Physics 521

Problem Set #9: Spin Angular Momentum

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Problem 1: Landau Levels: Sakurai Chapter 2, #6 page 150

An electron moves in *x*-*y* plane the presence of a uniform magnetic field in the *z*-direction ($\mathbf{B}=B\mathbf{e}_z$).

(a) Evaluate $\left[\hat{\pi}_x, \hat{\pi}_y\right]$, where $\hat{\vec{\pi}} = \hat{\mathbf{p}} - \frac{q}{c}\mathbf{A}(\hat{\mathbf{x}})$ is the “kinetic momentum”.

(b) By comparing that Hamiltonian and the commutation relation obtained in (a), show that the Hamiltonian has the form on a 1D simple harmonic oscillator, and thus the energy eigenvalues are

$$E_n = \hbar\omega_c\left(n+\frac{1}{2}\right), \text{ where } \omega_c = \frac{qB}{mc} \text{ is the cyclotron frequency.}$$

These energy levels are known as “Landau levels”. The oscillator is a “pseudo-oscillator”, since there is no harmonic potential.

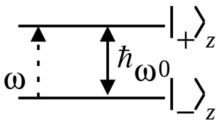
Problem 2 and 3: Magnetic Resonance (20 Points)

We have considered the problem of a spin in a static magnetic field B_0 for some time now - its effect is to cause a precession of the spin about its direction with Larmor frequency $\omega_0 = \gamma_s B_0$. Now suppose we have two magnetic fields: A static field in the –*z*-direction of magnitude B_0 and a *field* B_1 *rotating in the x-y plane* at frequency ω:

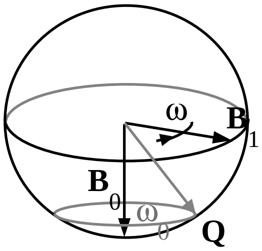
$$\mathbf{B} = -B_0\mathbf{e}_z + B_1(\cos\omega t\,\mathbf{e}_x + \sin\omega t\,\mathbf{e}_y).$$

The resonance phenomenon that ensues can be understood in two ways:

(i) Resonant excitation between quantized energy levels: In the absence of the oscillating field, the energy eigenstates are $\left|\pm_z\right\rangle$ with eigenvalues $\pm\frac{\hbar\omega_0}{2}$. The effect of the oscillating magnetic field is to drive population between $\left|-_z\right\rangle \leftrightarrow \left|+_z\right\rangle$. If $\omega=\omega_0$ the effect is “resonant”.



(2) Bloch vector picture: The spin precesses around the instantaneous combination of the static and rotating transverse field. If the latter rotates at the frequency ω_0 it follows the precession induced by the static field. In this case the transverse field is static in the rotating frame and the spin oscillates between the $\left|\pm_z\right\rangle$ states.



In this problem we will analyze the spin-resonance phenomenon from these two different points of view in order to contrast different problem solving techniques.

(a) Preliminaries: Show that the Hamiltonian for the spin interacting with the field is

$$\hat{H}(t)=\frac{1}{2}\hbar\omega_0\,\hat{\sigma}_z-\frac{1}{2}\hbar\Omega_R\left(\hat{\sigma}_+e^{-i\omega t}+\hat{\sigma}_-e^{+i\omega t}\right)$$

where $\Omega_R=\gamma_s B_1$ is known as the "Rabi frequency" after I. I. Rabi who first studied spin resonances in molecular beam experiments. Notice that the interaction with the transverse field involves the $\hat{\sigma}_\pm$ operators which flips the spin between $\left|\pm_z\right\rangle$.

Brute-Force Schrödinger Picture.

(b) Let us expand the time dependent state in the basis $\left|\pm_z\right\rangle$

$$\left|\psi(t)\right\rangle=c_+(t)\left|+_z\right\rangle+c_-(t)\left|-_z\right\rangle \text{ with the initial condition } \left|\psi(0)\right\rangle=\left|-_z\right\rangle.$$

Use the time-dependent Schrödinger equation to derive the coupled differential equations:

$$\begin{aligned}\dot{c}_+&=-i\frac{\omega_0}{2}c_++i\frac{\Omega_R}{2}e^{-i\omega t}c_-\\ \dot{c}_-&=+i\frac{\omega_0}{2}c_-+i\frac{\Omega_R}{2}e^{+i\omega t}c_+\end{aligned}$$

(c) Now we can remove the explicit time dependence by making the substitution $c_+ = \tilde{c}_+ e^{-i\omega t/2}$, $c_- = \tilde{c}_- e^{+i\omega t/2}$. Show that the new coefficients evolve according to

$$\frac{d}{dt}\begin{bmatrix}\tilde{c}_+\\ \tilde{c}_-\end{bmatrix} = \frac{i}{2}\begin{bmatrix}\Delta & \Omega_R\\ \Omega_R & -\Delta\end{bmatrix}\begin{bmatrix}\tilde{c}_+\\ \tilde{c}_-\end{bmatrix}$$

where $\Delta \equiv \omega - \omega_0$ is the detuning of the driving field from resonance.

(d) Finally we can solve these coupled equations by noting

$$\begin{bmatrix}\Delta & \Omega_R\\ \Omega_R & -\Delta\end{bmatrix} = \Delta\hat{\sigma}_z + \Omega_R\hat{\sigma}_x = -\Omega\,\mathbf{e}_n\cdot\hat{\vec{\sigma}},$$

where we have defined the "generalized Rabi frequency" $\Omega \equiv \sqrt{\Omega_R^2 + \Delta^2}$ and the unit vector $\mathbf{e}_n = -\frac{\Omega_R}{\Omega}\mathbf{e}_x - \frac{\Delta}{\Omega}\mathbf{e}_x$. Show then the time evolution is:

$$\begin{bmatrix}\tilde{c}_+(t)\\ \tilde{c}_-(t)\end{bmatrix} = e^{-i\Omega t/2\,\mathbf{e}_n\cdot\hat{\vec{\sigma}}}\begin{bmatrix}0\\ 1\end{bmatrix} = \begin{bmatrix} i\frac{\Omega_R}{\Omega}\sin\left(\frac{\Omega t}{2}\right)\\ \cos\left(\frac{\Omega t}{2}\right) + i\frac{\Delta}{\Omega}\sin\left(\frac{\Omega t}{2}\right)\end{bmatrix}.$$

Plot the probability to be find $|+_z\rangle$ as a function of time for $\Delta = 0$ and also for $\Delta >> \Omega_R$.

Using the rotation operator:

(e) Using the geometric picture of the rotating spin, we can obtain the solution of part (d) in a more direct and (perhaps) intuitive way. First let us transform into a frame rotating with the transverse field at ω, so that it appears static. This is can be achieved via a unitary transformation which *removes,* the rotation about the *z*-axis by an amount ωt.

Let $|\psi'\rangle = \hat{D}_z^\dagger(\omega t)|\psi\rangle$ (state vector in the "rotating frame").

Show that: $i\hbar\frac{\partial}{\partial t}|\psi'\rangle = \hat{H}_{eff}|\psi'\rangle$, where the effective Hamiltonian in the rotating frame is

$$\hat{H}_{eff} = \hat{D}_z^\dagger(\omega t)\hat{H}(t)\hat{D}_z(\omega t) - \frac{\hbar\omega}{2}\hat{\sigma}_z = -\hat{\vec{\mu}}\cdot\mathbf{B}_{eff}\text{, where, }\mathbf{B}_{eff} = (-B_0 + \hbar\omega/\gamma_s)\mathbf{e}_z + \mathbf{B}_1\,\mathbf{e}_x$$

Thus show that the solution in the rotating frame represents rotation of the spin with effective Larmor frequency vector $\vec{\Omega}_{eff} = -\gamma_s\mathbf{B}_{eff} = -\Delta\,\mathbf{e}_z - \Omega_R\,\mathbf{e}_x$ whose solution is as in part (d).

The Bloch vector:

(f) In Problem Set # 3, we defined the “Bloch vector” $\mathbf{Q} = Tr(\hat{\rho}\,\hat{\vec{\sigma}})$, where $\hat{\rho}$ is the density operator for the spin (note I reserve the notation, **R,** here for the rotation tensor). With the notation $\mathbf{Q} = u\,\mathbf{e}_x + \mathrm{v}\,\mathbf{e}_\mathrm{y} + w\,\mathbf{e}_z$ and the notation of part (a), show that in the rotating frame

$$\tilde{\mathbf{Q}} = \tilde{u}\,\mathbf{e}_x + \tilde{\mathrm{v}}\,\mathbf{e}_\mathrm{y} + \tilde{w}\,\mathbf{e}_z, \text{ with } \tilde{u} = 2\,\mathrm{Re}(\tilde{c}_+^*\tilde{c}_-)\ ,\ \ \tilde{\mathrm{v}} = 2\,\mathrm{Im}(\tilde{c}_+^*\tilde{c}_-),\ \ \tilde{\mathrm{w}} = |\tilde{c}_+|^2 - |\tilde{c}_-|^2 = \mathrm{w}.$$

Thus show that the equation of motion for the Bloch vector (the so-called Bloch equations) is:

$$\frac{d}{dt}\begin{bmatrix}\tilde{u}\\ \tilde{\mathrm{v}}\\ \tilde{\mathrm{w}}\end{bmatrix} = \begin{bmatrix}0 & \Delta & 0\\ -\Delta & 0 & \Omega_R\\ 0 & -\Omega_R & 0\end{bmatrix}\begin{bmatrix}\tilde{u}\\ \tilde{\mathrm{v}}\\ \tilde{\mathrm{w}}\end{bmatrix} \quad \text{or} \quad \frac{d}{dt}\tilde{\mathbf{Q}} = \vec{\Omega}_{eff}\times\tilde{\mathbf{Q}}.$$

Show that the solution (with the given initial condition) is

$$\begin{bmatrix}\tilde{u}(t)\\ \tilde{\mathrm{v}}(\mathrm{t})\\ \tilde{\mathrm{w}}(\mathrm{t})\end{bmatrix} = \begin{bmatrix}\dfrac{\Delta\Omega_R}{\Omega^2}(1-\cos\Omega t)\\ \dfrac{\Omega_R}{\Omega}\sin\Omega t\\ \dfrac{(\Delta^2+\Omega_R^2\cos\Omega t)}{\Omega^2}\end{bmatrix} \quad \text{Sketch the trajectory of } \tilde{\mathbf{Q}}.$$