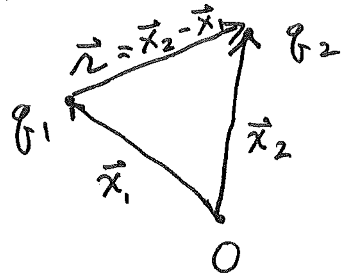
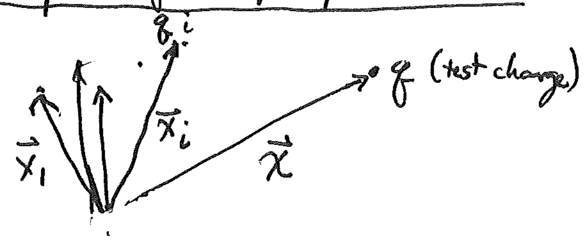
Physics 511 Lecture # 3 - Review Electrostatics I

Coulomb's Law



$$\vec{F}_{2,1} = \frac{q_1 q_2}{r^2}\hat{r} = \frac{q_2 q_1}{r^3}\vec{r} = q_1 q_2 \frac{(\vec{x}_2 - \vec{x}_1)}{|\vec{x}_2 - \vec{x}_1|^3}$$

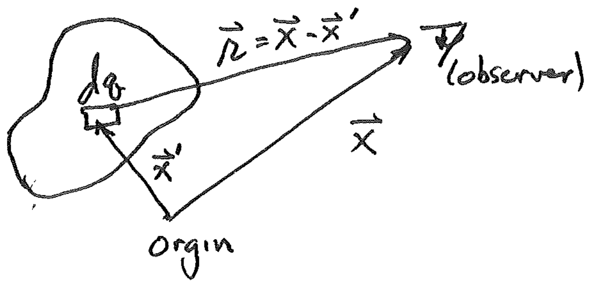
Principle of superposition



$$\vec{F}(\vec{x}) = \sum_i \frac{q q_i}{|\vec{x} - \vec{x}_i|^3}(\vec{x} - \vec{x}_i)$$

$$= q \underbrace{\sum_i \frac{q_i}{|\vec{x} - \vec{x}_i|^3}(\vec{x} - \vec{x}_i)}_{\vec{E}(\vec{x}) \ \left(\text{Electric field of charge distribution}\right)}$$

$$\vec{E}(\vec{x}) = \sum_i \frac{q_i}{|\vec{x} - \vec{x}_i|^3}(\vec{x} - \vec{x}_i) \underset{\uparrow \text{ continuous}}{=} \int \frac{dq(\vec{x}')}{|\vec{x} - \vec{x}'|^3}(\vec{x} - \vec{x}')$$

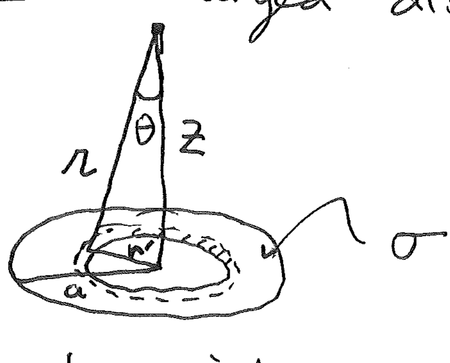


Distribution

- Volume : $dq = \rho(\vec{x}')\, d^3\vec{x}'$

  $d^3\vec{x} = dx\, dy\, dz$
- Surface: $dq = \sigma(\vec{x}')\, da'$
- line $dq = \lambda(\vec{x}')\, dl'$

Example: Charged disk, uniform surface density σ



$$dq = (2\pi r' dr')\sigma$$

$$r = \sqrt{r'^2 + z^2}$$

Observation point on axis

L3.2

Use symmetry: $\vec{E}(z) = \vec{e}_z E_z(z)$ (must point in z-direction)

$$dE_z = \frac{dq}{r^2}\cos\theta = \frac{dq}{r^3} z$$

$$\Rightarrow E_z(z) = 2\pi\sigma z \int_0^a \frac{r' dr'}{(r'^2+z^2)^{3/2}} = 2\pi\sigma z \left[-(r'^2+z^2)^{-1/2} \right]_0^a$$

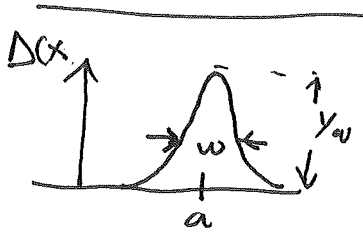
$$= 2\pi\sigma z \left[\frac{1}{z} - \frac{1}{\sqrt{a^2+z^2}} \right] = 2\pi\sigma \left[1 - \frac{1}{\left(1+\frac{a^2}{z^2}\right)^{1/2}} \right]$$

lim $\frac{a}{z} \to 0$ (infinitely far away) $\left(1+\frac{a^2}{z^2}\right)^{-1/2} \approx 1 - \frac{1}{2}\frac{a^2}{z^2}$

(recall $(1+x)^\delta \approx 1 + \delta x \quad \delta \ll 1$

$$\Rightarrow E_z(z) \approx \frac{\pi a^2 \sigma}{z^2} = \frac{Q}{z^2} = \frac{Q}{r^2}$$

Dirac Delta function for "point" distributions



$$\delta(x) = \lim_{w \to 0} \Delta(x)$$

$$\delta(x-a) = \begin{cases} 0 & x \neq a \\ \infty & x = a \end{cases}$$ "Distribution"

Properties

(i) $\int_{-\infty}^{\infty} dx\, \delta(x-a) = 1$ (ii) $\int_{-\infty}^{\infty} dx\, \delta(x-a) f(x) = f(a)$

(iii) $\delta(f(x)) = \sum_{\substack{i \\ f(x_i)=0}} \frac{\delta(x-x_i)}{\left|\frac{df}{dx}(x_i)\right|}$ (iv) δ has units $\frac{1}{[x]}$

Higher dimensional delta funct.

$$\delta^{(3)}(\vec{x}-\vec{x}') = \delta(x-x')\,\delta(y-y')\,\delta(z-z')$$: Cartesian expansion

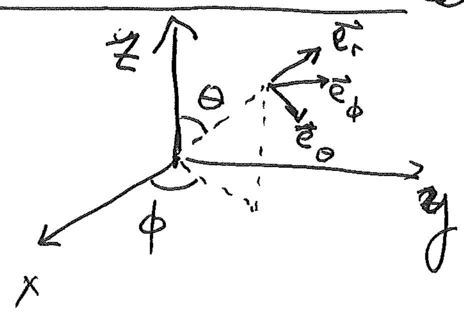
$\therefore$ density of a point

$$\int d^3x\, \delta^{(3)}(\vec{x}-\vec{x}_0) = 1$$

- Charge density of a point charge at $\vec{x}'$: $\rho(\vec{x}) = q\,\delta^{(3)}(\vec{x}-\vec{x}')$

- Charge density of an infinite plane of surface charge σ (say the x-y plane at z_0) $\rho(\vec{x}) = \sigma\,\delta(z-z_0)$

<u>Curvilinear coordinate</u> (e.g. spherical)



$$d^3x = r^2 \sin\theta\, d\theta\, d\phi\, dr$$

$\vec{e}_r, \vec{e}_\theta, \vec{e}_\phi$: <u>depend on position</u>

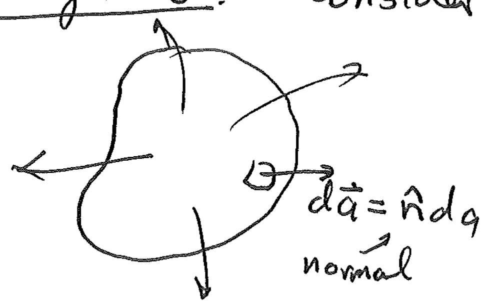
$$\vec{x} = r\,\vec{e}_r$$

$$\Rightarrow \quad \delta^{(3)}(\vec{x}-\vec{x}') = \frac{\delta(r-r')\,\delta(\theta-\theta')\,\delta(\phi-\phi')}{r^2\sin\theta}$$

<u>Field equations</u>

Any vector field defined by its divergence + curl (plus boundary condition at ∞)

<u>Divergence</u>: Consider flux of field $\vec{A}$ through closed surface



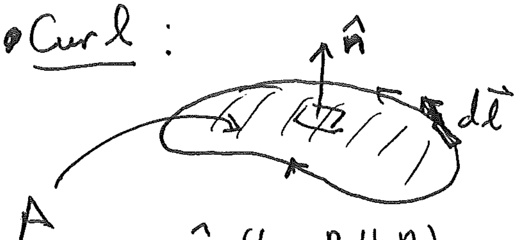
$$\oint_S \vec{V}\cdot d\vec{a} = \text{flux of } \vec{\ }$$

<u>define</u> $\displaystyle \operatorname{div}\vec{V} = \lim_{\Delta V\to 0} \frac{\oint_S \vec{V}\cdot d\vec{a}}{\Delta V}$ (Flux into or out of a point)

$\Rightarrow$ Divergence theorem

$$\boxed{\int_V (\mathrm{div}\,\vec{V})\, d^3x = \oint_S \vec{V}\cdot d\vec{a}}$$

• Curl :



$\hat{n}$ (by R.H.R.)

$$\oint_C \vec{V}\cdot d\vec{l} = \text{Circulation of } \vec{A}$$

Define $\hat{n}\cdot\overrightarrow{\mathrm{curl}\,V} = \lim_{\Delta a\to 0} \frac{\oint_C \vec{V}\cdot d\vec{l}}{\Delta a}$

$$\Rightarrow \boxed{\int_S d\vec{a}\cdot(\overrightarrow{\mathrm{curl}\,V}) = \oint_C \vec{V}\cdot d\vec{l}}$$ Stoke's theorem

Note general form of integral calculus

$$\underbrace{\int_{\mathcal{O}} DF\, d^{(N)}\tau}_{\text{integral of DF on object } \mathcal{O}} = \underbrace{\int_{\partial\mathcal{O}} F\, d^{N-1}\tau}_{\text{antiderivate on the boundary } \partial\mathcal{O}}$$

"Del operator" $\vec{\nabla} = \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \frac{\partial}{\partial z} = e_i\,\partial_i$

Can Show
$$\begin{cases} \mathrm{div}\,\vec{V} = \vec{\nabla}\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = \partial_i V_i \\ \overrightarrow{\mathrm{curl}\,V} = \vec{\nabla}\times\vec{V} = \vec{e}_i\, \epsilon_{ijk}\, \partial_j V_k \end{cases}$$

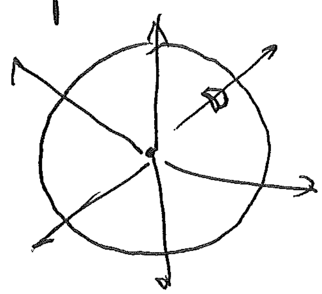
• Gradient : Given scalar field ϕ : $\overrightarrow{\mathrm{grad}}\,\phi = \vec{\nabla}\phi$

Integral $\int_{\vec{x}_a}^{\vec{x}_b} \vec{\nabla}\phi\cdot d\vec{l} = \phi(\vec{x}_b) - \phi(\vec{x}_a)$ (independent of path)

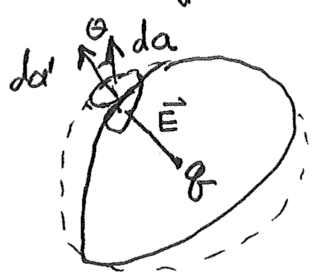
Field equations for Electrostatic fields

$\vec{E} = \int \frac{dq}{\imath^2}\hat{\imath}$, so by superposition need only consider point charge

e.g. point charge at the origin of a sphere, radius R



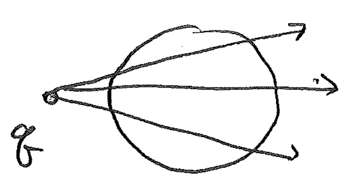
$$\oint \vec{E}\cdot d\vec{a} = \oint \frac{q}{R^2}\, da = \left(\frac{q}{R^2}\right) 4\pi R^2$$
$$= 4\pi q$$



Deformed sphere

$$d\vec{a} = \hat{e}_r\, da \qquad d\vec{a}' = \hat{n}'\, da' = \hat{n}' \frac{da}{\cos\theta}$$

$$\Rightarrow \vec{E}\cdot d\vec{a}' = \underbrace{(\vec{E}\cdot\hat{n})}_{E\cancel{\cos\theta}} \frac{da}{\cancel{\cos\theta}} = \vec{E}\cdot d\vec{a}$$



$\oint \vec{E}\cdot da = 0$

$$\rightarrow \boxed{\oint_S \vec{E}\cdot d\vec{a} = 4\pi Q_{enc}}$$ Gauss' Law (integral form)

$$Q_{enc} = \int_V \rho(\vec{x})\, d^3\vec{x}$$

$$\oint_S \vec{E}\cdot d\vec{a} = \int_V (\vec{\nabla}\cdot\vec{E})\, d^3x$$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{E} = 4\pi\rho(\vec{x})}$$ Gauss' Law (differential form)

- Using Gauss' Law to determine $\vec{E}$

Requires very symmetric situation (1) Know form of $\vec{E}$
(2) $|\vec{E}|$ constant on $\mathcal{S}$

$$\Rightarrow \oint \vec{E}\cdot d\vec{a} = E(\xi) \oint da = E(\xi)\, A(\xi) = 4\pi Q_{enc}$$

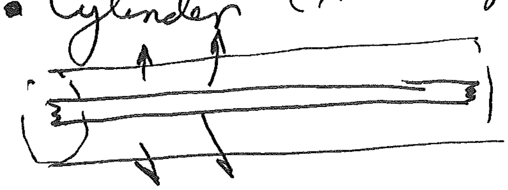
Examples: Spheres, infinite cylinders, infinite slabs
$\Downarrow$ points, $\Downarrow$ lines, $\Downarrow$ surfaces

- Sphere : $\vec{E} = E(r)\vec{e}_r$

$$\oint \vec{E}\cdot d\vec{a} = E(r)\, 4\pi r^2 = 4\pi Q_{enc}$$

$$E(r) = \frac{Q_{enc}}{r^2}$$ (like point charge at origin)

- Cylinder (λ = charge/length) $\vec{E} = E(\rho)\,\vec{e}_\rho$

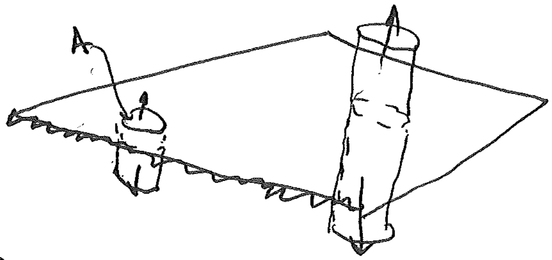


$$\oint \vec{E}\cdot d\vec{a} = E(\rho)\,(2\pi\rho L) = 4\pi Q_{enc}$$

$$\Rightarrow E(\rho) = \frac{2\lambda}{\rho}$$

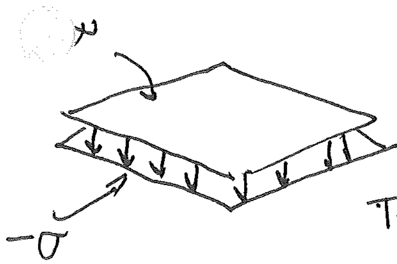
- Slab/surface (σ = charge/area) $E = E(z)\,\vec{e}_z$

$$E(z) = -E(-z)$$



$$\oint \vec{E}\cdot d\vec{a} = 2EA = 4\pi Q_{enc}$$

$$\Rightarrow \boxed{E = 2\pi\sigma}$$ Uniform



Two opposite infinite plane $E = \begin{cases} 4\pi\sigma & \text{inside} \\ 0 & \text{outside} \end{cases}$