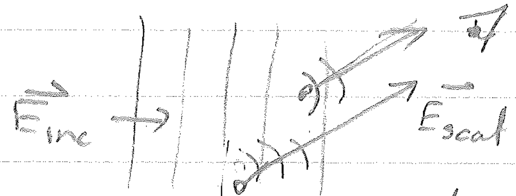
Lecture 21

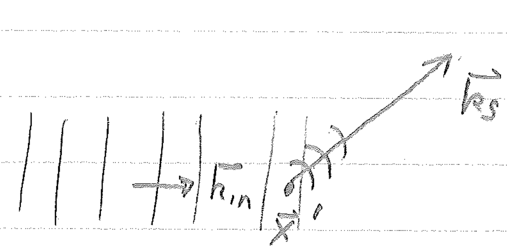
Rayleigh scattering and the "structure factor"

When we derived Beer's Law we implicitly assumed that we could add the scattered power from each radiating dipole. But we should have added the fields not the intensity. How did we get away with this? Should we have gotten away with this?



Different scatterers see different phases of the incident wave. Also, the path length from the scatterer to the observer is different, so accumulated phase is different.

Consider a scatterer @ $\vec{x}'$, observation point $\vec{x}$



Dipole radiation: $\underset{\sim}{\vec{E}}_{scat} = k^2 \underset{\sim}{\vec{p}}_\perp \dfrac{e^{ik_s|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}$

In the far-field $|\vec{x}| \gg |\vec{x}'|$ $\quad |\vec{x}-\vec{x}'| \approx r - \hat{r}\cdot\vec{x}'$

$$\frac{e^{ik_s|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \simeq \frac{e^{ik_s r}}{r}\; e^{-ik_s\hat{r}\cdot\vec{x}'}$$

Spherical wave

Extra phase accumulated

The induced dipole moment

$$\vec{p} = \underset{\sim}{\alpha} \vec{E}_{inc}(\vec{x}') = \underset{\sim}{\alpha} \vec{E}_0 \, \underbrace{e^{i\vec{k}_i \cdot \vec{x}'}}_{\Uparrow}$$

phase of oscillation @ position of dipole

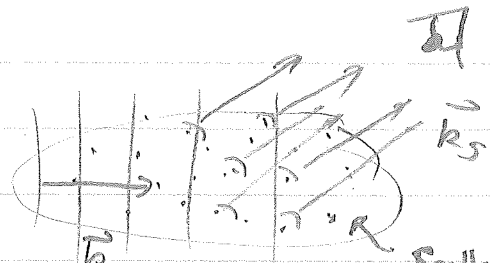
$$\Rightarrow \boxed{\vec{E}_{scat} = k^2 \underset{\sim}{\alpha} \vec{E}_{0\perp} \, e^{i(\vec{k}_i - \vec{k}_s)\cdot \vec{x}'} \, \frac{e^{ik_s r}}{r}}$$

<u>Forward scattering</u>: When $\vec{k}_s = \vec{k}_i$

$$\vec{E}_{scat} = k^2 \underset{\sim}{\alpha} \vec{E}_{0\perp} \, \frac{e^{ik_s r}}{r}$$

independent of $\vec{x}'$ $\Rightarrow$ All scattered waves in the forward direct add in phase

<u>Collection of scatterers</u>



Scatterer @ $\vec{x}_j$

$$\vec{E}_{scat} = k^2 \underset{\sim}{\alpha} \vec{E}_{0\perp} \frac{e^{ikr}}{r} \underbrace{\left(\sum_j e^{i\vec{q}\cdot\vec{x}_j} \right)} \qquad (\vec{q} = \vec{k}_s - \vec{k}_i)$$

Phasor addition depending on positions of scatterers and direction of scattered wave relative to incident wave.

We can turn the discrete sum over scatterers into a continuous integral by coarse-graining

$$\boxed{\sum_j e^{-i\vec{q}\cdot\vec{x}_j} = \int d^3x' \underset{\substack{\uparrow \\ \text{density of scatterers}}}{n(\vec{x}')} e^{-i\vec{q}\cdot\vec{x}'} = \tilde{n}(\vec{q})}$$

$\Rightarrow$ Scattered field proportional to Fourier transform of the density distribution @ $\vec{q} = \vec{k}_s - \vec{k}_i$

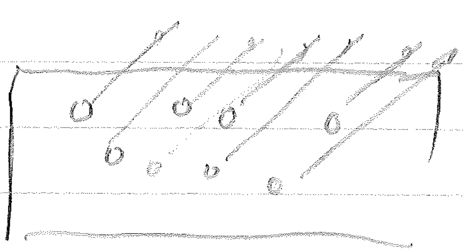
Example: Continuous smooth, uniform dielectric (glass, water)

$$n(\vec{x}) = n_0 \qquad \text{(inside glass)}$$

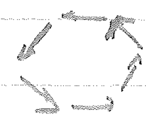
$$\Rightarrow \tilde{n}(\vec{q}) = n_0 \int d^3x' \, e^{-i\vec{q}\cdot\vec{x}'} = n_0 \delta^{(3)}(\vec{q}) = n_0 \delta^{(3)}(\vec{k}_s - \vec{k}_i)$$

$$\Rightarrow \boxed{\vec{E}_{scat} = 0 \text{ except in forward direction}}$$

In all directions other than $\vec{k}_s = \vec{k}_i$ (forward) the scattered waves have every possible phase

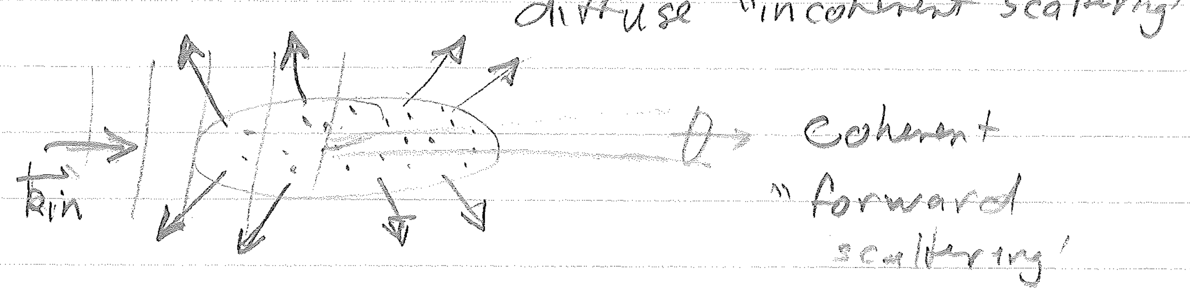


$$\sum_j e^{-i\vec{q}\cdot\vec{x}_j} = 0 \qquad \vec{q} \neq 0$$



All possible phases

So diffuse Rayleigh scattering is due to <u>density fluctuations</u>



If we write $n(\vec{x}) = \bar{n} + \delta n(\vec{x})$

(mean density) (fluctuation around mean)

$$\Rightarrow \tilde{n}(\vec{q}) = \bar{n}\,\delta^{(3)}(\vec{q}) + \delta\tilde{n}(\vec{q})$$

$\Rightarrow$ Diffuse scattering into 4π due solely to density fluctuations

<u>Structure factor</u>

From our expression for the scattered field

$$I_{scat}(\vec{q}) \propto \left|\sum_j e^{-i\vec{q}\cdot\vec{x}_j}\right|^2$$

$$= \sum_{i,j} e^{-i\vec{q}\cdot(\vec{x}_j - \vec{x}_i)}$$

If positions of scatterers are random then we average over some probability distribution of position

$$\Rightarrow I_{scat}(\vec{q}) \propto \underbrace{\left\langle \sum_{i,j}{}' e^{-i\vec{q}\cdot(\vec{x}_j - \vec{x}_i)} \right\rangle}$$

$\equiv S(\vec{q})$: "Structure factor"
(sometimes normalized by N scatters)

Note: If we break up the sum for $i=j$
$i \neq j$

$$\Rightarrow S(\vec{q}) = \underset{\uparrow}{N} + \underbrace{\left\langle \sum_{i \neq j} e^{-i\vec{q}\cdot(\vec{x}_j - \vec{x}_i)} \right\rangle}$$

<u>incoherent</u>
addition of scatters

<u>coherent</u>
<u>contribution</u>
(interference)

For $\vec{q} = 0$ $\sum_{i \neq j} e^{-i\vec{q}\cdot(\vec{x}_j - \vec{x}_i)} = N(N-1)$

$\Rightarrow$ $\boxed{S(\vec{q}=0) = N^2 \Rightarrow \text{In first Born scattered intensity } \propto N^2 \text{: interference}}$

For $\vec{q} \neq 0$, if <u>positions are random</u>

then $\left\langle \sum_{i \neq j}{}' e^{-i\vec{q}\cdot(\vec{x}_j - \vec{x}_i)} \right\rangle = 0$

$\Rightarrow$ $\boxed{S(\vec{q} \neq 0) = N \text{ for random scatterers} \Rightarrow \text{Beer's Law}}$

Said another way, for random positions, the fluctuation in density $\propto \sqrt{\bar{n}}$ ← mean density

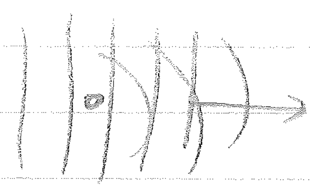
Since $\vec{E}_{scat}(\vec{q} \neq 0) \propto$ density fluctuations $\propto \sqrt{\bar{n}}$

$$I_{scat} \propto |\vec{E}_{scat}|^2 \propto \bar{n} \propto N$$

"incoherent"

Index of refraction

The forward scattered wave can interfere with the incident wave



This interference can phase shift the arrival of a peak or valley of the wave. This changes the velocity of a phase of the wave ⇒ index of refraction

The imaginary part of the index corresponds to destructive interference of the forward wave with the incident wave.

(See homework)

General theory of E/M scattering by dielectrics

Begin with the macroscopic Maxwell's Eqns. in the frequency domain (no free charge, current)

$$\vec{\nabla}\cdot\vec{D} = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = i\frac{\omega}{c}\vec{B} \qquad \vec{\nabla}\times\vec{B} = -i\frac{\omega}{c}\vec{D}$$

With a linear dielectric $\vec{D} = \epsilon\vec{E}$

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E} = \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2 E = i\frac{\omega}{c}\vec{\nabla}\times\vec{B} = \frac{\omega^2}{c^2}\epsilon\vec{E}$$

We consider $\epsilon(\vec{x}) = \underset{\substack{\uparrow \\ \text{Background mean}}}{\bar{\epsilon}} + \delta\epsilon(\vec{x})$ ← spatially dependent variation

e.g. dilute gas with density fluctuations:

$$\bar{\epsilon} = 1 + 4\pi\underset{\substack{\uparrow \\ \text{mean density}}}{\bar{n}}\alpha, \qquad \delta\epsilon = 4\pi\underset{\substack{\uparrow \\ \text{fluctuation}}}{\delta n(\vec{x})}\alpha$$

Aside: $\vec{\nabla}\cdot\vec{D} = \vec{\nabla}\cdot\left[(\bar{\epsilon}+\delta\epsilon)\vec{E}\right] = \bar{\epsilon}\vec{\nabla}\cdot\vec{E} + \vec{\nabla}\cdot(\delta\epsilon\vec{E})$
$= 0$

$$\Rightarrow \vec{\nabla}\cdot\vec{E} = -\nabla\left(\frac{\delta\epsilon}{\bar{\epsilon}}\vec{E}\right)$$

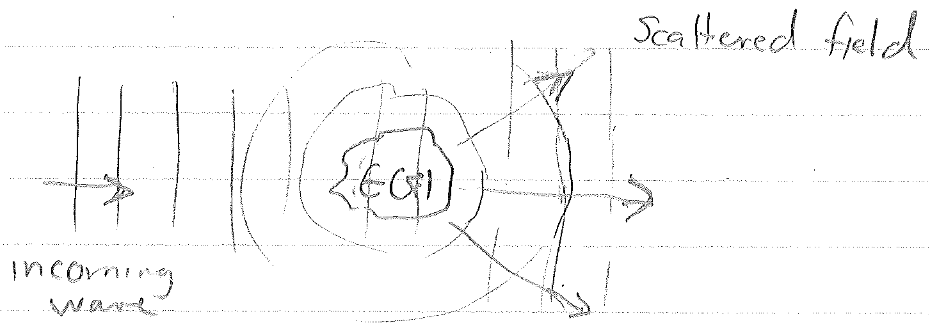
$$\Rightarrow \boxed{\left(\nabla^2 + \frac{\omega^2}{c^2}\bar{\epsilon}\right)\vec{E} = \underbrace{-\frac{\omega^2}{c^2}\delta\epsilon\vec{E} - \nabla\left(\vec{\nabla}\cdot\left(\frac{\delta\epsilon}{\bar{\epsilon}}\vec{E}\right)\right)}_{\vec{S}(\vec{x},\omega)}}$$

Formal solution

$$\vec{E}(\vec{x},\omega) = \underbrace{\vec{E}_{Hom}(\vec{x},\omega)}_{\text{Homogeneous solution}} + \underbrace{\int G(\vec{x}-\vec{x}';\omega)\, \vec{\mathcal{S}}(\vec{x}')\, d^3x'}_{\text{Particular solution}}$$

Green's function for Helmholtz Eqn.

By choosing $\vec{E}_{Hom}(\vec{x},\omega)$ with appropriate boundary conditions as an incomming wave, asymptotically far from the scatterer, and $G(\vec{x}-\vec{x}',\omega)$ as the retarded solution, this solution describes scattering



Green's function: $G^{(+)}(\vec{x}-\vec{x}';\omega) = -\frac{1}{4\pi}\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}$

$$k = \frac{\omega}{c}\sqrt{\epsilon}$$

$$\Rightarrow \vec{E}(\vec{x},\omega) = \vec{E}_{in}(\vec{x},\omega) - \frac{1}{4\pi}\int d^3x' \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\vec{\mathcal{S}}(\vec{x},\omega)$$

Asymptotically far from the scatterer $|\vec{x}| \gg |\vec{x}'|$

$$\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|} \cong e^{-i\vec{k}_s\cdot\vec{x}'}\frac{e^{ikr}}{r} \qquad \vec{k}_s = k\hat{r}$$

$$\Rightarrow \quad \vec{E}(\vec{x},\omega) = \underbrace{\vec{E}_{in}(\vec{x},\omega)}_{\text{incoming wave}} + \underbrace{\vec{A}_{scat}(\omega)\frac{e^{ikr}}{r}}_{\text{outgoing spherical wave}}$$

<u>Scattering amplitude:</u>

$$\vec{A}_{scat} = -\frac{1}{4\pi}\int d^3x'\, \vec{\mathcal{S}}(\vec{x}')\, e^{-i\vec{k}_s\cdot\vec{x}'} = -\frac{1}{4\pi}\underset{\sim}{\vec{\mathcal{S}}}(\vec{k}_s)$$

spatial Fourier transform

$$= \frac{1}{4\pi\epsilon}\int d^3x'\, e^{-i\vec{k}_s\cdot\vec{x}'}\left\{k^2\,\delta\epsilon(\vec{x}') + \vec{\nabla}'(\vec{\nabla}'\cdot\delta\epsilon\vec{E}(\vec{x}'))\right\}$$

$$\underset{\text{integration by parts}}{=} \frac{1}{4\pi\epsilon}\int d^3x'\, e^{-i\vec{k}_s\cdot\vec{x}'}\left\{k^2\vec{E}(\vec{x}') - \vec{k}_s(\vec{k}_s\cdot\vec{E}(\vec{x}'))\right\}\delta\epsilon(\vec{x}')$$

$$\Rightarrow \quad \boxed{\vec{A}_{scat}(\omega) = \frac{k^2}{4\pi\epsilon}\int d^3x'\, e^{-i\vec{k}_s\cdot\vec{x}'}\,\vec{E}_\perp(\vec{x}')\,\delta\epsilon(\vec{x}') \qquad \text{where } \vec{E}_\perp(\vec{x}') = \vec{E}(\vec{x}') - \hat{r}(\hat{r}\cdot\vec{E}(\vec{x}'))}$$

This solution in purely formal since the scattering amplitude depends the total field $\vec{E}$, including the scattered field itself.

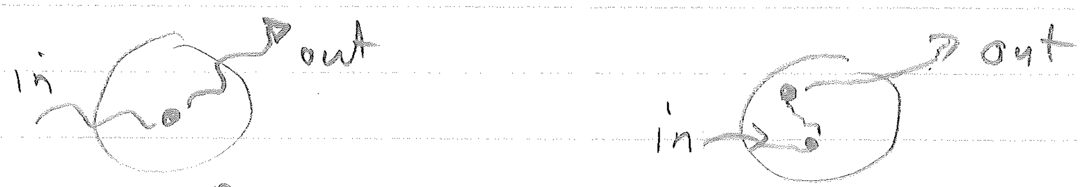
Born approximation (familiar in Quantum Mechanics)

Iterate the solution:

(1) Assume that the incident field only contributes to the scattering amplitude $\Rightarrow$ first Born approx

(2) Next use $\vec{E}^{(2)} = \vec{E}^{(0)} + \vec{E}^{(1)}$ in source term $\Rightarrow$ second Born approximation

This leads to a hierarchy of terms $\Rightarrow$ Multiple scattering



First Born

Second Born

First Born Approximation: Take $\vec{E}^{(0)} = \hat{e}_i E_0 e^{i\vec{k}_i \cdot \vec{x}}$

$$\Rightarrow \vec{A}_{scat} = \frac{k^2}{4\pi\epsilon} \int d^3x\, e^{-i\vec{q}\cdot\vec{x}'} \left(\hat{e}_i - \hat{r}(\hat{r}\cdot\hat{e}_i)\right) E_0\, \delta\epsilon(\vec{x}')$$

$$= \frac{k^2}{4\pi\epsilon} E_0 \underset{\substack{1 \quad\quad \cos\theta \\ \sigma\text{-pol} \quad \pi\text{-pol}}}{\left(\hat{e}_{i_\perp}\right)} \underbrace{\int d^3x'\, e^{-i\vec{q}\cdot\vec{x}'}\, \delta\epsilon(\vec{x}')}_{\delta\tilde{\epsilon}(\vec{q})}$$

$\delta\tilde{\epsilon}(\vec{q}) \leftarrow$ leads to Structure factor

For dilute gas $\delta\epsilon(\vec{x}) = 4\pi\alpha\, \delta n(\vec{x})$

$$\Rightarrow \boxed{\vec{A}_{scat} = \left(\frac{\omega}{c}\right)^2 \alpha\, E_0 \left(\hat{e}_{i_\perp}\right) \delta\tilde{n}(\vec{q}) \quad \text{As before}}$$