Lecture 19 : Multipole Radiation II

For higher multipole moments we well consider only the radiation terms.

Most easily done in the Fourier domain:
Sinusoidal sources:

$$\rho(\vec{x},t) = \tilde{\rho}(\vec{x})\, e^{-i\omega t} \qquad \vec{J}(\vec{x},t) = \underset{\sim}{\vec{J}}(\vec{x})\, e^{-i\omega t}$$

(Real part in the end)

$$\Rightarrow \phi(\vec{x},t) = \int \frac{\rho(\vec{x}', t_{ret})}{|\vec{x}-\vec{x}'|}\, d^3x' = \int \frac{\tilde{\rho}(\vec{x}')}{|\vec{x}-\vec{x}'|}\, e^{-i\omega\left(t - \frac{|\vec{x}-\vec{x}'|}{c}\right)}$$

$$= \underbrace{\int \tilde{\rho}(\vec{x}')\, \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}}_{\tilde{\phi}(\vec{x})}\, e^{-i\omega t}$$

Similarly $\vec{A}(\vec{x},t) = \underset{\sim}{\vec{A}}(\vec{x})\, e^{-i\omega t}$

$$\underset{\sim}{\vec{A}}(\vec{x}) = \frac{1}{c} \int \underset{\sim}{\vec{J}}(\vec{x}')\, \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}$$

Keeping only radiation terms $\left(\sim \frac{1}{r}\right)$

$$\tilde{\phi}(\vec{x}) = \frac{1}{r} \int \tilde{\rho}(\vec{x}')\, e^{ik|\vec{x}-\vec{x}'|}$$

$$\underset{\sim}{\vec{A}}(\vec{x}) = \frac{1}{cr} \int \underset{\sim}{\vec{J}}(\vec{x}')\, e^{ik|\vec{x}-\vec{x}'|}$$

In the radiation zone $|\vec{x}| >> |\vec{x}'| \Rightarrow |\vec{x}-\vec{x}'| \approx r - \hat{r}\cdot\vec{x}'$

Far field: $e^{ik|\vec{x}-\vec{x}'|} \approx e^{ikr}\, e^{-i(k\hat{r})\cdot\vec{x}'}$

$$\left.\begin{aligned} \Rightarrow \tilde{\phi}(\vec{x}) &= \frac{e^{ikr}}{r}\int \underset{\sim}{\rho}(\vec{x}')\, e^{-i\vec{k}\cdot\vec{x}'}\, d^3x' \\ \vec{A}(\vec{x}) &= \frac{e^{ikr}}{cr}\int \underset{\sim}{\vec{J}}(\vec{x}')\, e^{-i\vec{k}\cdot\vec{x}'}\, dx' \end{aligned}\right\} \qquad \vec{k} = \frac{\omega}{c}\hat{r}$$

$$\Rightarrow \boxed{\tilde{\phi}(\vec{x}) = \frac{e^{ikr}}{r}\underset{\sim}{\rho}(\vec{k}), \qquad \vec{A}(\vec{x}) = \frac{e^{ikr}}{cr}\underset{\sim}{\vec{J}}(\vec{k}) \quad \text{Fourier transform}}$$

In the radiation zone the potentials depend on the Fourier components of the sources

<u>Fields:</u>

$$\underset{\sim}{\vec{B}}(\vec{x}) = \vec{\nabla}\times\underset{\sim}{\vec{A}}(\vec{x}) = \frac{1}{c}\vec{\nabla}\left(\frac{e^{ikr}}{r}\right)\times\underset{\sim}{\vec{J}}(\vec{k})$$

Keeping only radiation term

$$\underset{\sim}{\vec{B}}_{rad}(\vec{x}) = \frac{ik\hat{r}}{c}\times\underset{\sim}{\vec{J}}(\vec{k})\,\frac{e^{ikr}}{r} = i\vec{k}\times\frac{\underset{\sim}{\vec{J}}(\vec{k})}{c}\,\frac{e^{ikr}}{r}$$

$$\underset{\sim}{\vec{E}}_{rad}(\vec{x}) = i\frac{\omega}{c}\underset{\sim}{\vec{A}}(\vec{x}) - \vec{\nabla}\tilde{\phi}(\vec{x})$$

$$= \left(i\frac{k}{c}\underset{\sim}{\vec{J}}(\vec{k}) - i\vec{k}\,\tilde{\rho}(\vec{k})\right)\frac{e^{ikr}}{r}$$

Now $\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0 \Rightarrow -i\omega\,\tilde{\rho}(\vec{k}) + i\vec{k}\cdot\underset{\sim}{\vec{J}}(\vec{k}) = 0$

$$\Rightarrow \tilde{\rho}(\vec{k}) = \frac{\hat{k}}{c}\cdot\underset{\sim}{\vec{J}}(\vec{k})$$

$$\Rightarrow \underset{\sim}{\vec{E}}_{rad} = \frac{ik}{c}\left(\underset{\sim}{\vec{J}}(\vec{k}) - \hat{k}\,(\hat{k}\cdot\underset{\sim}{\vec{J}}(\vec{k}))\right)\frac{e^{ikr}}{r} \qquad (\hat{k}=\hat{r})$$

$$\Rightarrow \underset{\sim}{\vec{E}}_{rad} = ik\,\underset{\sim}{\vec{J}}_{\perp}(\vec{k})\,\frac{e^{ikr}}{cr}$$

Thus in general, in radiation zone

$$\boxed{\begin{aligned} \underset{\sim}{\vec{B}}_{rad}(\vec{x},\omega) &= \frac{i\omega}{c}\left(\hat{r}\times\underset{\sim}{\vec{J}}(\vec{k})\right)\frac{e^{ikr}}{cr} \\ \underset{\sim}{\vec{E}}_{rad}(\vec{x},\omega) &= \frac{i\omega}{c}\,\underset{\sim}{\vec{J}}_{\perp}(\vec{k})\,\frac{e^{ikr}}{cr} = \frac{i\omega}{c}\,\hat{r}\times\left(\underset{\sim}{\vec{J}}(\vec{k})\times\hat{r}\right)\frac{e^{ikr}}{cr} \\ &= \underset{\sim}{\vec{B}}_{rad}\times\hat{r} \end{aligned}}$$

Multipole contributions to $\underset{\sim}{\vec{J}}(\vec{k})$

$$\underset{\sim}{\vec{J}}(\vec{k}) = \int d^3x'\,\vec{J}(\vec{x}')\,e^{-i\vec{k}\cdot\vec{x}'}$$

Expand in power series in $\vec{k}\cdot\vec{x}'$ $\qquad (\lambda \gg d)$

$$\Rightarrow \underset{\sim}{\vec{J}}(\vec{k}) = \underbrace{\int d^3x'\,\vec{J}(\vec{x}')}_{\vec{J}^{(1)}(\vec{k})} + \underbrace{\int d^3x'\,(-i\vec{k}\cdot\vec{x}')\,\vec{J}(\vec{x}')}_{\vec{J}^{(2)}(\vec{k})} + \ldots$$

$$\vec{J}^{(1)}(\vec{k}) = -i\omega\,\vec{p}_0$$ Electric dipole contribution $\quad \vec{p} = \vec{p}_0 e^{-i\omega t}$

$$\Rightarrow \underset{\sim}{\vec{E}}^{(1)}_{rad}(\vec{x},\omega) = +\frac{\omega^2}{c^2}\,\vec{p}_{0\perp}\,\frac{e^{ikr}}{r}, \qquad \underset{\sim}{\vec{B}}^{(1)}_{rad} = -\frac{\omega^2}{c^2}\,(\hat{r}\times\vec{p}_0)\,\frac{e^{ikr}}{r}$$

$$= \hat{r}\times\underset{\sim}{\vec{E}}_{rad}$$

Magnetic Dipole + Electric Quadrupole Radiation

$$\underset{\sim}{\vec{J}}^{(2)}(\vec{k}) = -i\int(\vec{k}\cdot\vec{x}')\,\underset{\sim}{\vec{J}}(\vec{x}')\,d^3x' = -i\vec{k}\cdot\int\vec{x}'\,\underset{\sim}{\vec{J}}(\vec{x}')\,d^3x'$$

$$\tilde{J}_i^{(2)}(\vec{k}) = -ik_j\int x_j'\,\tilde{J}_i(\vec{x}')\,d^3x'$$

Aside: $$\int x_j'\tilde{J}_i\,d^3x' = \underbrace{\frac{1}{2}\int\left(x_j'\tilde{J}_i - x_i\tilde{J}_j\right)}_{\text{Magnetic dipole contribution}} + \underbrace{\frac{1}{2}\int x_j'\tilde{J}_i + x_i'\tilde{J}_j}_{\text{Electric Quad Contribution}}$$

Magnetic dipole: $$\tilde{J}_i^{(M1)}(\vec{k}) = ik_j\epsilon_{ijk}\int\left(\frac{\vec{x}'\times\tilde{\vec{J}}(\vec{x}')}{2}\right)_k d^3x' = i\epsilon_{ijk}\,ck_j\,m_{0k}$$

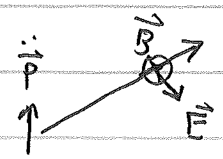
$$\Rightarrow \vec{J}^{(M1)}(\vec{k}) = +ic\vec{k}\times\vec{m}_0 = i\omega\,\hat{r}\times\vec{m}_0$$

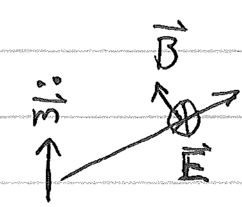
$\Rightarrow$ Oscillating magnetic dipole $\vec{m}_0 e^{-i\omega t}$

$$\Rightarrow \boxed{\begin{aligned}\vec{B}_{rad}^{(M1)} &= -\frac{\omega^2}{c^2}\hat{r}\times(\hat{r}\times\vec{m}_0)\frac{e^{ikr}}{r} = -\frac{\omega^2}{c^2}\vec{m}_\perp\frac{e^{ikr}}{r}\\ \vec{E}_{rad}^{(M1)} &= \hat{B}_{rad}\times\hat{r} = \frac{\omega^2}{c^2}(\vec{m}_0\times\hat{r})\frac{e^{ikr}}{r}\end{aligned}}$$

Magnetic dipole contribution like Electric dipole

$\vec{p}\to\vec{m}$ $\qquad\vec{E}\to\vec{B}$ $\qquad\vec{B}\to-\vec{E}$





(E1)

Power radiated in <u>magnetic dipole</u>

<u>Angular distribution:</u>

$$\frac{d\langle P_{rad}\rangle}{d\Omega} = \langle \vec{S}\rangle \cdot r^2 = \frac{c}{8\pi}|\underset{\sim}{\vec{E}}_{rad}|^2 r^2 = \frac{c}{8\pi}|\underset{\sim}{\vec{B}}_{rad}| r^2$$

$$= \frac{c}{8\pi}k^4|\vec{m}_{0\perp}|^2 = \frac{ck^4}{8\pi} m_0^2 \sin^2\theta$$

Total radiated power in 4π-steradians

$$\boxed{\langle P_{rad}\rangle = \frac{ck^4}{3} m_0^2}$$

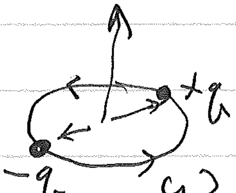
Suppose system has both a time varying <u>electric</u> and <u>magnetic dipole</u>. How does the power (E1) compare to (M1)?

$$\frac{\text{Power (E1)}}{\text{Power (M1)}} = \frac{p_0^2}{m_0^2} \sim \frac{(qd)^2}{\left(\frac{q}{cT}d^2\right)^2} \sim \left(\frac{cT}{d}\right)^2 \sim \left(\frac{c}{v}\right)^2 \sim \left(\frac{\lambda}{d}\right)^2$$

Expansion parameter

So for charges moving slowly compared to c electric dipole radiation dominates except if the dipole contribution vanishes due to symmetry

<u>Example:</u>



$$p_0 \sim qR$$

$$m_0 \sim q\frac{\omega}{c}R^2 = I\frac{\text{Area}}{c}$$

$$\Rightarrow \quad \frac{m_0}{p_0} \sim \frac{\omega}{c}R \ll 1$$

Electric Quadrapole Contribution

$$\underset{\sim}{\vec{J}}^{(E2)}(\vec{k}) = -i\frac{\vec{k}}{2}\cdot\int\left(\vec{x}'\,\vec{J}(\vec{x}') + \vec{J}(\vec{x}')\,\vec{x}'\right)\,d^3x'$$

$$= i\frac{\vec{k}}{2}\cdot\int\vec{x}'\,\vec{x}'\,\underbrace{\left(\vec{\nabla}'\cdot\vec{J}(\vec{x}')\right)}_{\to\, i\omega\rho(\vec{x}')}\,d^3x'$$

$$= -\frac{\omega^2}{2c}\,\hat{r}\cdot\underbrace{\int\vec{x}'\,\vec{x}'\,\rho(\vec{x}')\,d^3x'}_{\frac{1}{3}\left(\overleftrightarrow{Q}_0 + \overleftrightarrow{1}\int(r')^2\rho(\vec{x}')\right)d^3x}$$

Electric Quadrapole moment: $\overleftrightarrow{Q}(t) = \overleftrightarrow{Q}_0\,e^{-i\omega t}$

$$\overleftrightarrow{Q}_0 = \int\left(3\vec{x}'\vec{x}' - \overleftrightarrow{1}\right)\rho(\vec{x}')\,d^3x'$$

$$\underset{\sim}{\vec{J}}^{(E2)}(\vec{k}) = -\frac{\omega^2}{6c}\,\hat{r}\cdot\overleftrightarrow{Q}_0 \quad - \frac{\omega^2}{6}\,\hat{r}\left(\int(r')^2\rho(\vec{x}')\,d^3x'\right)$$

Fields $\quad \vec{B}_{rad}^{(E2)} = i\frac{\omega}{c^2}\left(\hat{r}\times\underset{\sim}{\vec{J}}^{(E2)}(\vec{k})\right)\frac{e^{ikr}}{r}$

$$\boxed{\begin{aligned}
&\Rightarrow \vec{B}_{rad}^{(E2)} = -i\frac{\omega^3}{6c^3}\,\hat{r}\times\left(\hat{r}\cdot\overleftrightarrow{Q}_0\right)\frac{e^{ikr}}{r}\\[1em]
&\vec{E}_{rad}^{(E2)} = \vec{B}_{rad}^{(E2)}\times\hat{r} = -i\frac{\omega^3}{6c^3}\left(\hat{r}\cdot\overleftrightarrow{Q}_0\right)_{\perp}\frac{e^{ikr}}{r}
\end{aligned}}$$

$$\frac{dP_{rad}}{d\Omega} = \frac{c}{8\pi}\left|\vec{B}_{rad}\right|^2 r^2 = \frac{c\,k^6}{288\pi}\left|\left(\hat{r}\cdot\overleftrightarrow{Q}_0\right)\times\hat{r}\right|^2$$

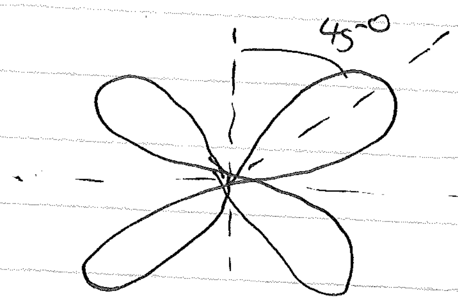
Example: Distribution with azimuthal symmetry

$$\overleftrightarrow{Q}_0 = Q_0 \begin{bmatrix} -\frac{1}{2} & & \\ & -\frac{1}{2} & \\ & & 1 \end{bmatrix} \qquad \hat{r} = \sin\theta(\cos\phi\,\hat{x} + \sin\phi\,\hat{y}) + \cos\theta\,\hat{z}$$

$$\Rightarrow \overleftrightarrow{Q}_0 \cdot \hat{r} = Q_0\left(-\tfrac{1}{2}\sin\theta\,(\cos\phi\,\hat{x} + \sin\phi\,\hat{y}) + \cos\theta\,\hat{z}\right)$$

$$\Longrightarrow \vec{B}_{rad} = \frac{-i\omega^3}{4c^3}\,\sin 2\theta\; Q_0\,\frac{e^{ikr}}{r}\,\hat{\phi}$$

$$\frac{dP}{d\Omega} \sim |\vec{B}_{rad}|^2 \sim \sin^2 2\theta$$



radiation pattern with 4 lobes

Example: Oscillating charge:

q ω_0

$z(t) = a\cos\omega_0 t$

$$\rho(z,t) = q_0\,\delta(z - a\cos\omega_0 t)$$

$$\Rightarrow Q_0(t) = q a^2(\cos 2\omega_0 t - 1)$$

radiates at 2ω!

Generally, all harmonics are radiated:

$$\rho(z,t) = \left(q_0 \int \frac{dk}{2\pi} e^{ikz}\right) e^{-ika\cos\omega t} \longrightarrow \sum_{n=0}^{\infty} (-ika)\, \underbrace{J_n(ka)}_{\text{Bessel fnct.}}\, e^{-in\omega_0 t}$$

$$\Longrightarrow \rho = \sum_{n=0}^{\infty} \boxed{\int_{-\infty}^{\infty} (-ika)\, e^{ikz}\, J_n(ka)}\, e^{-in\omega t}$$

expansion parameter

Fourier series