Physics 511

Problem Set #1 Solutions

(1) Vector identities

(a) $(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) \equiv \vec{V}\cdot\vec{W} = V_i W_i$

where $V_i = (\vec{A}\times\vec{B})_i = \epsilon_{ijk} A_j B_k$, $\quad W_i = (\vec{C}\times\vec{D})_i = \epsilon_{ilm} C_l D_m$

(Remember, repeated indicies are summed over. Never use the same dummy index for two different sums)

$$\Rightarrow (\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = \epsilon_{ijk}\epsilon_{ilm} A_j B_k C_l D_m = \epsilon_{jki}\epsilon_{ilm} A_j B_k C_l D_m$$

cyclic permutation

$$= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) A_j B_k C_l D_m$$

$$= A_j B_k C_j D_k - A_j B_k C_k D_j$$

$$\boxed{(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})}$$

(b) $\vec{\nabla}\times(\psi\vec{A}) \equiv \vec{V}$

(product rule of derivatives)

$$\Rightarrow V_i = \epsilon_{ijk}\partial_j(\psi A_k) = \epsilon_{ijk}\left(A_k(\partial_j\psi) + \psi(\partial_j A_k)\right)$$

$$= -\epsilon_{ikj} A_k \partial_j \psi + \psi\,\epsilon_{ijk}\partial_j A_k$$

non-cyclic permutation of ijk

$$\Rightarrow \boxed{\vec{V} \equiv \vec{\nabla}\times(\psi\vec{A}) = -(\vec{A}\times\vec{\nabla})\psi + \psi(\vec{\nabla}\times\vec{A})}$$

(1) continued

(c) $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) \equiv \vec{V}$

$$\Rightarrow V_i = \epsilon_{ijk} \underset{\nabla_j}{\partial_j} \Big(\underbrace{\epsilon_{klm} \partial_l A_m}_{(\vec{\nabla} \times \vec{A})_k} \Big) = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \partial_j \partial_l A_m$$

$$= \partial_j \partial_i A_j - \partial_j \partial_j A_i = \partial_i (\vec{\nabla} \cdot \vec{A}) - \nabla^2 A_i$$

$$\therefore \boxed{\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}}$$

What about the "old fashion" way, proving by explicitly evaluation of curl, div, grad in Cartesian coordinates

Let $\vec{F} = \vec{\nabla} \times \vec{A} = \hat{x}(\partial_y A_z - \partial_z A_y) + \hat{y}(\partial_z A_x - \partial_x A_z) + \hat{z}(\partial_x A_y - \partial_y A_x)$

$$\Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \hat{x}(\partial_y F_z - \partial_z F_y) + \hat{y}(\partial_z F_x - \partial_x F_z) + \hat{z}(\partial_x F_y - \partial_y F_x)$$

$$= \hat{x}(\partial^2_{xy} A_y - \partial^2_y A_x - \partial^2_z A_x - \partial^2_{xz} A_z)$$
$$+ \hat{y}(\partial^2_{yz} A_z - \partial^2_z A_y - \partial^2_x A_y + \partial^2_{xy} A_x)$$
$$+ \hat{z}(\partial^2_{zx} A_x - \partial^2_x A_z - \partial^2_y A_z + \partial^2_{zy} A_y)$$

$$= \hat{x}\partial_x(\partial_y A_y + \partial_z A_z) + \hat{y}\partial_y(\partial_x A_x + \partial_z A_z) + \hat{z}\partial_z(\partial_x A_x + \partial_y A_y)$$
$$- (\partial_y^2 + \partial_z^2) A_x \hat{x} \quad (\partial_x^2 + \partial_z^2) A_y \hat{y} \quad (\partial_x^2 + \partial_y^2) A_z \hat{z}$$

Now:

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = (\hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z)(\partial_x A_x + \partial_y A_y + \partial_z A_z)$$
$$- (\partial_x^2 + \partial_y^2 + \partial_z^2)(A_x \hat{x} + A_y \hat{y} + A_z \hat{z})$$

The terms $\partial_x^2 A_x$, $\partial_y^2 A_y$, $\partial_z^2 A_z$ cancel, reproducing the the result above. OK, which method is easier?

(2) Irreducible tensors

(a) Consider the second rank tensor $U_{ij} = A_i B_j$ (outer product)

This can be decomposed into a sum of "irreducible tensors"

$$U_{ij} = U_{ij}^{(0)} + U_{ij}^{(1)} + U_{ij}^{(2)}$$

$$U_{ij}^{(0)} \equiv \frac{1}{3} \operatorname{Tr}(U)\, \delta_{ij} = \frac{1}{3}\Big(\sum_i U_{ii}\Big)\delta_{ij} = \frac{1}{3}(A_i B_i)\,\delta_{ij} = \boxed{\frac{1}{3}(\vec{A}\cdot\vec{B})\,\delta_{ij}}$$

$$U_{ij}^{(1)} \equiv \frac{1}{2}(U_{ij} - U_{ji}) = \frac{1}{2}(A_i B_j - A_j B_i) = \frac{1}{2}(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) A_l B_m$$

$$= \frac{1}{2}\epsilon_{ijk}\epsilon_{klm} A_l B_m = \boxed{\frac{1}{2}\epsilon_{ijk}\,(\vec{A}\times\vec{B})_k}$$

$$U_{ij}^{(2)} \equiv \frac{1}{2}(U_{ij} + U_{ji}) - \frac{1}{3}\operatorname{Tr}(U)\,\delta_{ij} =$$

(b) Consider the contraction of two Cartesian tensors

$$U_{ij} W_{ij} = (U_{ij}^{(0)} + U_{ij}^{(1)} + U_{ij}^{(2)})(W_{ij}^{(0)} + W_{ij}^{(1)} + W_{ij}^{(2)})$$

We must show that $U_{ij}^{(l)} W_{ij}^{(l')} = 0$ if $l \neq l'$

- $l=0,\ l'=1$ $\quad U_{ij}^{(0)} W_{ij}^{(1)} = \frac{\operatorname{Tr}(U)}{3}\delta_{ij} W_{ij}^{(1)} = \frac{\operatorname{Tr}(U)}{3} W_{ii}^{(1)}$

$$= 0 \quad \text{since } W_{ii}^{(1)} = 0 \ \text{(traceless)}$$

- $l=0,\ l'=2$ $\quad U_{ij}^{(0)} W_{ij}^{(2)} = \frac{\operatorname{Tr}(U)}{3} W_{ii}^{(2)}$

$$= 0 \quad \text{since } W_{ii}^{(2)} = 0 \ \text{(Traceless by construction)}$$

- $l=1,\ l'=2$ $\quad U_{ij}^{(1)} W_{ij}^{(2)} = \left[\frac{1}{2}(U_{ij} - U_{ji})\right]\left[\frac{1}{2}(W_{ij} + W_{ji})\right]$

$$= \frac{1}{4}\left[U_{ij} W_{ij} - U_{ji} W_{ij} + U_{ij} W_{ji} - U_{ji} W_{ji}\right]$$

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Problem (2) continued

• $\ell' = 1, \ell' = 2$ $\quad U_{ij}^{(1)} W_{ij}^{(2)} = \frac{1}{4}\left[U_{ij}W_{ij} - U_{ji}W_{ij} + U_{ij}W_{ji} - U_{ji}W_{ji}\right]$

$$= \frac{1}{4}\left[\mathrm{Tr}(U^T W) - \mathrm{Tr}(UW) + \mathrm{Tr}(UW) - \mathrm{Tr}(U^T W)\right]$$

$$\Rightarrow U_{ij}^{(1)} W_{ij}^{(2)} = 0$$

(c) For $\ell \geq 2$ the irreducible tensors are traceless and symmetric

Consider

$$[x_i x_j x_k]^{(3)} \equiv x_i x_j x_k - \frac{1}{5}(x_i \delta_{jk} + x_k \delta_{ij} + x_j \delta_{ki}) r^2$$

This is clearly symmetric with respect to exchange of indicies

Now consider the trace over any two indicies, say $i = j$

$$\delta_{ij} [x_i x_j x_k]^{(3)} = r^2 x_k - \frac{1}{5}(x_i \delta_{ik} + 3x_k + x_i \delta_{ik}) r^2$$

(having used $\quad x_i x_j \delta_{ij} = \vec{x} \cdot \vec{x} = r^2, \quad \delta_{ii} = 3$)

$$= r^2 x_k - \frac{1}{5}(x_k + 3x_k + x_k) r^2 = 0 \checkmark$$

<u>What are the independent components?</u>

There are a total of $3^3 = 27$ components of any rank three tensor in 3 dimensions. However they are not all independent.

First the tensor is symmetric with respect to exchange of any two indicies

Our first question is, how many independent components does a rank-ℓ tensor have in d-dimensions

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Problem 2 (continued)

Since the tensor is symmetric

$$[x_1 x_2 x_3]^{(3)} = [x_2 x_1 x_3]^{(3)} = [x_1 x_3 x_2]^{(3)} \text{ etc}$$

Thus, order of any of the indicies doesn't matter.

The number of independent components is then ~~equi~~ equivalent to the number of distinct "rolls" of l dice with d-faces, where order doesn't matter. This is equivalent to counting how many ways of putting l-balls into d different colored boxes

(each ball represents one of the indices, and each box represents a digit between 1 and d)

This is just the famous Bose-Einstein distribution for l-particles, d-states. The number of different ways of putting l-particles in d-states is

$$\#_{sym} \equiv \binom{l+d-1}{d-1} = \frac{(l+d-1)!}{(d-1)!\, l!}$$

For $d=3$ (three dimensions)

$$\#_{sym} = \frac{(l+2)!}{2!\, l!} = \frac{(l+2)(l+1)}{2}$$

For the irreducible tensor, we must substract off the number of different traces (in 3dimension)

$$\#_{traces} = \frac{l(l-1)}{2} \quad (\text{Put } l-2 \text{ balls in 3 boxes})$$

$\Rightarrow$ Number of independent components of rank l irreducible tensor $= \frac{(l+2)(l+1)}{2} - \frac{l(l-1)}{2} = 2l+1$

Problem 2 continued

For the particular case of an irreducible rank-3 tensor in 3-D, $T^{(3)}_{ijk}$, the independent components (before accounting for traces) are

$T^{(3)}_{111}$, $T^{(3)}_{112}$, $T^{(3)}_{113}$, $T^{(3)}_{122}$, $T^{(3)}_{123}$, $T^{(3)}_{133}$, $T^{(3)}_{222}$, $T^{(3)}_{223}$, $T^{(3)}_{233}$, $T^{(3)}_{333}$

(any other component is the same as one of these)

Then the trace rule $T^{(3)}_{ijj} = 0$, $T^{(3)}_{jij} = 0$, $T^{(3)}_{jji} = 0$

$$\Rightarrow \begin{cases} T^{(3)}_{111} + T^{(3)}_{122} + T^{(3)}_{133} = 0 \\ \underset{\underset{T^{(3)}_{112}}{\Updownarrow}}{T^{(3)}_{211}} + T^{(3)}_{222} + T^{(3)}_{233} = 0 \\ T^{(3)}_{113} + T^{(3)}_{223} + T^{(3)}_{333} = 0 \end{cases}$$

Thus three of the elements can be eliminated in terms of others

e.g.
$$T^{(3)}_{111} = -\left(T^{(3)}_{122} + T^{(3)}_{133}\right)$$
$$T^{(3)}_{222} = -\left(T^{(3)}_{211} + T^{(3)}_{233}\right)$$
$$T^{(3)}_{333} = -\left(T^{(3)}_{113} + T^{(3)}_{223}\right)$$

We are then left with $7 = 2\ell+1$ $(\ell=3)$ independent comp

For the specific case $[x_i x_j x_k]^{(3)}$

$$[x_1 x_1 x_2]^{(3)} = x^2 y - \frac{y r^2}{5} = \frac{4}{5}x^2 y - \frac{1}{5}(y^3 + yz^2)$$
$$[x_1 x_1 x_3]^{(3)} = x^2 z - \frac{z r^2}{5} = \frac{4}{5}x^2 z - \frac{1}{5}(zy^2 + z^3)$$
$$[x_1 x_2 x_2]^{(3)} = xy^2 - \frac{x r^2}{5} = \frac{4}{5}xy^2 - \frac{1}{5}(xz^2 + x^3)$$
$$[x_1 x_2 x_3]^{(3)} = xyz$$
$$[x_1 x_3 x_3]^{(3)} = xz^2 - \frac{x r^2}{5} = \frac{4}{5}xz^2 - \frac{1}{5}(x^3 + xy^2) \quad \text{etc.}$$

(3) Helmholtz's Theorem

Consider $\vec{\nabla} \times \left(\vec{\nabla} \times \int_{\text{Volume}} \frac{\vec{V}(\vec{x}')}{|\vec{x}-\vec{x}'|} \, d^3x' \right) \equiv I$

$$I = \vec{\nabla}\left(\vec{\nabla} \cdot \int \frac{\vec{V}(\vec{x}')}{|\vec{x}-\vec{x}'|} \, d^3x'\right) - \nabla^2 \int \frac{\vec{V}(\vec{x}')}{|\vec{x}-\vec{x}'|} \, d^3x'$$

Note that $\vec{\nabla}$ here acts on the <u>nonprimed</u> variable

let $\frac{1}{\mathscr{r}} \equiv \frac{1}{|\vec{x}-\vec{x}'|}$: Bring differential operators inside the integral

$$\Rightarrow I = \vec{\nabla} \int \vec{V}(\vec{x}') \cdot \left(\vec{\nabla} \frac{1}{\mathscr{r}}\right) d^3x' - \int \vec{V}(\vec{x}') \left(\nabla^2 \frac{1}{\mathscr{r}}\right) d^3x'$$

Aside: $\nabla^2 \frac{1}{\mathscr{r}} = \nabla^2 \frac{1}{|\vec{x}-\vec{x}'|} = -4\pi\,\delta^{(3)}(\vec{x}-\vec{x}')$ (from class)

"Trick" (used often): $\vec{\nabla} \frac{1}{\mathscr{r}} = -\frac{\vec{\mathscr{r}}}{\mathscr{r}^3} = -\frac{\vec{x}-\vec{x}'}{\mathscr{r}^3} = -\vec{\nabla}' \frac{1}{\mathscr{r}}$ ← gradient w.r.t. primed variables

$$\Rightarrow I = -\vec{\nabla} \int_{\text{Volume}} \left(\vec{V}(\vec{x}') \cdot \vec{\nabla}' \frac{1}{\mathscr{r}}\right) d^3x' + 4\pi\,\vec{V}(\vec{x})$$

Finally we want to do an integration by parts to move $\vec{\nabla}'$ onto $\vec{V}(\vec{x}')$

Use $\vec{\nabla}' \cdot \left(f(\vec{x}')\,\vec{A}(\vec{x}')\right) = \vec{A} \cdot \vec{\nabla}' f + f\,\vec{\nabla}' \cdot \vec{A}$

$\therefore$ [illegible] Next Page

Thus: $\vec{V}(\vec{x}') \cdot \vec{\nabla}' \frac{1}{r} = \vec{\nabla}' \cdot \left(\frac{\vec{V}(\vec{x}')}{r} \right) - \frac{\vec{\nabla}' \cdot \vec{V}(\vec{x}')}{r}$

$$\Rightarrow \int_{\text{Volume}} \vec{V}(\vec{x}') \cdot \left(\vec{\nabla}' \frac{1}{r} \right) d^3x' = \int_{\text{Volume}} - \frac{\vec{\nabla}' \cdot \vec{V}(\vec{x}')}{r} \, d^3x' + \underbrace{\oint_{S} \frac{\vec{V}(\vec{x}')}{r} \, d^3x'}_{\text{"Surface term"}}$$

where I have used the divergence theorem on the last integral

Now assume $V(\vec{x}) \to 0$ faster than $\frac{1}{r}$

Then as $S \to \infty$ the (surface integral) will go to 0

Putting it all together

$$\vec{V}(\vec{x}) = \frac{1}{4\pi} \vec{\nabla} \int_{\text{allspace}} \vec{V}(\vec{x}') \cdot \vec{\nabla}' \frac{1}{r} \, d^3x' + \frac{1}{4\pi} I$$

$$\Rightarrow \vec{V}(\vec{x}) = \underbrace{-\vec{\nabla} \left[\frac{1}{4\pi} \int_{\substack{\text{all} \\ \text{space}}} \frac{\vec{\nabla}' \cdot \vec{V}(\vec{x}')}{|\vec{x} - \vec{x}'|} \, d^3x' \right]}_{\vec{V}_L(\vec{x})} + \underbrace{\vec{\nabla} \times \left[\frac{1}{4\pi} \int_{\substack{\text{all} \\ \text{space}}} \frac{\vec{\nabla}' \times \vec{V}(\vec{x}')}{|\vec{x} - \vec{x}'|} \, d^3x' \right]}_{\vec{V}_T(\vec{x})}$$

("Longitudinal", "Irrotational") part

("Transverse", "Solenoidal") part