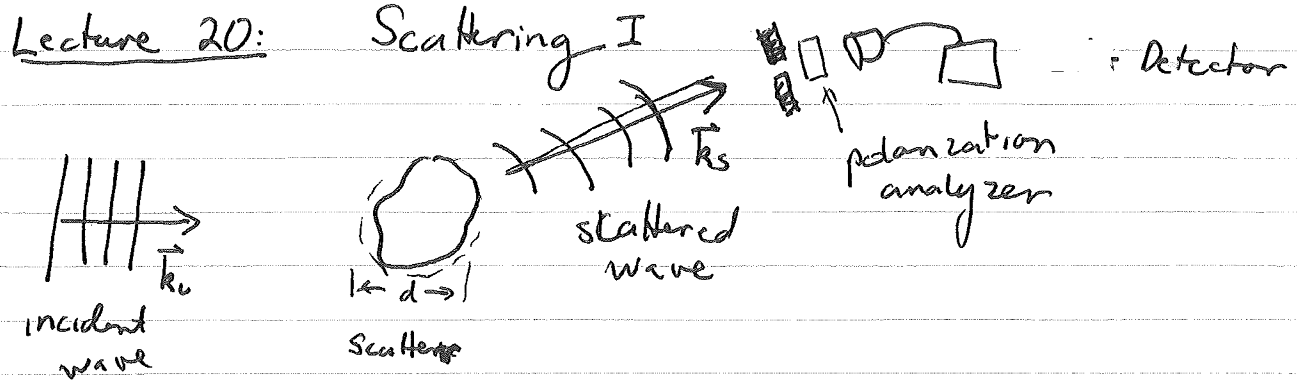
Lecture 20: Scattering I



Definition:

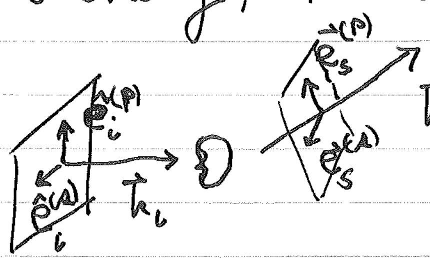
Scattering cross-section $\sigma = \dfrac{\text{Total stuff scattered/time}}{\text{Incident Flux density of stuff}}$

Example: For E/M waves $\sigma = \dfrac{\text{Total Power scattered}}{\text{Incident intensity}}$

Differential scattering cross-section

$$\frac{d\sigma}{d\Omega}^{(\hat{k}_i,\hat{k}_s)} \equiv \frac{\text{Power scattered / solid angle into direction } \hat{k}_s}{\text{Incident intensity in direction } \hat{k}_i}$$

Generally, the scattering will be polarization dependent



$\underset{\pi}{P}$-polarization (in the plane of $\hat{k}_i$ and $\hat{k}_s$)

$\underset{\sigma}{A}$-polarization ($\perp$ to the plane)

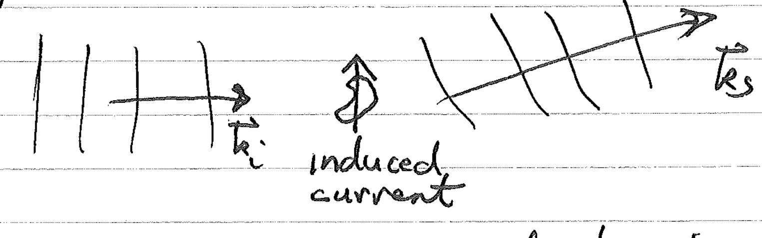
$\Rightarrow$ Define polarization dependent scattering cross-section

$$\frac{d\sigma}{d\Omega}(\hat{k}_i, \hat{e}_i ; \hat{k}_s, \hat{e}_s) = \frac{\dfrac{d\langle \text{Power}\rangle_{scat}}{d\Omega}}{\langle \vec{S}\rangle_{inc}} = r^2 \frac{\langle \vec{S}\rangle_{scat}\cdot\hat{k}_s}{\langle \vec{S}\rangle_{inc}\cdot\hat{k}_i}$$

$$\hat{k}_i\cdot\langle\vec{S}\rangle_{inc} = \frac{c}{8\pi}\left|\hat{e}_i^{*}\cdot\vec{E}_i\right|^2, \quad \hat{k}_s\cdot\langle\vec{S}\rangle_{scat} = \frac{c}{8\pi}\left|\hat{e}_s^{*}\cdot\vec{E}_s\right|^2$$

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega}(\hat{k}_i, \hat{e}_i \,;\, \hat{k}_s, \hat{e}_s) = r^2 \frac{|\hat{e}_s^* \cdot \vec{E}_s|^2}{|\hat{e}_i^* \cdot \vec{E}_i|^2}}$$

Long-wavelength scattering $\lambda \gg d$



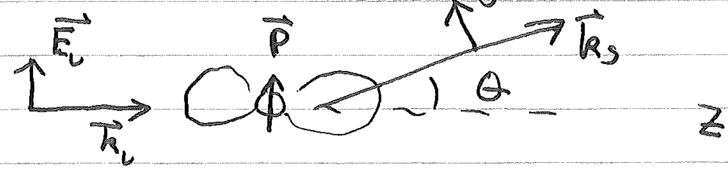
Induced currents dominated by lowest order multipoles:
Unless symmetry dictates otherwise,
Electric dipole dominates : $\vec{p}$: induced by $\vec{E}_i$

$$\Rightarrow \vec{E}_{scat} = k^2 \vec{p}_\perp \frac{e^{ikr}}{r} \qquad \vec{p}_\perp = \hat{r} \times (\vec{p} \times \hat{r})$$

$$\Rightarrow \frac{d\sigma}{d\Omega}(\hat{k}_i, \hat{e}_i \,;\, \hat{k}_s, \hat{e}_s) = k^4 \frac{|\hat{e}_s^* \cdot \vec{p}_\perp|^2}{|\hat{e}_i^* \cdot \vec{E}_i|^2}$$

Example: linear (scalar) polarizable particle
(e.g. small dielectric sphere $d \ll \lambda$)

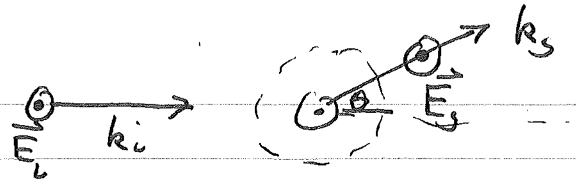
~~TE~~ polarization



(Ignoring back action of radiated field on the particle
induced dipole $\vec{p} = \tilde{\alpha} \vec{E}_i$)

p-polarization $|\vec{p}_\perp| = p \cos\theta = \tilde{\alpha} E_i \cos\theta$

<u>σ-polarization</u>



$$\Rightarrow \quad \vec{P}_\perp = \vec{p} \quad \text{for all } \theta$$

$$\Rightarrow \boxed{\begin{array}{c} \dfrac{d\sigma}{d\Omega}(\hat{k}_i, \hat{e}_i^{(\pi)}; \hat{k}_s, \hat{e}_s^{(\pi)}) = k^4 |\tilde{\alpha}|^2 \cos^2\theta \\ \dfrac{d\sigma}{d\Omega}(\hat{k}_i\, \hat{e}_i^{(\sigma)}; \hat{k}_s, \hat{e}_s^{(\sigma)}) = k^4 |\tilde{\alpha}|^2 \\ \text{where } \cos\theta = \hat{k}_i \cdot \hat{k}_s \end{array}}$$

For isotropic polarizability scattering $\sigma \leftrightarrow \pi$ is zero

Often the incident field is <u>unpolarized</u> (incoherent superposition of σ + π polarization)

$\Rightarrow$ Average over incident polarization

<u>Unpolarized, Scattering into p-polarization</u>

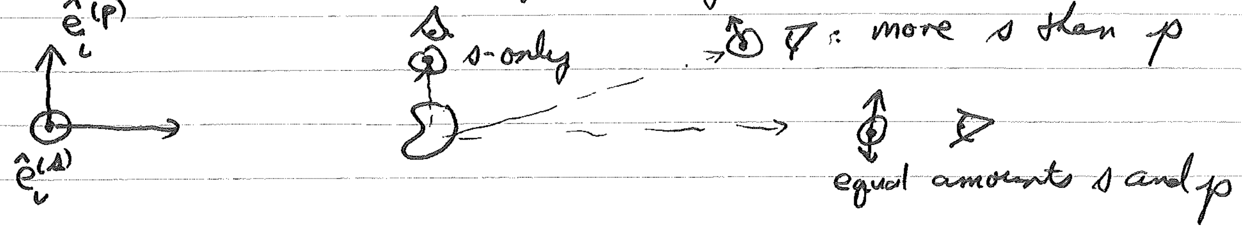
$$\frac{d\sigma}{d\Omega}(\hat{e}_s^{(p)}, \cos\theta) = \frac{1}{2}\left[\cancel{\frac{d\sigma}{d\Omega}(\hat{k}_i, \hat{e}_i^{(s)}; \hat{k}_s, \hat{e}_s^{(p)})}^{\,0} + \frac{d\sigma}{d\Omega}(\hat{k}_i, \hat{e}_i^{(p)}; \hat{k}_s\, \hat{e}_s^{(p)})\right]$$

$$= \frac{1}{2} k^4 |\tilde{\alpha}|^2 \cos^2\theta$$

Unpolarized scattering into s-polarization

$$\frac{d\sigma}{d\Omega}(\hat{e}_s^{(s)}, \cos\theta) = \frac{1}{2} k^4 |\tilde{\alpha}|^2$$

$\Rightarrow$ Scattered wave is generally <u>partially</u> polarized

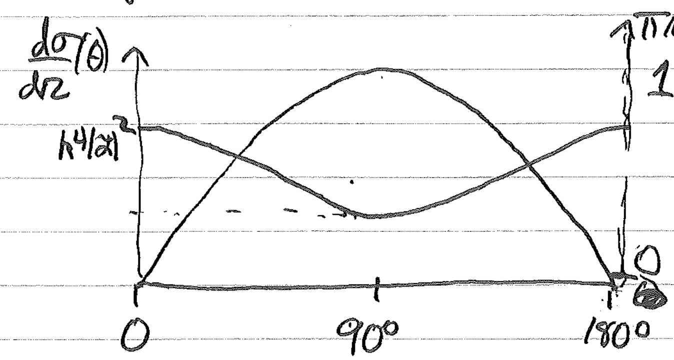


Degree of polarization (how polarized in scattered wave)

$$\Pi(\theta) = \frac{\left|\frac{d\sigma}{d\Omega}(\hat{e}_s^{(s)}, \theta) - \frac{d\sigma}{d\Omega}(\hat{e}_s^{(p)}, \theta)\right|}{\left| \quad '' \quad + \quad '' \quad \right|} = 1 - \cos^2\theta$$
$$= \sin^2\theta$$

Total differential cross section into any polarization: $\frac{d\sigma}{d\Omega}(\theta) = \frac{d\sigma}{d\Omega}(\hat{e}_s^{(s)}, \theta) + \frac{d\sigma}{d\Omega}(\hat{e}_s^{(p)}, \theta)$

$$= \frac{1}{2} k^4 |\alpha|^2 (1 + \cos^2\theta)$$



- Scattering is peaked into the forward and backward directions (unpolarized)
- Scattering at 90° is decreased by a factor of two (polarized)

Example: Lorentz oscillator (charge on spring)

$$\alpha = \frac{e^2/m}{\omega_0^2 - \omega^2 - i\omega\Gamma}$$

• $\omega \gg \omega_0 \Rightarrow$ Free electrons $\alpha = -\frac{e^2}{m\omega^2}$

Thompson scattering

(Unpolarized into all $\hat{e}_s$) $$\frac{d\sigma}{d\Omega} = k^4\left(\frac{e^2}{m\omega^2}\right)^2\left(\frac{1+\cos^2\theta}{2}\right) = r_c^2\left(\frac{1+\cos^2\theta}{2}\right)$$

$r_c = \frac{e^2}{mc^2}$: Classical electron radius: 2.8×10^{-13} cm

Total Thompson cross-section

$$\sigma_T = \int\frac{d\sigma}{d\Omega}d\Omega = 2\pi r_c^2 \underbrace{\int\left(\frac{1+\cos^2\theta}{2}\right)d(\cos\theta)}_{4/3}$$

$$\Rightarrow \boxed{\sigma_T = \frac{8\pi}{3}r_0^2 = 0.665\times10^{-24}\,\text{cm}^2 = 0.665 \text{ barns}}$$

Note: Thompson formula is valid only when $\hbar\omega \ll mc^2$, i.e. $\lambda \gg \lambda_{compton} = \frac{\hbar}{mc}$, otherwise Compton scattering

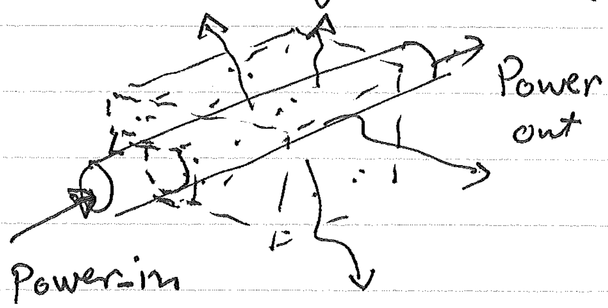
• $\omega \ll \omega_0 \Rightarrow$ Rayleigh scattering

$$\Rightarrow \alpha \approx \frac{e^2}{m\omega_0^2} = \frac{\omega^2}{\omega_0^2}\alpha_{Thompson}$$

$$\Rightarrow \boxed{\sigma_R = \frac{8\pi}{3}r_0^2\left(\frac{\omega}{\omega_0}\right)^4 = \sigma_T\left(\frac{\omega}{\omega_0}\right)^4}$$

High frequencies scatter more than low!

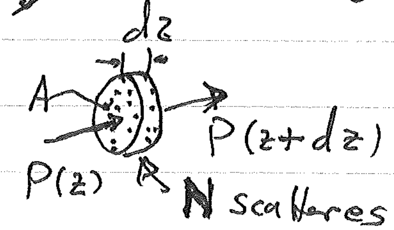
Rayleigh scattering and why the sky is blue



Attenuation by scattering: Power scattered out of beam

Beer's law: Differential eqn ~~by~~ for attenuation by scattering

Consider a differential slab:



$$dP = P(z+dz) - P(z) = A\,dI = -dP_{scat} = -N\sigma I$$

$$N = n A\,dz \quad \left(n = \frac{\#\ \text{scatterers}}{\text{Volume}}\right)$$

$$\Rightarrow dI = -n\sigma\,dz\,I \Rightarrow \boxed{\frac{dI}{dz} = -n\sigma I}$$

Exponential attenuation: $I(z) = e^{-n\sigma z} I$

"Attenuation length" $\frac{1}{n\sigma}$ = mean-free path for scattering

For a total path ~~the~~ length L

$$I(L) = e^{-OD}\, I(0)$$

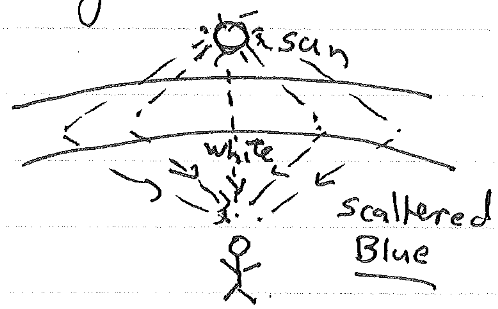
"Optical density" $OD = n\sigma L$

Why the sky is blue

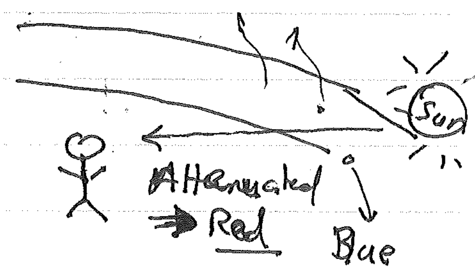
- The atmosphere is dominated by nitrogen molecules N_2.
- Main resonance @ $\lambda_0 = 75\,\text{nm}$ (ultraviolet)
- Visible $400 \lesssim \lambda \lesssim 600\,\text{nm}$ $\Rightarrow$ $\omega \ll \omega_0$ $\Rightarrow$ Rayleigh

$$\Rightarrow \sigma_R(\omega) = \left(\frac{\omega}{\omega_0}\right)^4 \sigma_T \Rightarrow$$ Blue scattered much more than red

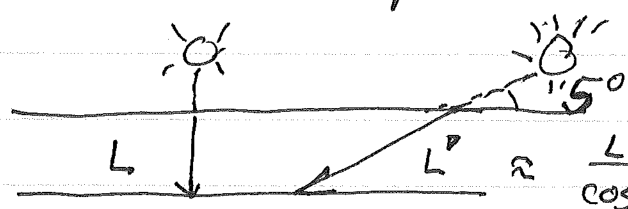
Day time @ noon



Sun set



Sunsets are red because the effective length of the atmosphere is larger @ sunset



$$L' \approx \frac{L}{\cos 5°} = 11.5\,L$$

$$\text{Fraction of Power scattered} = 1 - e^{-n\sigma_R z}$$

Atmospher @ zenith $nL \cong 1.7 \times 10^{25}\,\text{cm}^{-2}$

For blue @ $\lambda = 450\,\text{nm}$ $\sigma_R = \sigma_T \left(\frac{\lambda_0}{\lambda}\right)^4 = 2.17 \times 10^{-16}\,\text{cm}^2$

$\Rightarrow$ Fraction scattered @ zenith $= 1 - e^{-nL\sigma_R} = 8.6 \times 10^{-3}$

" " @ sunset $= 1 - e^{-nL'\sigma_R} = 0.1$