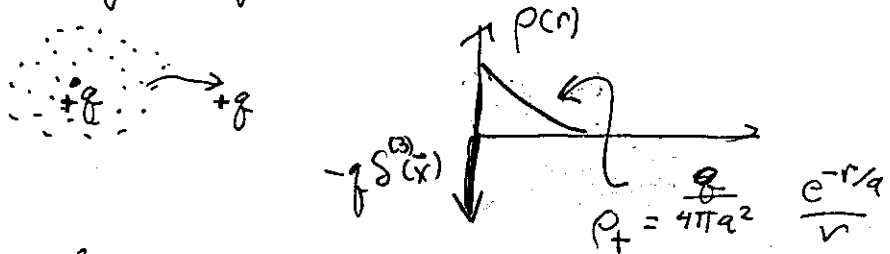


# Physics 511

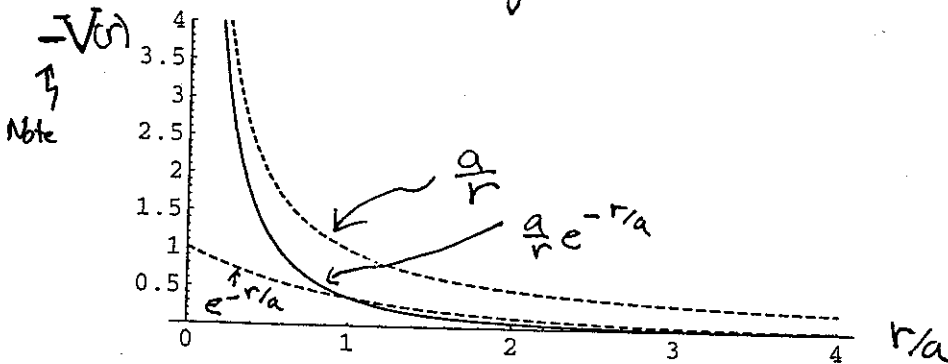
## P.S. #2 Solutions

### (1) The Yukawa Potential

Consider a charge distribution consisting of a point charge at the origin  $-q$ , and a "smeared out" charge  $+q$  which "screens" the long range force of the point



(a) We want to show that this charge distribution yields the Yukawa potential  $V(r) = G \frac{e^{-r/a}}{r/a}$



For short distances

(i.e.  $r \ll a$ )

$$e^{-r/a} \approx 1$$

$$\Rightarrow V(r) \propto \frac{1}{r}$$

For long distance

(i.e.  $r \gg a$ )

the exponential dominates

Thus, the Yukawa potential looks like a Coulomb potential for  $r \ll a$ , but falls off very rapidly to zero for  $r \gg a$ .

(b) The total charge is neutral: Proof

$$Q_{\text{total}} = -q + \int d^3r \rho_+(\vec{r}) \quad , \quad \rho_+(\vec{r}) = -\frac{q}{4\pi a^2} \frac{e^{-r/a}}{r}$$

(1.b) continued

$$\bullet Q_{\text{total}} = -q \left[ 1 - \int_0^{\infty} \frac{e^{-r/a}}{4\pi a^2 r} 4\pi r^2 dr \right]$$

Here I used the fact that  $\rho$  is spherically symmetric, so the volume element is just a spherical shell  $d^3r = 4\pi r^2 dr$

Now change variables: let  $u = r/a$

$$\Rightarrow Q_{\text{total}} = -q \left[ 1 - \int_0^{\infty} u e^{-u} du \right]$$

Using integration by parts:  $\frac{d}{du}(u e^{-u}) = e^{-u} - u e^{-u}$

$$\Rightarrow \int u e^{-u} du = \int \left( \frac{d}{du}(u e^{-u}) + e^{-u} \right) du$$

$$= -e^{-u} - u e^{-u} = -e^{-u}(1+u)$$

$$\therefore Q_{\text{total}} = -q \left[ 1 + e^{-u}(1+u) \Big|_0^{\infty} \right]$$

$$= -q [1 + 0 - 1] = 0 \quad \checkmark$$

This makes sense physically since  $V(r)$  falls off at far distances much faster than  $1/r$  so there is no monopole moment. In fact this

charge distribution creates a potential which goes to zero at infinity faster than any multipole moment for a confined set of charges.

(1.c) Since this charge distribution is spherically symmetric, Gauss' Law calls out to us.

First, from pure symmetry arguments, we know

$$\vec{E}(\vec{r}) = E(r) \hat{r} \quad \left( \begin{array}{l} \text{i.e. the magnitude of } \vec{E} \text{ depends} \\ \text{only on } r, \text{ not } \theta + \phi \\ \text{and points in radial direction} \end{array} \right)$$

⇒ If we consider a sphere centered at the origin

$$\oint_S \vec{E} \cdot \hat{n} dA = 4\pi r^2 E(r) \quad \text{where } r \text{ is the radius}$$

$$\text{Gauss' Law: } \oint_S \vec{E} \cdot \hat{n} dA = 4\pi Q_{\text{enclosed}} \Rightarrow E(r) = \frac{Q_{\text{enclosed}}}{r^2}$$

All we need now is the charge enclosed in the sphere

$$Q_{\text{enclosed}} = \int_V \rho(\vec{r}) d^3r = \int_0^r \rho(r') 4\pi r'^2 dr' \quad (r' = \text{dummy})$$

$$= -q \left( 1 - \int_0^r \frac{e^{-r'/a}}{4\pi a^2 r'} 4\pi r'^2 dr' \right)$$

$$= -q \left( 1 - \int_0^{r/a} u e^{-u} du \right) \quad (\text{some change of variables})$$

$$= -q \left( 1 + e^{-u} (1+u) \Big|_0^{r/a} \right)$$

$$= -q \left( 1 + e^{-r/a} \left( 1 + \frac{r}{a} \right) - 1 \right)$$

$$\Rightarrow Q_{\text{enc}} = -q e^{-r/a} \left( 1 + \frac{r}{a} \right)$$

$$\therefore \vec{E}(r) = \frac{-q}{r^2} \left( e^{-r/a} \left( 1 + \frac{r}{a} \right) \hat{r} \right)$$

(Next Page)

(1c) Continued

Now, to find the potential, use  $\vec{E} = -\vec{\nabla}\phi$

$$\Rightarrow \phi(\vec{r}) = -\int_{\vec{r}_{\text{ground}}}^{\vec{r}} \vec{E}(\vec{r}') \cdot d\vec{l}$$

where  $\vec{r}_{\text{ground}}$  is the point where  $\phi = 0$  and the path of integration is arbitrary since  $\vec{\nabla} \times \vec{E} = 0$

For a confined charge distribution, choose  $\vec{r}_{\text{ground}} = \infty$ , and since  $\vec{E}$  is along the radial direction, choose  $d\vec{l} = \hat{r} dr$

$$\begin{aligned} \Rightarrow \phi(\vec{r}) &= -\int_{\infty}^r E(r') dr' = \int_r^{\infty} \frac{q}{r'^2} e^{-r'/a} \left(1 + \frac{r'}{a}\right) dr' \\ &= \frac{-q}{a} \int_{r/a}^{\infty} du e^{-u} \left(\frac{1}{u^2} + \frac{1}{u}\right) \quad \text{where } (u = r/a) \end{aligned}$$

The integrand is a perfect derivative:

$$\frac{d}{du} \left( \frac{e^{-u}}{u} \right) = -\frac{e^{-u}}{u} - \frac{e^{-u}}{u^2} = -e^{-u} \left( \frac{1}{u^2} + \frac{1}{u} \right)$$

$$\therefore \phi(\vec{r}) = \frac{-q}{a} \left[ -\frac{e^{-u}}{u} \right]_{r/a}^{\infty}$$

$$\boxed{\phi(\vec{r}) = -\frac{q}{a} \frac{e^{-r/a}}{r}}$$

Yukawa form  
with  $G = -\frac{q}{a}$

(d) The potential energy stored in the charge distribution  $\rho_c(\vec{r})$

In class we found the potential energy can be expressed as

$$U = \frac{1}{8\pi\epsilon_0} \int_{\text{all space}} |\vec{E}|^2 d^3\vec{x}$$

From part (c),  $|\vec{E}|^2 = \frac{q^2}{r^4} e^{-2r/a} \left(1 + \frac{r}{a}\right)^2$

Again, radial symmetry; choose  $d^3r = 4\pi r^2 dr$

$$\Rightarrow U = \frac{q^2}{2} \int_0^\infty e^{-2r/a} \left(1 + \frac{r}{a}\right)^2 \frac{dr}{r^2}$$

$$u = r/a$$

$$\Rightarrow U = \frac{q^2}{2a} \int_0^\infty e^{-2u} \left(\frac{1+u}{u}\right)^2 du$$

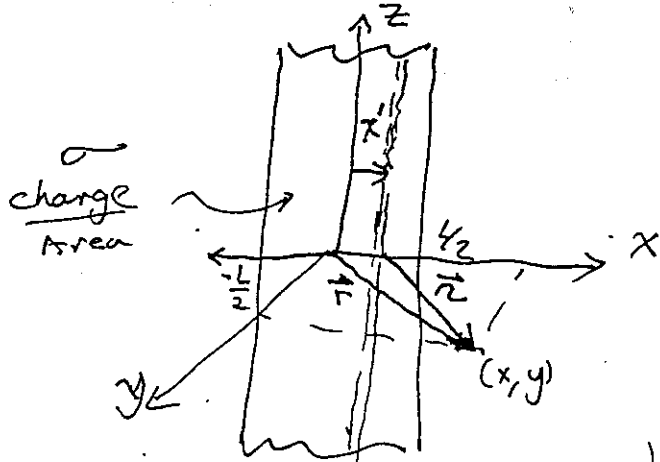
$$\Rightarrow U = \frac{q^2}{2a} \left[ -\left(\frac{1}{2} + \frac{1}{u}\right) e^{-2u} \right]_0^\infty$$

$$= \frac{q^2}{2a} \left[ \frac{1}{2} + \infty \right] !$$

Diverges!

Why? This is because of the point charge. The potential energy stored in a point singularity is infinite. It makes no sense to ask how much work it takes to push charge  $q$  into a point

## (2) Potential of a strip of charge



Break up the strip into substrip of thickness  $dx'$ : these are infinite line charges, with charge per unit length

$$\lambda = \sigma dx'$$

We then integrate the contribution of each line charge as  $x'$  goes from  $-\frac{L}{2} \rightarrow \frac{L}{2}$

Aside

What is the potential of an infinite line charge? This is most easily obtained from Gauss' Law:

The diagram shows a horizontal line charge with density  $\lambda$ . A cylindrical Gaussian surface of length  $L$  and radius  $r$  is drawn around it. The surface is labeled "Gaussian Surface". The electric field  $\vec{E}$  is shown pointing radially outwards from the line charge.

$$\oint \vec{E} \cdot d\vec{a} = E(r) 2\pi r L = 4\pi \lambda L$$

$$\Rightarrow E(r) = \frac{2\lambda}{r}$$

Now this is tricky. The potential is defined by

$$\Phi(\vec{x}) = -\int_{\vec{r}_0}^{\vec{x}} \vec{E} \cdot d\vec{l}, \text{ where } \vec{r}_0 \text{ is ground}$$

For an infinite line charge, we cannot take  $\vec{r}_0$  at  $\infty$ . However, the choice of  $\vec{r}_0$  is arbitrary. Choose a radial path  $d\vec{l} = \hat{r} dr$

$$\Phi(r) = -\int_{r_0}^r E(r) dr = [-\ln(r) + \ln(r_0)] 2\lambda$$

For convenience, choose  $r_0 = 1$  (arbitrary)  $\Phi(r) = -2\lambda \ln(r)$

OK, so now we have the potential of a line charge

$$\Phi(r) = -2\lambda \ln(r) \quad \text{where } r \text{ is the radial}$$

distance from the line to the point of observation.

Back to the problem at hand. The line charge at  $\vec{r}' = x'\hat{x}$  contribute a potential  $dV$  at the observation point  $\vec{r} = x\hat{x} + y\hat{y}$  of

$$d\Phi(x, y; x') = -2\lambda \ln(r), \quad \lambda = \sigma dx'$$

where

$$r = |\vec{r}| = |\vec{r} - \vec{r}'| = |(x-x')\hat{x} + y\hat{y}| \\ = \sqrt{(x-x')^2 + y^2}$$

$$\Rightarrow d\Phi(x, y; x') = -2\sigma dx' \ln[\sqrt{(x-x')^2 + y^2}]$$

Now integrate:

$$\Phi(x, y) = \int_{x'=-L/2}^{x'=L/2} d\Phi(x, y; x') = -2\sigma \int_{-L/2}^{L/2} dx' \ln[\sqrt{(x-x')^2 + y^2}]$$

$$\text{let } u = x' - x \\ x' = u + x$$

$$du = dx'$$

$$\Rightarrow \Phi(x, y) = -2\sigma \int_{x-L/2}^{x+L/2} du \ln\left[\sqrt{u^2 + y^2}\right] =$$

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$$-\sigma \int_{x-L/2}^{x+L/2} du \ln(u^2 + y^2)$$

From the integral

$$\int du \ln[u^2 + y^2] = -2u + 2y \tan^{-1}\left(\frac{u}{y}\right) + u \ln[u^2 + y^2]$$

$$\Rightarrow \Phi(x, y) = -\sigma \left[ -2u + 2y \tan^{-1}\left(\frac{u}{y}\right) + u \ln[u^2 + y^2] \right]_{x-L/2}^{x+L/2}$$

$$\Phi(x, y) = -\sigma \left[ -2L + 2y \left[ \tan^{-1}\left(\frac{x+L/2}{y}\right) - \tan^{-1}\left(\frac{x-L/2}{y}\right) \right] + (x+L/2) \ln[(x+L/2)^2 + y^2] - (x-L/2) \ln[(x-L/2)^2 + y^2] \right]$$

(b) The electric field

$$\vec{E} = -\vec{\nabla}\Phi = -\hat{x} \frac{\partial}{\partial x} - \hat{y} \frac{\partial}{\partial y} = E_x \hat{x} + E_y \hat{y}$$

$$-\frac{\partial \Phi}{\partial x} = \sigma \left\{ 2y \left[ \frac{1}{y} \left( \frac{1}{1 + \left(\frac{x+L/2}{y}\right)^2} \right) - \frac{1}{y} \left( \frac{1}{1 + \left(\frac{x-L/2}{y}\right)^2} \right) \right] + \frac{2(x+L/2)^2}{y^2 + (x+L/2)^2} - \frac{2(x-L/2)^2}{y^2 + (x-L/2)^2} + \ln[(x+L/2)^2 + y^2] - \ln[(x-L/2)^2 + y^2] \right\}$$

$$= \sigma \left\{ \frac{2y^2 + 2(x+L/2)^2}{y^2 + (x+L/2)^2} - \frac{2y^2 + 2(x-L/2)^2}{y^2 + (x-L/2)^2} + \ln[(x+L/2)^2 + y^2] - \ln[(x-L/2)^2 + y^2] \right\}$$

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$$\therefore E_x = -\frac{\partial \Phi}{\partial x} = \sigma \left\{ \ln[(x+L/2)^2 + y^2] - \ln[(x-L/2)^2 + y^2] \right\}$$

$$E_x = 2\sigma \left\{ \frac{1}{2} \ln \left[ \frac{(x+L/2)^2 + y^2}{(x-L/2)^2 + y^2} \right] \right\}$$

$$E_y = -\frac{\partial \Phi}{\partial y} = \sigma \left\{ 2 \left( \tan^{-1} \left( \frac{x+L/2}{y} \right) - \tan^{-1} \left( \frac{x-L/2}{y} \right) \right) \right. \\ \left. + 2y \left( -\frac{(x+L/2)}{y^2} \frac{1}{1 + \left( \frac{x+L/2}{y} \right)^2} + \frac{(x-L/2)}{y^2} \frac{1}{1 + \left( \frac{x-L/2}{y} \right)^2} \right) \right. \\ \left. + (x+L/2) \frac{2y}{y^2 + (x+L/2)^2} - (x-L/2) \frac{2y}{y^2 + (x-L/2)^2} \right\}$$

$$\Rightarrow E_y = 2\sigma \left\{ \tan^{-1} \left( \frac{x+L/2}{y} \right) - \tan^{-1} \left( \frac{x-L/2}{y} \right) \right. \\ \left. - \frac{y(x+L/2)}{y^2 + (x+L/2)^2} + \frac{y(x+L/2)}{y^2 + (x+L/2)^2} \right. \\ \left. + \frac{y(x-L/2)}{y^2 + (x-L/2)^2} - \frac{y(x-L/2)}{y^2 + (x-L/2)^2} \right\}$$

$$\Rightarrow E_y = 2\sigma \left\{ \tan^{-1} \left( \frac{x+L/2}{y} \right) - \tan^{-1} \left( \frac{x-L/2}{y} \right) \right\}$$

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(c) limit  $L \rightarrow \infty$  (i.e.  $x \ll L, y \ll L$ )

$$\lim_{L \rightarrow \infty} \left[ \tan^{-1} \left( \frac{x+L/2}{y} \right) - \tan^{-1} \left( \frac{x-L/2}{y} \right) \right] = \pi$$

Now things get tricky again

$$\lim_{L \rightarrow \infty} \left\{ (x+L/2) \ln [(x+L/2)^2 + y^2] - (x-L/2) \ln [(x-L/2)^2 + y^2] \right\}$$

$$= \lim_{L \rightarrow \infty} \left( 2L \ln(L/2) + L \ln(L/2) \right)$$

$$= \lim_{L \rightarrow \infty} \left[ 2L \ln(L/2) \right]$$

$$\text{So } \lim_{L \rightarrow \infty} \Phi(x,y) = -2\pi\sigma y + \underbrace{\lim_{L \rightarrow \infty} (-2L + 2L \ln(L/2))}_{?}$$

The term in brackets blows up, but that's really an artifact of the problem of where to put ground. In general we can add

or subtract any constant from  $V$  without affecting the physics

$$\text{So } \boxed{\lim_{L \rightarrow \infty} \Phi(x,y) = -2\pi\sigma y + \text{constant}}$$

Good: This is the potential of an infinite plane

$$\vec{E} = -\vec{\nabla}\Phi = 2\pi\sigma \hat{y} \quad (y > 0) \quad \checkmark$$

(c) continued

Now take the limit as  $x, y \rightarrow \infty$

$$\begin{aligned} & \lim_{x, y \rightarrow \infty} \left[ \tan^{-1} \left( \frac{x+L/2}{y} \right) - \tan^{-1} \left( \frac{x-L/2}{y} \right) \right] \\ &= \lim_{x, y \rightarrow \infty} \left[ \tan^{-1} \left( \frac{x}{y} \right) - \tan^{-1} \left( \frac{x}{y} \right) \right] = 0 \end{aligned}$$

$$\begin{aligned} & \lim_{x, y \rightarrow \infty} \left\{ (x+L/2) \ln[(x+L/2)^2 + y^2] - (x-L/2) \ln[(x-L/2)^2 + y^2] \right\} \\ &= L \ln(x^2 + y^2) = L \ln(r^2) = 2L \ln(r) \end{aligned}$$

$$\therefore \lim_{x, y \rightarrow \infty} V(x, y) = \sigma (-2L + 2L \ln(r))$$

$$\lim_{x, y \rightarrow \infty} V(x, y) = -2\sigma L \ln(r) + \text{const}$$

Good, this is the potential of an infinite line charge with

$\lambda = \sigma L$  the charge length

```
Needs["Graphics`PlotField`"]
```

```
In[21]:=
```

```
V[x_,y_] := 2y (ArcTan[(L/2-x)/y] + ArcTan[(L/2+x)/y]) -  
            (x-L/2) Log[(x-L/2)^2+y^2] + (x+L/2) Log[(x+L/2)^2+y^2]
```

```
In[22]:=
```

```
L=1 (* Choosing L=1 to set the scale *)
```

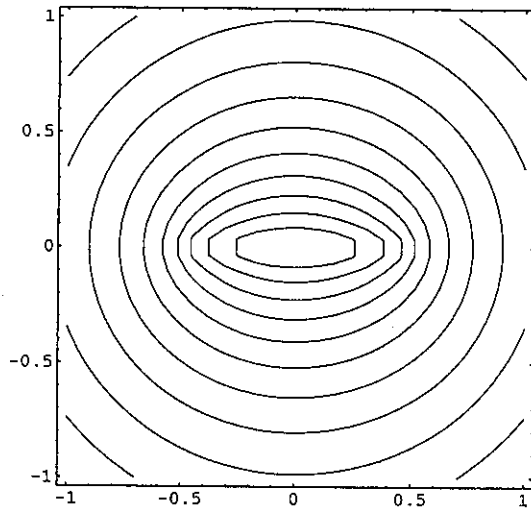
```
Out[22]=
```

```
1
```

## ■ Single Positively charged strip

### ■ x and y on the order of L

```
ContourPlot[V[x,y], {x,-1,1}, {y,-1,1}, ContourShading->False,  
            PlotPoints->30]
```

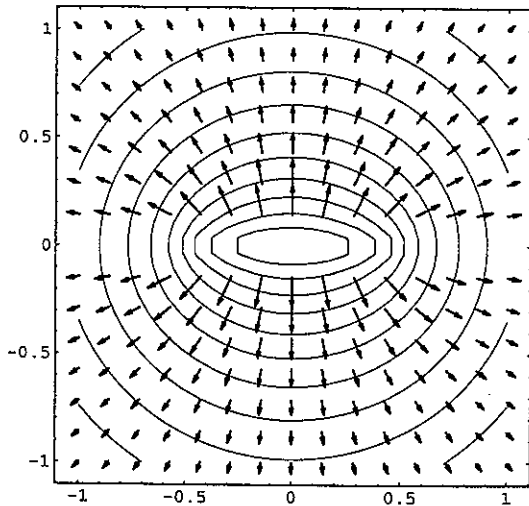


Mathematica  
Code

```
-ContourGraphics-
```

```
PlotGradientField[V[x,y], {x,-1,1}, {y,-1,1}]
```

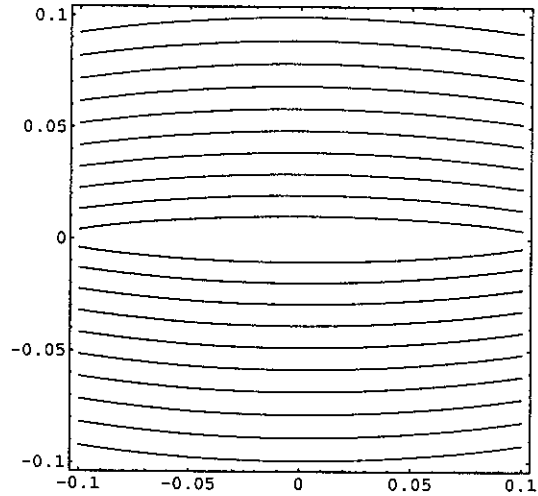
```
Show[%24,%25]
```



```
-Graphics-
```

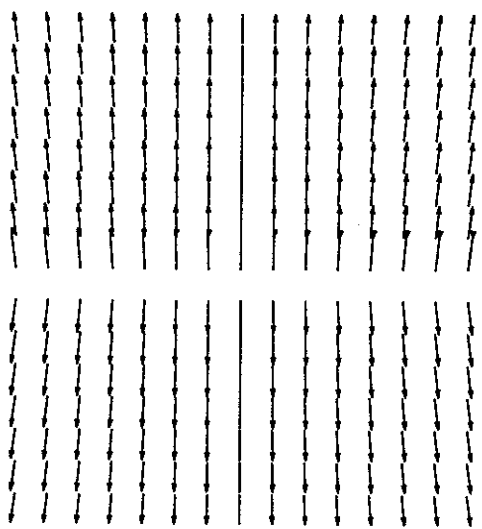
x and y on the small compared to L (looks more like a plane)

```
ContourPlot[V[x,y], {x,-.1,.1}, {y,-.1,.1}, ContourShading->False, PlotPoints->30]
```



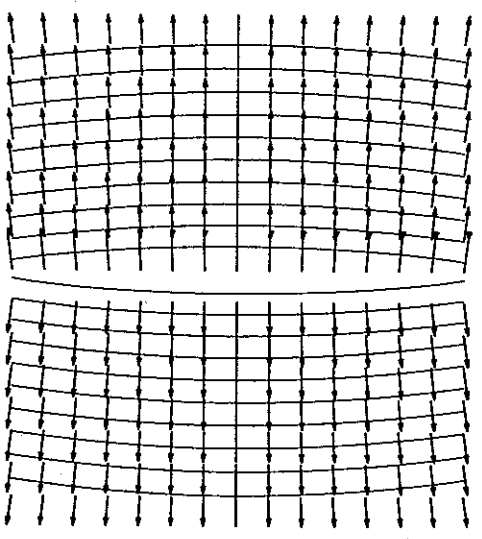
-ContourGraphics-

```
PlotGradientField[V[x,y], {x,-.1,.1}, {y,-.1,.1}]
```



-Graphics-

```
Show[%,%]
```

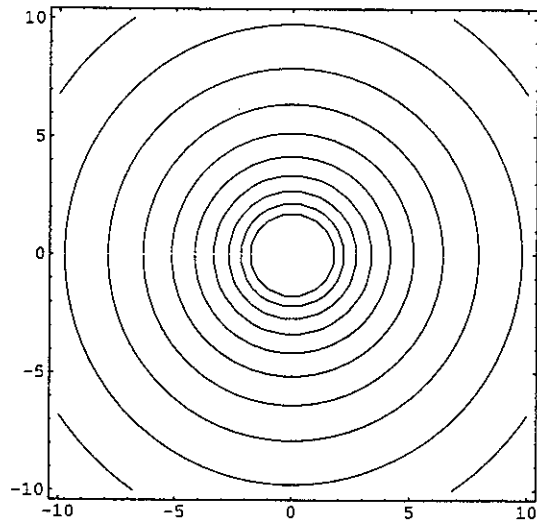


-Graphics-

■ x and y on the big compared to L (looks more like a line charge)

In[24]:=

```
ContourPlot[V[x,y], {x,-10,10}, {y,-10,10}, ContourShading->False, PlotPoints->30]
```

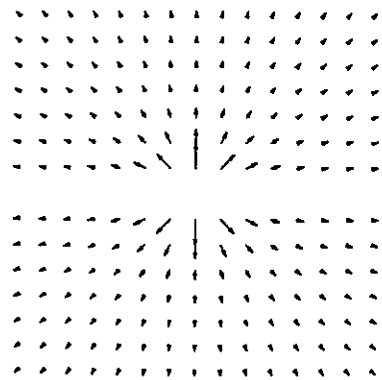


Out[24]=

-ContourGraphics-

In[26]:=

```
PlotGradientField[V[x,y], {x,-10,10}, {y,-10,10}]
```

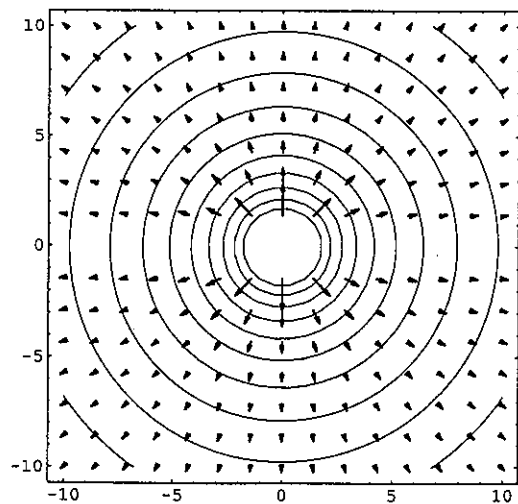


Out[26]=

-Graphics-

In[27]:=

```
Show[%24,%26]
```



Out[27]=

-Graphics-

## Two oppositely charged strips

In[28]:=

$$V2[x_, y_] = V[x, y] - V[x, y + s]$$

Out[28]=

$$2 y \left( \text{ArcTan}\left[\frac{\frac{1}{2} - x}{y}\right] + \text{ArcTan}\left[\frac{\frac{1}{2} + x}{y}\right] \right) - 2 (s + y) \left( \text{ArcTan}\left[\frac{\frac{1}{2} - x}{s + y}\right] + \text{ArcTan}\left[\frac{\frac{1}{2} + x}{s + y}\right] \right) -$$

$$\left(-\frac{1}{2} + x\right) \text{Log}\left[\left(-\frac{1}{2} + x\right)^2 + y^2\right] + \left(\frac{1}{2} + x\right) \text{Log}\left[\left(\frac{1}{2} + x\right)^2 + y^2\right] +$$

$$\left(-\frac{1}{2} + x\right) \text{Log}\left[\left(-\frac{1}{2} + x\right)^2 + (s + y)^2\right] - \left(\frac{1}{2} + x\right) \text{Log}\left[\left(\frac{1}{2} + x\right)^2 + (s + y)^2\right]$$

Principle of superposition!

## spacing s on the order of the thickness L

In[30]:=

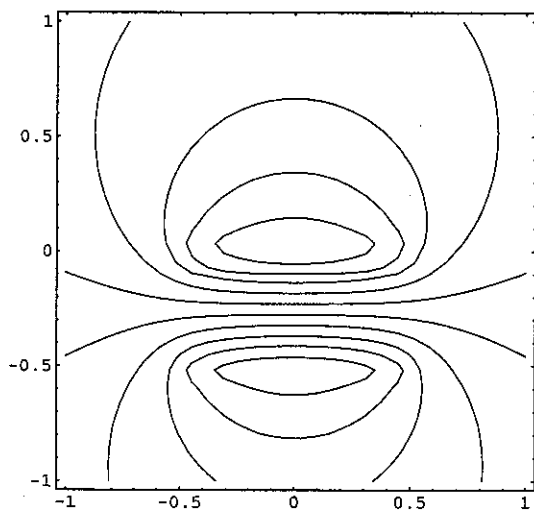
s=.5

Out[30]=

0.5

In[31]:=

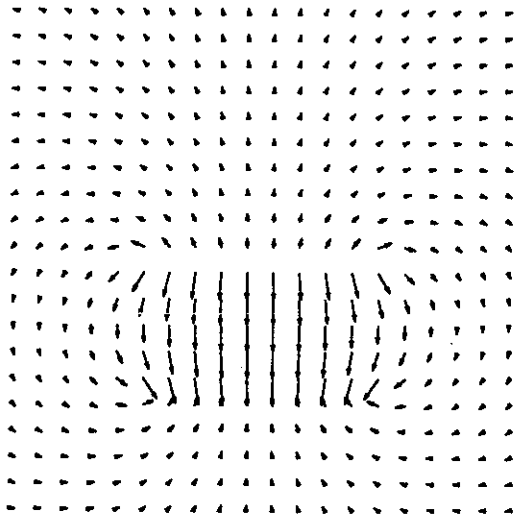
ContourPlot[V2[x, y], {x, -1, 1}, {y, -1, 1}, ContourShading->False,  
PlotPoints->30]



Out[31]=

-ContourGraphics-

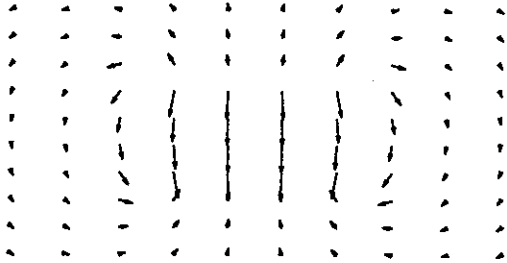
3  
PlotGradientField[V2[x,y], {x,-1,1}, {y,-1,1},  
PlotPoints->20]



Out[32]=

-Graphics-

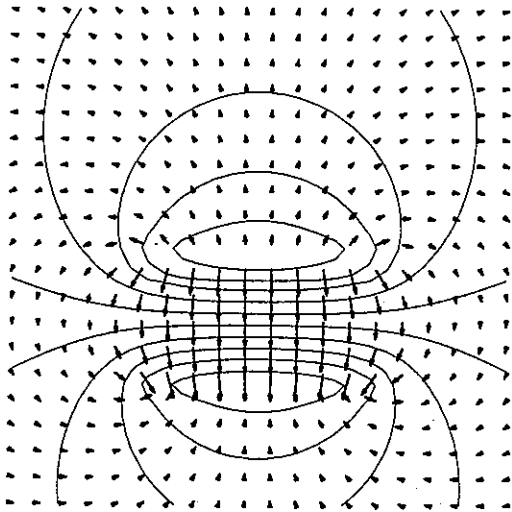
PlotGradientField[V2[x,y], {x,-1,1}, {y,-.75,.25},  
PlotPoints->10]



-Graphics-

Show[%,%%]

(\* Field somewhat uniform inside. Smaller fringing fields outside\*)



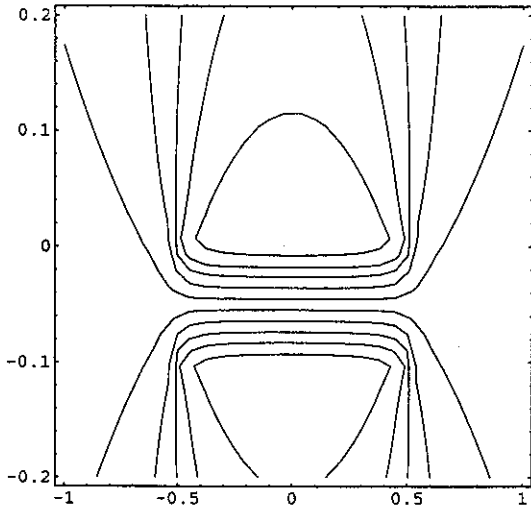
-Graphics-



s=.1

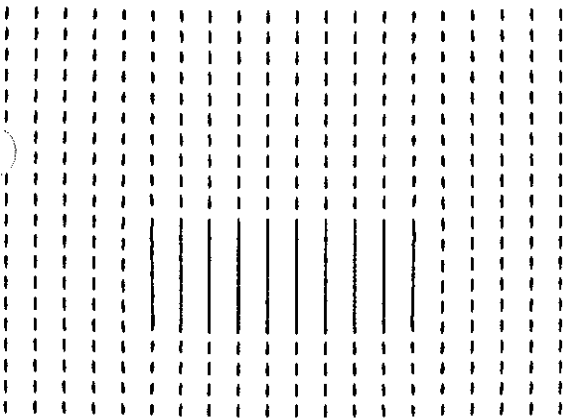
0.1

ContourPlot[V2[x,y], {x,-1,1}, {y,-.2,.2}, ContourShading->False, PlotPoints->30]



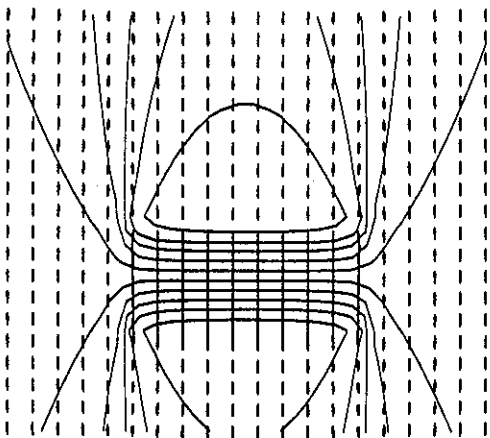
-ContourGraphics-

PlotGradientField[V2[x,y], {x,-1,1}, {y,-.2,.2}, PlotPoints->20]



-Graphics-

Show[%19,%20]  
(\* Field very uniform inside, very little fringing fields \*)



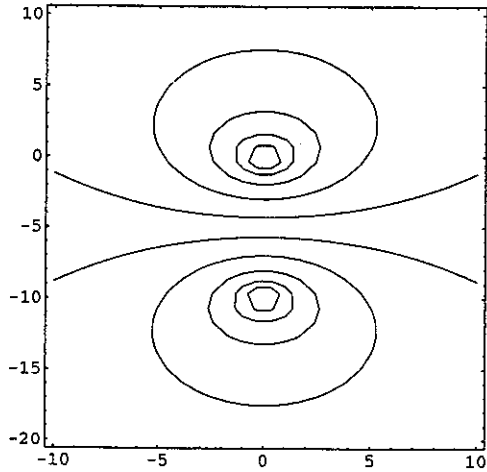
-Graphics-

■ Spacing  $s$  very large compared to width  $L$

```
In[33]:=
s=10
```

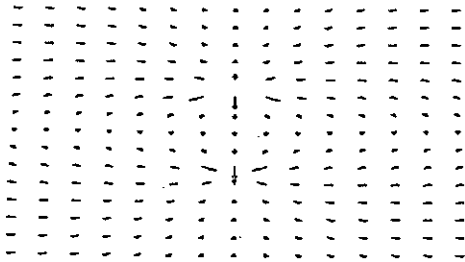
```
Out[33]=
10
```

```
In[34]:=
ContourPlot[V2[x,y],{x,-10,10}, {y,-20,10}, ContourShading->False,
PlotPoints->30]
```



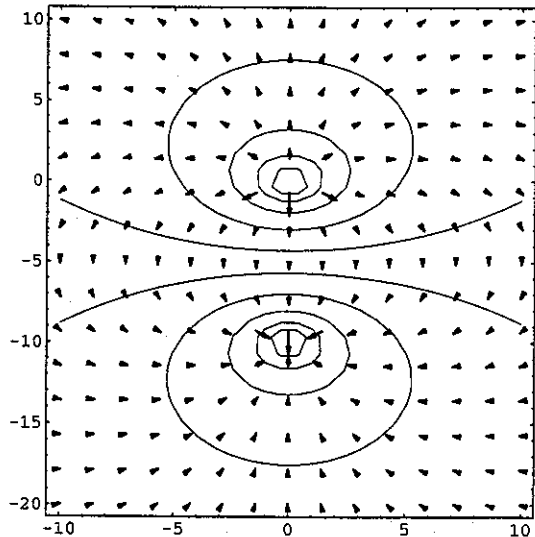
```
Out[34]=
-ContourGraphics-
```

```
In[36]:=
PlotGradientField[V2[x,y],{x,-10,10}, {y,-20,10}]
```



```
Out[36]=
-Graphics-
Show[%34,%36]
```

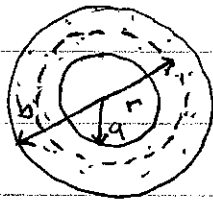
(\* Looks like the field of a dipole \*)



```
Out[37]=
-Graphics-
```

#### (4) Boundary condition at a surface charge

(i) Spherical shell:  $\vec{E} = E(r) \hat{r}$  by symmetry



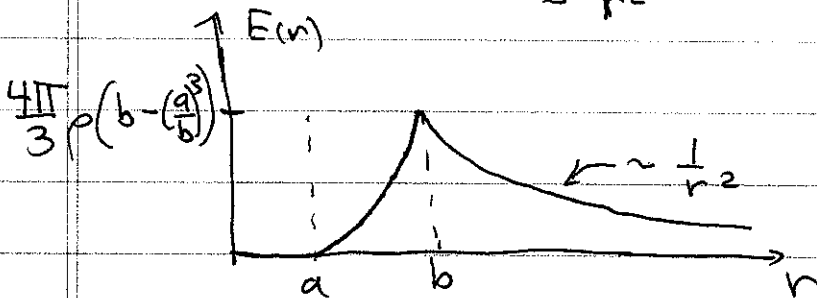
$$\oint \vec{E} \cdot d\vec{a} = 4\pi r^2 E(r) = 4\pi Q_{enc}(r)$$


$$\Rightarrow E(r) = \frac{Q_{enc}(r)}{r^2}$$

The enclosed charge

$$Q_{enc} = \int_0^r \rho(r') 4\pi r'^2 dr' = \begin{cases} 0 & r < a \\ \rho \int_a^r 4\pi r'^2 dr' = \frac{4\pi\rho}{3}(r^3 - a^3) & a < r < b \\ \frac{4\pi\rho}{3}(b^3 - a^3) & r > b \end{cases}$$

$$\Rightarrow E(r) = \begin{cases} 0 & r < a \\ \frac{4\pi\rho}{3} \frac{(r^3 - a^3)}{r^2} & a < r < b \\ \frac{4\pi\rho}{3} \frac{(b^3 - a^3)}{r^2} & r > b \end{cases}$$



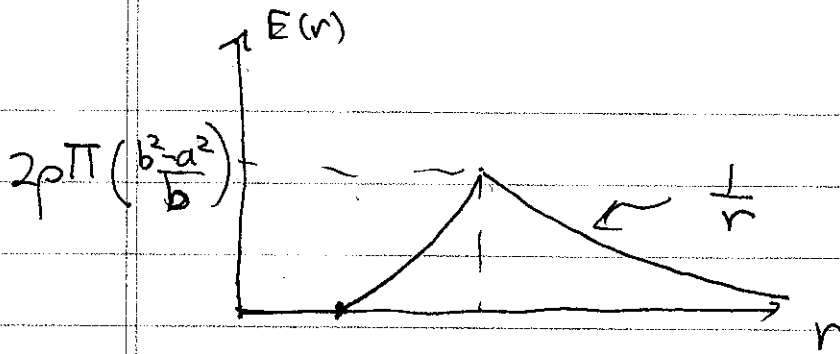
(ii) Cylindrical Shell: 

Again  $\oint \vec{E} \cdot d\vec{a} = (2\pi r L) E(r) = 4\pi Q_{enc}(r) \Rightarrow E(r) = \frac{2Q_{enc}(r)}{Lr}$

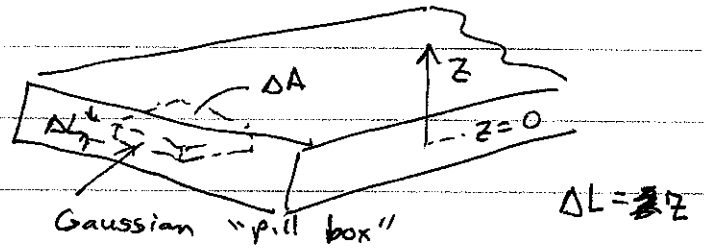
$$Q_{enc}(r) = \int_0^r \rho(r') 2\pi r' L dr' = \begin{cases} 0 & r < a \\ 2\pi L \rho \int_a^r r' dr' = \rho L \pi (r^2 - a^2) & a < r < b \\ \rho L \pi (b^2 - a^2) & r > b \end{cases}$$

$$\Rightarrow E(r) = \begin{cases} 0 & r < a \\ \frac{\rho L \pi (r^2 - a^2)}{Lr} & a < r < b \\ \frac{\rho L \pi (b^2 - a^2)}{Lr} & r > b \end{cases}$$

(ii) Cylindrical Shell (sketch)



(iii) Infinite slab



From symmetry, field

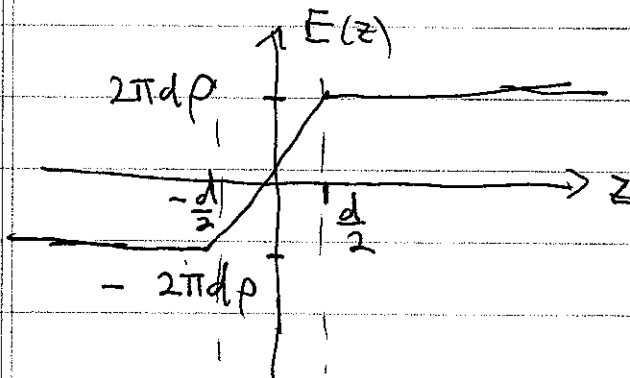
is uniform,  $\perp$  to slab, points away from slab

$$\Rightarrow \oint \vec{E} \cdot d\vec{a} = 2\Delta A E(z) = 4\pi Q_{enc}(z)$$

$$\Rightarrow |E(z)| = 2\pi \frac{Q_{enc}(z)}{\Delta A}$$

$$Q_{enc}(z) = \begin{cases} \rho(z) (\Delta A) & |z| < d \\ \rho(d) (\Delta A) & |z| > d \end{cases}$$

$$\Rightarrow E(z) = \begin{cases} -2\pi\rho(d) & z < -d \\ +2\pi\rho(z) & -d < z < d \\ 2\pi\rho(d) & z > d \end{cases}$$

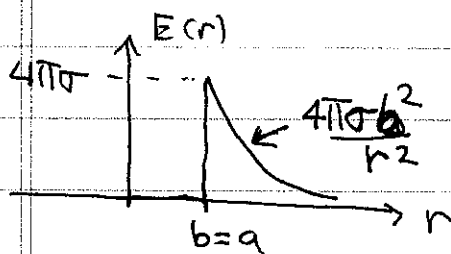


Now take limit as thickness goes to zero, while charge concentrated on surface goes to a constant

$$(i) \quad b \equiv a + \delta \quad \delta \rightarrow 0 \quad \rho \delta \rightarrow \sigma$$

$$E(r=b) = \frac{4\pi}{3} \frac{\rho}{a^2} ((a+\delta)^3 - a^3) \approx \frac{4\pi}{3} \frac{\rho a^3}{a^2} \left[ \left(1 + \frac{\delta}{a}\right)^3 - 1 \right]$$

$$\lim_{\delta \rightarrow 0} E(r=b) \approx \frac{4\pi}{3} \rho a \left[ 1 + \frac{3\delta}{a} - 1 \right] = 4\pi \rho \delta \rightarrow 4\pi \sigma$$

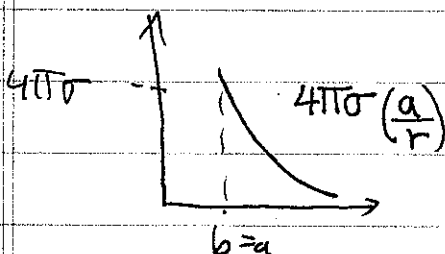


Discontinuity  $4\pi\sigma$

$$(ii) \quad b \equiv a + \delta \quad \delta \rightarrow 0 \quad \rho \delta \rightarrow \sigma$$

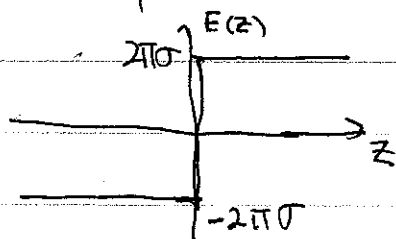
$$E(r=b) \equiv \frac{2\rho\pi}{a} [(a+\delta)^2 - a^2] = 2\pi\rho a \left[ \left(1 + \frac{\delta}{a}\right)^2 - 1 \right]$$

$$\lim_{\delta \rightarrow 0} E(r=b) \approx 2\pi\rho a \left[ 1 + \frac{2\delta}{a} - 1 \right] = 4\pi\rho\delta \rightarrow 4\pi\sigma$$



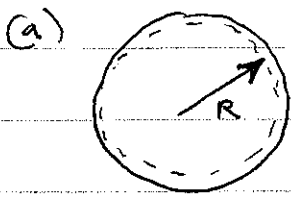
Discontinuity  $4\pi\sigma$

$$(iii) \quad \lim \rho d \rightarrow \sigma \quad d \rightarrow 0$$



Discontinuity  $4\pi\sigma$

(5) Jackson 1.3 : Delta function



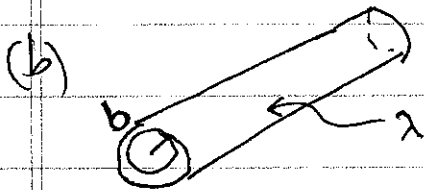
Spherical Shell, radius ~~was~~  $R$ , charge  $Q$  uniformly distributed on surface

$$\sigma = \frac{Q}{4\pi R^2} \quad \left( \frac{\text{Charge}}{\text{Area}} \right)$$

$$\rho(\vec{x}) = \sigma \delta(r-R) = \frac{Q}{4\pi R^2} \delta(r-R) \quad \left[ \begin{array}{l} \text{check units} \\ [S] = \frac{1}{\text{Length}} \end{array} \right]$$

Check

$$\begin{aligned} \int d^3x \rho(\vec{x}) &= \int_0^\infty r^2 dr \int_0^{2\pi} d\phi \int_{-1}^1 d(\cos\theta) \rho(r) \\ &= 4\pi \int_0^\infty dr r^2 \rho(r) = Q \int_0^\infty dr \frac{r^2}{R^2} \delta(r-R) = Q \checkmark \end{aligned}$$



Cylindrical shell, radius  $b$ , charge length  $\lambda$  (uniformly distributed)

$$\Rightarrow \frac{\lambda}{2\pi b} = \frac{\text{charge}}{\text{Area}} \equiv \sigma \quad (\text{uniform on surface of cylinder})$$

$$\Rightarrow \rho(\vec{x}) = \sigma \delta(\rho-b) = \frac{\lambda}{2\pi b} \delta(\rho-b) \quad \text{cylindrical radius}$$

Check

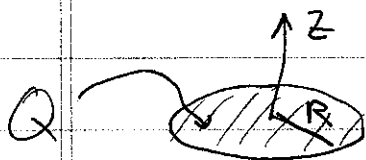
$$\text{Charge on Segment, length } L = \int_0^L dz \int_0^{2\pi} d\phi \int_0^\infty \rho d\rho \rho(\vec{x}) = \frac{\lambda}{2\pi b} [L] (2\pi) \int_0^\infty \rho d\rho \delta(\rho-b)$$

$$= \lambda L \quad \checkmark$$

(Next Page)

## Jackson 1.3 continued

(c) Disk, radius  $R$ , charge  $Q$  uniformly distributed



$$\Rightarrow \frac{\text{Charge}}{\text{Area}} = \sigma = \frac{Q}{\pi R^2}$$

In cylindrical coordinates, all charge is concentrated at  $z=0$

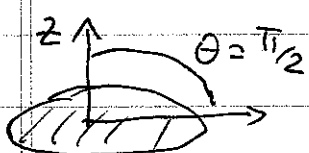
$$\Rightarrow \rho(\vec{x}) = \begin{cases} \sigma \delta(z) = \frac{Q}{\pi R^2} \delta(z) & \rho \leq R \\ 0 & \rho > R \end{cases}$$

cylindrical radius

Check:

$$\int_{\text{all space}} d^3x \rho(\vec{x}) = \int_{-\infty}^{\infty} dz \int_0^{2\pi} d\phi \int_0^R \rho d\rho \frac{Q}{\pi R^2} \delta(z) = \frac{Q}{\pi R^2} (2\pi) \left( \frac{1}{2} \rho^2 \right) \Big|_0^R = Q \quad \checkmark$$

(d) Same as (c) except in spherical coords



Now all the charge is concentrated at  $\theta = \pi/2$

$$\rho(\vec{x}) = \sigma \frac{\delta(\theta - \pi/2)}{r}$$

Remember, in spherical

$$\delta^{(3)}(\vec{x} - \vec{x}') = \delta(r - r') \frac{\delta(\theta - \theta')}{r} \frac{\delta(\phi - \phi')}{r \sin \theta}$$

$$\Rightarrow \rho(\vec{x}) = \frac{Q}{\pi R^2} \frac{\delta(\theta - \pi/2)}{r}$$

Check

$$\int d^3x \rho(\vec{x}) = \frac{Q}{\pi R^2} \int_0^{R^2} r^2 dr \int_0^{2\pi} d\phi \int_0^{\pi} \sin \theta d\theta \frac{\delta(\theta - \pi/2)}{r} = \frac{2Q}{R^2} \int_0^R r dr = Q \quad \checkmark$$