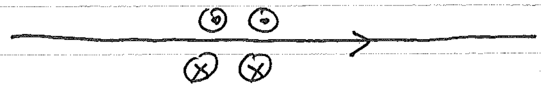
Physics 511 - Lecture #7

Magnetostatics I

Lorentz Force:



$\vec{F} = q\frac{\vec{v}}{c} \times \vec{B}$ (c ← speed of light)

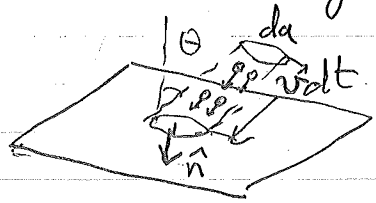
$c = 3\times 10^{10}$ cm/s

Magnetic field exerts force on moving charges: Currents
(Reference frame?)

Currents are the source of $\vec{B}$ - field

Single point charge $I = \frac{dq}{dt}$

Generally



$$dq = \rho\, da\, \vec{v}\cdot\hat{n}\, dt$$

$$dI = \frac{dq}{dt} = \rho\vec{v}\cdot\hat{n}\,da$$

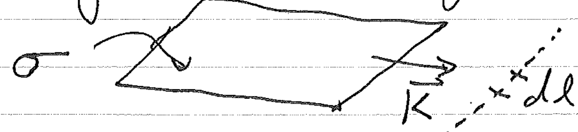
Current density $\vec{J} = \sum_\alpha q_\alpha\, \delta^{(3)}(\vec{x}-\vec{x}_\alpha)\, \vec{v}_\alpha$

$= \rho\vec{v}$ (for uniform flow)

$I = \int_S \vec{J}\cdot d\vec{a}$ flux of current through surface

$[\vec{J}] = \frac{\text{current}}{\text{Area}}$

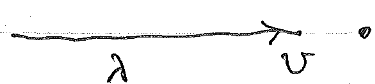
Flowing surface charge: $\vec{K} = \sigma\vec{v}$



Flux through line

$I = \int_C K\, dl$

Flowing line charge: $I = \lambda\vec{v}$



Flux through a point

General Lorentz force Law:

Collection of point charges $\vec{F} = \sum_\alpha q_\alpha \left(\vec{E}(\vec{x}_\alpha) + \frac{\vec{v}_\alpha}{c} \times \vec{B}(\vec{x}_\alpha) \right)$

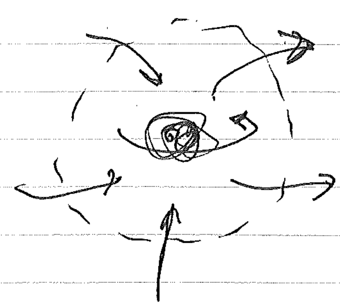
$\Rightarrow$ Continuous distribution, introduce delta functions

$$\vec{F} = \int d^3x \left[\left(\sum_\alpha q_\alpha \, \delta^{(3)}(\vec{x} - \vec{x}_\alpha) \right) \vec{E}(\vec{x}) + \frac{1}{c} \left(\sum_\alpha q_\alpha \, \vec{v}_\alpha \, \delta^{(3)}(\vec{x} - \vec{x}_\alpha) \right) \times \vec{B}(\vec{x}) \right]$$

$$\Rightarrow \boxed{\vec{F} = \int d^3x \left(\rho(\vec{x}) \, \vec{E}(\vec{x}) + \frac{1}{c} \vec{J}(\vec{x}) \times \vec{B}(\vec{x}) \right)}$$

<u>Lexicon:</u> $\vec{J} \, d^3x$ (volume current) $\rightarrow$ $\vec{K} \, da$ (surface current) $\rightarrow$ $I \, d\vec{l}$ (line current) $\rightarrow q \, \vec{v}$ (use delta function) ↑ point charge

Conservation of charge and steady currents



Change inside change because of <u>flow</u>

$$\frac{dQ_{inside}}{dt} = - \left(\begin{array}{c} \text{Flux out of} \\ \text{surface} \end{array} \right)$$

$$\frac{d}{dt} \int_V \rho \, d^3x = - \oint_S \vec{J} \cdot d\vec{a}$$

Divergence theorem $\Rightarrow$ $$\int_V \frac{\partial \rho}{\partial t} d^3x = \int_V \left(-\vec{\nabla} \cdot \vec{J} \right) d^3x$$

$$\therefore \quad \boxed{\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{J}}$$

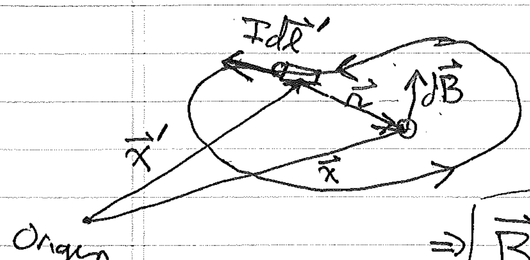
Charge "continuity equation"
Local density charge density changes if current divergences

Magnetostatics $\vec{\nabla} \cdot \vec{J} = 0 \equiv \frac{\partial \rho}{\partial t} = 0$: Electrostatics

<u>Steady current</u>

Currents are the source of $\vec{B}$

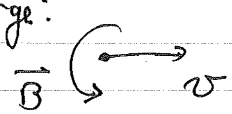
Steady current $\Rightarrow$ Biot - Savart (empirical)



$$d\vec{B} = \frac{1}{c} \frac{I\, d\vec{\ell}' \times \hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{I d\vec{\ell}'}{c} \times \frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

$$\Rightarrow \boxed{\vec{B} = \frac{1}{c}\int_{\text{line}} I\, d\vec{\ell}' \times \frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} = \frac{1}{c}\int_{\text{surface}} \vec{K}\, da' \times \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} \qquad \vec{B} = \frac{1}{c}\int_{\text{volume}} d^3x'\; \vec{J}(\vec{x}') \times \frac{(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}}$$

Point Charge:

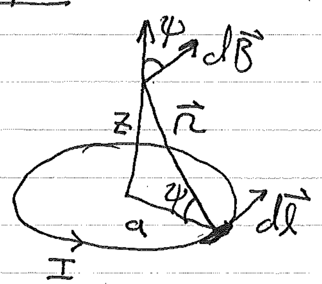


$$\vec{B} \overset{?}{=} \frac{1}{c}\, q\, \frac{\vec{v} \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

No: <u>Not steady current</u>

Only approximate for $v << c$

<u>Example:</u> Current Loop, steady current I



Only z-component survives <u>on axis</u>

$$dB_z = |d\vec{B}| \cos\psi = \frac{I\, d\ell}{c\mathscr{r}^2} \cos\psi$$

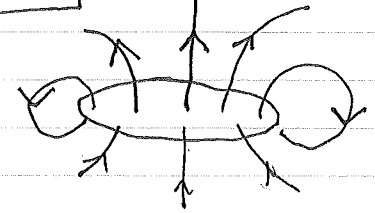
$$d\ell = a\, d\phi\,, \quad \cos\psi = \frac{a}{\mathscr{r}}\,, \quad \mathscr{r} = \sqrt{z^2+a^2}$$

$$\Rightarrow \vec{B}(z) = \left(\oint dB_z\right)\vec{e}_z = \oint \frac{I a^2 d\phi}{c(z^2+a^2)^{3/2}}\, \vec{e}_z$$

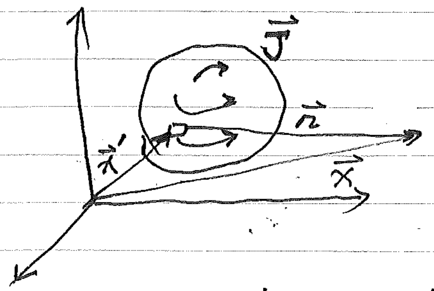
on axis $\Rightarrow$ $\boxed{\vec{B}(z) = \frac{2\pi a^2 I}{c(z^2+a^2)^{3/2}}\, \vec{e}_z}$

$$\lim_{z>>a} \vec{B} \to \frac{2\pi a^2 I}{c z^3}\, \vec{e}_z$$

Dipole field



Field equations:



$$\vec{B} = \frac{1}{c}\int d^3x'\, \vec{J}(\vec{x}') \times \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}$$

$$= \frac{1}{c}\int d^3x'\, \vec{J}(\vec{x}') \times \left(-\vec{\nabla}\frac{1}{|\vec{x}-\vec{x}'|}\right)$$

Aside: $\vec{\nabla}\times(\underset{\text{constant}}{\vec{V}} f(\vec{x})) = -\vec{V}\times\vec{\nabla}f(\vec{x})$

$$\therefore\ \vec{B}(\vec{x}) = \vec{\nabla}\times\left(\frac{1}{c}\int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right) = \vec{\nabla}\times\vec{A}$$

$$\boxed{\vec{A} = \frac{1}{c}\int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} = \text{Vector potential}}$$

Differential form of magnetostatic field eqns.

$\vec{B} = \vec{\nabla}\times\vec{A} \Rightarrow \boxed{\vec{\nabla}\cdot\vec{B} = 0}$ No magnetic monopoles

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2\vec{A}$$

$$= \underbrace{\frac{1}{c}\vec{\nabla}\int \vec{J}(\vec{x}')\cdot\vec{\nabla}\frac{1}{|\vec{x}-\vec{x}'|}}_{d_1} - \underbrace{\frac{1}{c}\int \vec{J}(\vec{x}')\,\nabla^2\frac{1}{|\vec{x}-\vec{x}'|}\,d^3x'}_{d_2}$$

$$d_1 = \frac{1}{c}\vec{\nabla}\int \vec{J}(\vec{x}')\cdot\left(-\vec{\nabla}'\frac{1}{|\vec{x}-\vec{x}'|}\right) \underset{\substack{\uparrow\\ \text{Integrate}\\ \text{by parts}}}{=} \frac{1}{c}\vec{\nabla}\left(\int\cancelto{0}{\frac{\vec{\nabla}'\cdot\vec{J}}{|\vec{x}-\vec{x}'|}} + \underset{\substack{\text{surface}\\ \text{at }\infty}}{\oint}\cancelto{0}{\frac{\vec{J}}{|\vec{x}-\vec{x}'|}}\right)$$

$$d_2 = -\frac{1}{c}\int \vec{J}(\vec{x}')\left(-4\pi\delta^{(3)}(\vec{x}-\vec{x}')\right) = \frac{4\pi}{c}\vec{J}(\vec{x})$$

$$\therefore\ \boxed{\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J}(\vec{x})}\quad \text{Ampère's Law}$$