r}} = \alpha\, \underset{\sim}{\vec{E}} \quad \text{where} \quad \alpha = \frac{-e/m}{\omega_0^2 - \omega^2 - i\omega\Gamma}$$

Consider rate at which angular momentum is transfered to charge = torque on charge by field

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times (-e\vec{E}) \Rightarrow \langle \vec{\tau} \rangle = -\frac{e}{2}\,\mathrm{Re}\left(\underset{\sim}{\vec{r}} \times \underset{\sim}{\vec{E}}^*\right)$$

$$\Rightarrow \langle \vec{\tau} \rangle = -\frac{e}{2}\,\mathrm{Re}\left(\alpha\, \underset{\sim}{\vec{E}} \times \underset{\sim}{\vec{E}}^*\right)$$

Aside: $\underset{\sim}{\vec{E}} \times \underset{\sim}{\vec{E}}^* = E_0^2\left(\vec{e}_\pm \times \vec{e}_\pm^{\,*}\right) = \frac{E_0^2}{2}(\hat{x} \pm i\hat{y}) \times (\hat{x} \mp i\hat{y})$

$$= -i\frac{E_0^2}{2}(\pm 2\hat{z}) = \mp i E_0^2\, \hat{z}$$

$$\Rightarrow \langle \vec{\tau} \rangle = -\frac{e}{2} E_0^2\, \mathrm{Re}(-i\alpha)\hat{z} = \mp\frac{e}{2} E_0^2\, \mathrm{Im}(\alpha)\,\hat{z}$$

$$\Rightarrow \langle \vec{\tau} \rangle = \pm\left(\frac{\frac{e^2}{2m}\omega\Gamma}{(\omega_0^2 - \omega^2)^2 + (\omega\Gamma)^2}\right) E_0^2\, \hat{z}$$

$\Rightarrow$ $\pm$ helicity circularly polarized light carries angular momentum in $\pm\hat{k}$ direction

But for the wave rate of energy transfer (work)

$$\left\langle \frac{d}{dt} W \right\rangle = \frac{e\omega}{2} E_0^2 \, \mathrm{Im}(\alpha)$$

and $\langle \vec{\tau} \rangle = \left\langle \frac{dL_z}{dt} \right\rangle \hat{z}$

$$\Rightarrow \boxed{|\langle L_z \rangle| = \frac{\langle W \rangle}{\omega}}$$

Photons: Single circular polarized photon carries angular momentum $\pm\hbar \hat{k}$

direction of propagation

Energy: $\hbar\omega$

Momentum: $\frac{\hbar\omega}{c}\hat{k} = \hbar\vec{k}$