Physics 511, [redacted]

Problem Set # 3 Solutions

Problem # 1

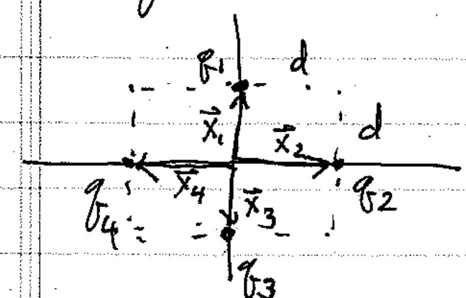
(a)

For a discrete set of charges, the charge density is $\rho(\bar{x}) = \sum_\alpha q_\alpha \delta^{(3)}(\vec{x}-\vec{x}_\alpha)$, where α labels the particle

$$\Rightarrow Q^{(\ell)}_{i,j,\ldots,\ell} \cong \int d^3x\, \rho(\vec{x})\, [x_i x_j \cdots x_\ell]^{(\ell)} = \sum_\alpha q_\alpha [x_i(\alpha) x_j(\alpha) \cdots x_\ell(\alpha)]^{(\ell)}$$

i^{th} coordinate of α^{th} particle

Configuration (i)



$q_1 = 3q,\ q_2 = q_3 = -2q,\ q_4 = q$

$\vec{x}_1 = d\hat{z},\quad \vec{x}_2 = d\hat{x}$

$\vec{x}_3 = -d\hat{x},\quad \vec{x}_4 = -d\hat{z}$

Monopole moment: $q_{net} = \sum_\alpha q_\alpha = 0$

Dipole moment: $\vec{p} = \sum_\alpha \vec{x}_\alpha q_\alpha = q_1\vec{x}_1 + q_2\vec{x}_2 + q_3\vec{x}_3 + q_4\vec{x}_4$

$$\Rightarrow \vec{p} = 3qd\hat{z} - 2qd\hat{x} - 2q(-d\hat{x}) + q(-d\hat{z}) = \boxed{2qd\hat{z}}$$

Quadrapole: $Q_{ij} = \sum_\alpha [3x_i(\alpha)x_j(\alpha) - r_\alpha^2\delta_{ij}]\, q_\alpha$

$$\Rightarrow Q_{xx} = \sum_\alpha (3x^2(\alpha) - r_\alpha^2) q_\alpha = \{(3\cdot 0 - d^2)(3q) + $$
$$+ (3\cdot d^2 - d^2)(-2q) + (3\cdot d^2 - d^2)(-2q) + (3\cdot 0 - d^2)q\}$$

$$\Rightarrow \boxed{Q_{xx} = -12qd^2}$$ Next page

$$Q_{yy} = \sum_\alpha (3y_{(\alpha)}^2 - r_\alpha^2)\, q_\alpha = -\sum_\alpha r_\alpha^2 q_\alpha \quad \left(\begin{array}{l}\text{Since } y\text{-coord} \\ \text{is zero } \forall \text{ charges}\end{array}\right)$$

$$= -d^2 \sum_\alpha q_\alpha \quad (\text{since } r_\alpha^2 = d^2 \ \forall \text{ charges})$$

$$\Rightarrow \boxed{Q_{yy} = 0} \quad (\text{since } \sum q_\alpha = q_{net} = 0)$$

Since $Q_{xx} + Q_{yy} + Q_{zz} = 0 \Rightarrow \boxed{Q_{zz} = -Q_{xx} = 12 q d^2}$

Since the x-axis and z-axis are the "principle axes" (think about moment of inertia) Q_{ij} is diagonal:

Check: $Q_{xy} = Q_{yx} = Q_{yz} = Q_{zy} = 0$ since y-coordinate $= 0$

$$Q_{xz} = Q_{zx} = 3\sum_\alpha q_\alpha x_\alpha z_\alpha = 0 \quad \left(\begin{array}{l}\text{Since } z\text{-coord} = 0 \text{ when} \\ x \text{ nonzero \& vice versa}\end{array}\right)$$

$$\therefore Q_{ij} = \begin{bmatrix} -12qd^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 12qd^2 \end{bmatrix} = +12qd^2 \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Spherical moments

$$q_{\ell,m} = \sum_\alpha q_\alpha r_\alpha^\ell \, Y^*_{\ell,m}(\theta_\alpha, \phi_\alpha)$$

We can use spherical ~~coordinates~~ coordinates of charge position, or express $r^\ell Y_{\ell,m}(\theta,\phi)$ in cartesian coordinates as in Jackson (4.4) – (4.6). Here I'll show the latter

(next page)

(1a) Spherical moments, distribution (i)

Monopole: $q_{0,0} = 0$ (net charge)

Dipole: $q_{1,1} = \sum_\alpha q_\alpha r_\alpha Y^*_{1,1}(\theta_\alpha, \phi_\alpha) = -\sqrt{\frac{3}{8\pi}} \sum_\alpha q_\alpha (x_\alpha + i y_\alpha)^*$

$= -\sqrt{\frac{3}{8\pi}} (p_x - i p_y) = 0 = q_{1,-1}$

$q_{1,0} = \sum_\alpha q_\alpha r_\alpha Y^*_{1,0}(\theta_\alpha, \phi_\alpha) = \sqrt{\frac{3}{4\pi}} \sum_\alpha q_\alpha z_\alpha = 0$

Quadrapole: $q_{2,2} = \sum_\alpha q_\alpha r_\alpha^2 Y^*_{2,2}(\theta_\alpha, \phi_\alpha) = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sum_\alpha q_\alpha \left[(x_\alpha + i y_\alpha)^2\right]^*$

$= \frac{1}{4}\sqrt{\frac{15}{2\pi}} \left(3q(0) - 2q(d^2) + q(0) - 2q(d^2)\right)$

$= -q d^2 \sqrt{\frac{15}{2\pi}} = q_{2,-2} = \frac{1}{12}\sqrt{\frac{15}{2\pi}} (Q_{xx} - 2i Q_{xy} - Q_{yy})$

$q_{2,1} = \sum_\alpha q_\alpha r_\alpha^2 Y^*_{2,1}(\theta_\alpha, \phi_\alpha) = -\sqrt{\frac{15}{8\pi}} \sum_\alpha q_\alpha z_\alpha (x_\alpha - i y_\alpha)^*$

$= 0$ since either $x, y,$ or z coordinate is zero

$= -q_{2,-1}$

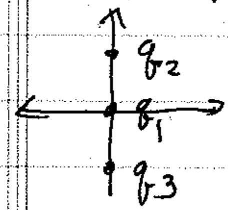
$q_{2,0} = \sum_\alpha q_\alpha r_\alpha^2 Y^*_{2,0}(\theta_\alpha, \phi_\alpha) = \frac{1}{2}\sqrt{\frac{5}{4\pi}} \sum_\alpha q_\alpha (3z_\alpha^2 - r_\alpha^2)$

$= \frac{1}{2}\sqrt{\frac{5}{4\pi}} Q_{zz} = \sqrt{\frac{5}{4\pi}} 6 q d^2$

Net page

(1a)

For configuration (i)



$q_1 = -q$ $\quad$ $q_2 = q$ $\quad$ $q_3 = q$

$\vec{x}_1 = \vec{0}$ $\quad$ $\vec{x}_2 = d\hat{z}$ $\quad$ $\vec{x}_2 = -d\hat{z}$

$$\Rightarrow \boxed{q_{net} = q} \qquad \boxed{\vec{p} = 0} \quad \text{(average position)}$$

$$\left.\begin{array}{l} \bullet\ Q_{xy} = Q_{yz} = Q_{xz} = 0 \\ \bullet\ Q_{xx} = Q_{yy} = -\tfrac{1}{2} Q_{zz} \end{array}\right\} \text{ since distribution is symmetric about z-axis}$$

$$Q_{zz} = \sum_\alpha q_\alpha (3z_\alpha^2 - r_\alpha^2) = q(3d^2 - d^2) + q(3(-d)^2 - d^2) = \boxed{4qd^2}$$

$$Q_{ij} = 4qd^2 \begin{bmatrix} -\tfrac{1}{2} & 0 & 0 \\ 0 & -\tfrac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{"Quadrapole moment"}$$

In spherical moments

$$\boxed{q_{0,0} = \frac{q}{\sqrt{4\pi}}, \quad q_{1,m} = 0, \quad q_{22} = q_{2,1} = 0, \quad q_{2,0} = 2qd^2\sqrt{\frac{5}{4\pi}}}$$

(1b) $$\Phi(\vec{x}) \cong \frac{q_{net}}{r} + \frac{\vec{x}\cdot\vec{p}}{r^3} + \frac{\vec{x}\cdot\overleftrightarrow{Q}\cdot\vec{x}}{2r^5} \quad \left(\text{up to order } \frac{d^3}{r^3}\right)$$

Config. (i): $$\Phi(\vec{x}) \cong \frac{z p_z}{r^3} + \frac{Q_{xx}}{2}\frac{x^2}{r^5} + \frac{Q_{yy}}{2}\frac{y^2}{r^5}$$

$$\boxed{\Phi(\vec{x}) \cong 2qd\frac{z}{r^3} - 6qd^2\left(\frac{x^2 - z^2}{r^5}\right)}$$

Config (ii): $$\Phi(\vec{x}) \cong \frac{q_{net}}{r} + \frac{Q_{zz}}{2}\frac{z^2}{r^5} - \frac{Q_{zz}}{4}\left(\frac{x^2}{r^5} + \frac{y^2}{r^5}\right)$$

$$\Rightarrow \boxed{\Phi(\vec{x}) \cong \frac{q_{net}}{r} + \frac{Q_{zz}}{4}\frac{3z^2 - r^2}{r^5} = \frac{q}{r} + qd^2\left(\frac{3z^2 - r^2}{r^5}\right)}$$

(c) The exact potential at position $\vec{x}$

$$\Phi(\vec{x}) = \sum_{\alpha} \frac{q_\alpha}{|\vec{x}-\vec{x}_\alpha|} \quad \text{(superposition)}$$

Configuration (i)

$$\Phi(\vec{x}) = \frac{3q}{|\vec{x}-d\hat{z}|} + \frac{-2q}{|\vec{x}-d\hat{x}|} + \frac{-2q}{|\vec{x}+d\hat{x}|} + \frac{q}{|\vec{x}+d\hat{z}|}$$

$$= \frac{3q}{\sqrt{r^2-2zd+d^2}} - \frac{2}{\sqrt{r^2-2xd+d^2}} - \frac{2}{\sqrt{r^2+2xd+d^2}} + \frac{1}{\sqrt{r^2+2zd+d^2}}$$

$$= \frac{q}{r}\Big\{3\Big(1-\frac{2zd}{r^2}+\frac{d^2}{r^2}\Big)^{-1/2} - 2\Big(1-\frac{2xd}{r^2}+\frac{d^2}{r^2}\Big)^{-1/2}$$

$$-2\Big(1+\frac{2xd}{r^2}+\frac{d^2}{r^2}\Big)^{-1/2} + \Big(1+\frac{2zd}{r^2}+\frac{d^2}{r^2}\Big)^{-1/2}\Big\}$$

Now expand using $(1+\delta)^{-1/2} \simeq 1 - \frac{1}{2}\delta + \frac{3}{8}\delta^2$, $\delta \ll 1$

Thus, to order $\left(\frac{d}{r}\right)^3$

$$\Phi(\vec{x}) \simeq \frac{q}{r}\Big\{3\Big(1+\frac{zd}{r^2} - \frac{d^2}{2r^2} + \frac{3}{8}\Big(-\frac{2zd}{r^2}\Big)^2\Big)$$

$$-2\Big(1+\frac{xd}{r^2} - \frac{d^2}{2r^2} + \frac{3}{8}\Big(-\frac{2xd}{r^2}\Big)^2\Big)$$

$$-2\Big(1-\frac{xd}{r^2} - \frac{d^2}{2r^2} + \frac{3}{8}\Big(\frac{2xd}{r^2}\Big)^2\Big)$$

$$+\Big(1-\frac{zd}{r^2} - \frac{d^2}{2r^2} + \frac{3}{8}\Big(\frac{2zd}{r^2}\Big)^2\Big)$$

(Note: I have dropped all terms, order $\left(\frac{d}{r}\right)^4$ or smaller)

$$\Rightarrow \Phi(\vec{x}) \cong 2qd\frac{z}{r^3} - 6qd^2\left(\frac{x^2-z^2}{r^5}\right) \quad \text{(as before)}$$

(1c) Continued

Configuration (ii)

$$\Phi(\vec{x}) = -\frac{q}{r} + \frac{q}{|\vec{x}-d\hat{z}|} + \frac{q}{|\vec{x}+d\hat{z}|}$$

$$= -\frac{q}{r} + \frac{q}{\sqrt{r^2-2zd+d^2}} + \frac{q}{\sqrt{r^2+2zd+d^2}}$$

$$= \frac{q}{r}\left\{-1 + \left(1-\frac{2zd}{r^2}+\frac{d^2}{r^2}\right)^{-1/2} + \left(1+\frac{2zd}{r^2}+\frac{d^2}{r^2}\right)^{1/2}\right\}$$

$$\cong \frac{q}{r}\left\{-1 + \left(1-\frac{zd}{r^2}-\frac{d^2}{2r^2}+\frac{3}{8}\left(\frac{-2zd}{r^2}\right)^2\right)\right.$$

$$\left. + \left(1+\frac{zd}{r^2}-\frac{d^2}{2r^2}\right)+\frac{3}{8}\left(\frac{2zd}{r^2}\right)^2\right)$$

$$\Rightarrow \Phi(\vec{x}) \cong \frac{q}{r}\left\{1-\frac{d^2}{r^2}+\frac{3z^2d^2}{r^4}\right\} = \frac{q}{r} + qd^2\left(\frac{3z^2-r^2}{r^5}\right)$$

✓ As before

(d) In order to plot the equipotentials, let us put the potential in dimensionless form

Configuration (i)

$$\Phi(\vec{x}) = \frac{q}{d}\left\{2\left(\frac{z}{d}\right)\left(\frac{d}{r}\right)^3 - 6\left(\frac{x^2}{d^2}-\frac{z^2}{d^2}\right)\left(\frac{d}{r}\right)^5\right\}$$

Configuration (ii)

$$\Phi(\vec{x}) = \frac{q}{d}\left\{\frac{d}{r} + 3\left(\frac{z^2}{d^2}\right)\left(\frac{d}{r}\right)^5 - \left(\frac{d}{r}\right)^3\right\}$$

All plots in units $\frac{q}{d}$, with distances in units d

Multipole Expansions of Discrete Charge Distributions

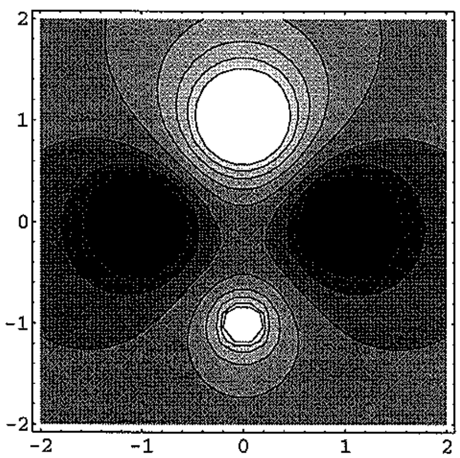
Configuration (i)

Definitions

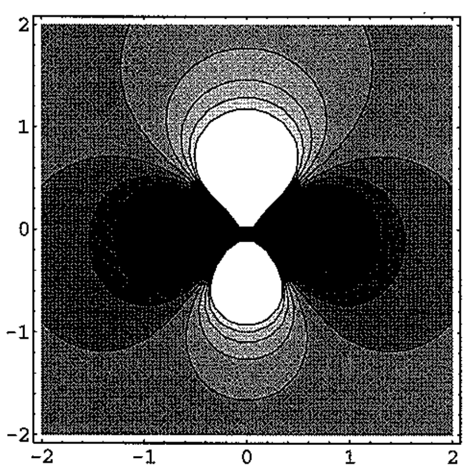
```
Norm[vector_] := Sqrt[vector.vector] (* norm of vector *)

V0[r_,rp_] := 1/Norm[r-rp]
      (* potenial of a unit point charge at rp *)

Vtruei[x_,z_] :=
        Module[{r={x,z}},
              3 V0[r,{0,1}] + V0[r,{0,-1}] -
              2 (V0[r,{1,0}] + V0[r,{-1,0}])]
       (*The exact potential  *)

Vi[x_,z_] := Module[{r=Sqrt[x^2+z^2]},
                     2 z/r^3 - 6 (x^2-z^2)/r^5]
(* Approximate potential including quadrapole correction  *)
```

Close up to the charges

```
(* Exact potential  *)
ContourPlot[Vtruei[x,z],{x,-2,2},{z,-2,2},PlotPoints->30]
```

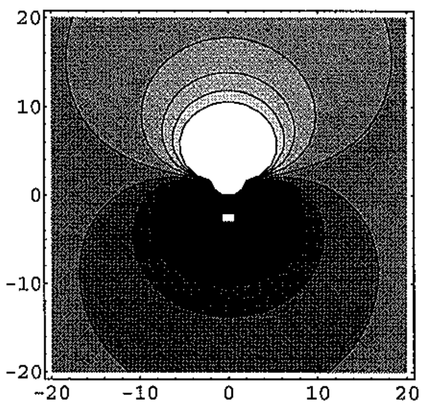


```
(*  Approximate Potential   *)
ContourPlot[Vi[x,z],{x,-2,2},{z,-2,2},PlotPoints->30]
```



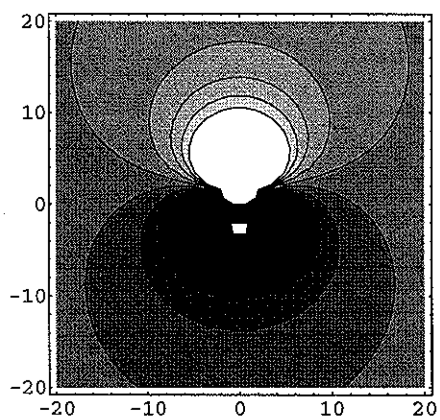
■ Far away from the charges (Dipole term dominates)

```
(* Exact potential  *)
ContourPlot[Vtruei[x,z],{x,-20,20},{z,-20,20},PlotPoints->30]
```



-ContourGraphics-

```
(*  Approximate Potential   *)
ContourPlot[Vi[x,z],{x,-20,20},{z,-20,20},PlotPoints->30]
```



-ContourGraphics-

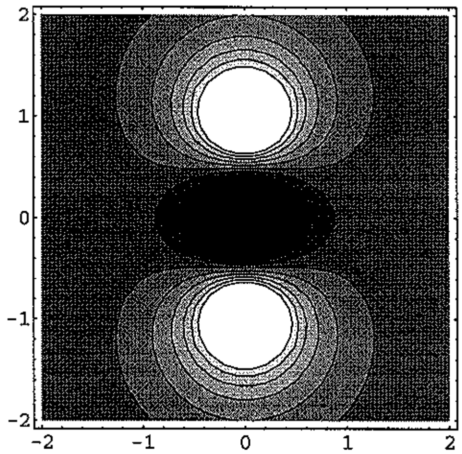
Configuration (ii)

■ Definitions

```
Vtrueii[x_,z_] :=
        Module[{r={x,z}},
                VO[r,{0,1}] + VO[r,{0,-1}] - VO[r,{0,0}]]
       (*The exact potential  *)

Vii[x_,z_] := Module[{r=Sqrt[x^2+z^2]},
                       1/r + 3 z^2/r^5 - 1/r^3]
(* Approximate potential including quadrapole correction  *)
```

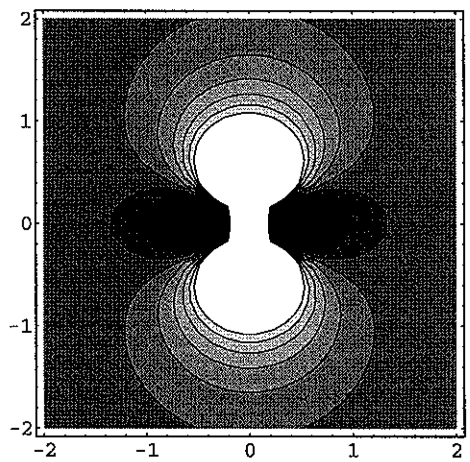
■ Close up to the charges

```
(* Exact potential  *)
ContourPlot[Vtrueii[x,z],{x,-2,2},{z,-2,2},PlotPoints->30]
```



-ContourGraphics-

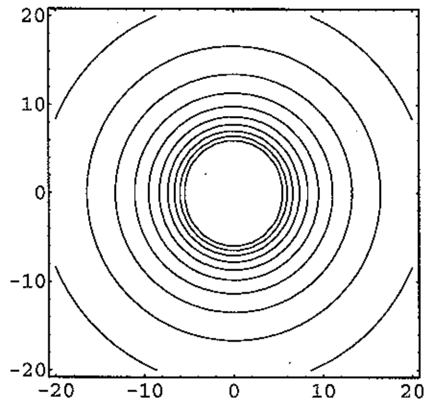
```
(*  Approximate Potential   *)
ContourPlot[Vii[x,z],{x,-2,2},{z,-2,2},PlotPoints->30]
```



-ContourGraphics-

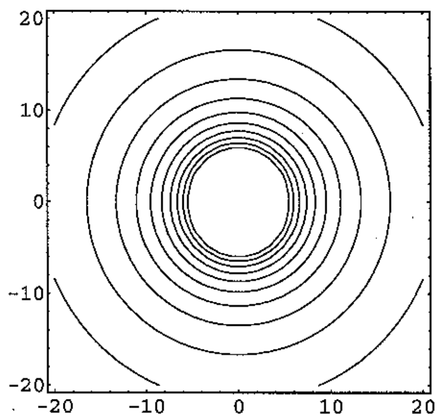
■ Far away from the charges (Monopole term dominates)

```
(* Exact potential  *)
ContourPlot[Vtrueii[x,z],{x,-20,20},{z,-20,20},PlotPoints->30,
            ContourShading->False]
```



-ContourGraphics-

```
(*  Approximate Potential   *)
ContourPlot[Vii[x,z],{x,-20,20},{z,-20,20},PlotPoints->30,
            ContourShading->False]
```



-ContourGraphics-

Problem 2. (Jackson 4.4)

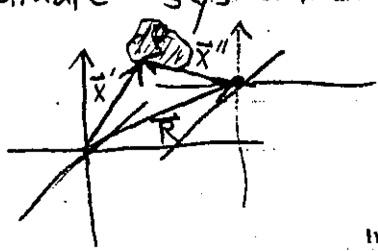
(a) Prove that the first $(2l+1)$ moments of a distribution $\rho(\vec{x})$ are independent of origin, but higher moments are in general dependent on origin:

To prove this theorem we need only show that the first non-vanishing Cartesian moment Q_l be independent of origin (with higher moments dependent) since the $2l+1$ spherical moments $q_{l,m}$ are linear combinations of the Q_l's.

$$Q^{(l)} = \int d^3x' \, \rho(\vec{x}') \left[x_i' x_j' \cdots x_l' \right]^{(l)}$$

where $[x_i' x_j' \cdots x_l']^{(l)}$ are the "solid harmonics"

Now let's find $Q^{(l)}$ in a new coordinate system $(Q''^{(l)})$ whose origin is displaced by the $\vec{R}$ w.r.t. the old coordinate system:



$$\vec{x}' = \vec{x}'' + \vec{R}$$

$\vec{x}'$: position of source in old frame; $\vec{x}''$: position in new frame; $\vec{R}$: displacement of frame

Before we evaluate Q_l', note that

$$d^3\vec{x}'' = d^3\vec{x}' \quad \text{(Volume elements equivalent in both coordinate systems)}$$

$$\rho''(\vec{x}'') = \rho(\vec{x}') \quad \text{(The charge density is a scalar w.r.t. coordinate transformations)}$$

$$\therefore\ Q'^{(\ell)} = \int d^3\vec{x}''\ \rho''(\vec{x}'')\ [x_i'' x_j'' \cdots x_\ell'']^{(\ell)}$$

$$Q'^{(\ell)} = \int d^3\vec{x}'\ \rho(\vec{x}')\left[(x_i'-R_i)(x_j'-R_j)\cdots(x_\ell'-R_\ell)\right]^{(\ell)}$$

Now, it is clear that upon multiplying out

$$[(x_i'-R_i)(x_j'-R_j)\cdots(x_\ell'-R_\ell)]^\ell$$

$$= [x_i'\cdots x_\ell']^\ell + \sum_{\ell'=0}^{\ell-1} f_{\ell'}(\vec{R})\ [x_k'\cdots x_{\ell'}']^{(\ell')}$$

$$\Rightarrow Q'^{(\ell)} = \int d^3\vec{x}'\ \rho(\vec{x}')\ [x_i'\cdots x_\ell']^{(\ell)} + \sum_{\ell=0}^{\ell-1} f_{\ell'}(\vec{R}) \int d^3\vec{x}'\ \rho(\vec{x}')\ [x_k'\cdots x_{\ell'}']^{(\ell')}$$

$$\Rightarrow \boxed{Q'^{(\ell)} = Q^{(\ell)} + \sum_{\ell=0}^{\ell-1} f_{\ell'}(\vec{R})\ Q^{(\ell')}} \quad (*)$$

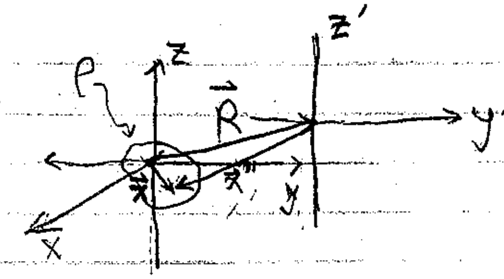
Now if $Q^{(\ell)}$ is the first non-vanishing moment, all the terms in the sum are zero since the moments $Q^{(\ell')}$ are of smaller order than ℓ (they go up to $\ell-1$)

$\Rightarrow$ $Q'^{(\ell)} = Q^{(\ell)}$: The first $2\ell+1$ $q_{\ell,m}$ are independent of origin if $Q^{(\ell)}$ is the first non-vanshing moment

If $Q^{(\ell)}$ is not the first non-vanishing order, then there will in general be non-zero terms in the sum of equation (*)

$\Rightarrow$ $Q'^{(\ell)} \underset{\text{in general}}{\neq} Q^{(\ell)}$ if $Q^{(\ell)}$ is not the first non-vanishing moment

(b)



$$\vec{x}' = \vec{x}'' + \vec{R}$$

$$\Rightarrow \vec{x}'' = \vec{x}' - \vec{R}$$

$$q^{old} = \int \rho(\vec{x}')\, d^3\vec{x}'$$

$$q^{new} = \int \rho''(\vec{x}'')\, d^3\vec{x}'' = \int \rho(\vec{x}')\, d^3\vec{x}' \quad \text{(Since } \rho \text{ is a scalar } \rho''(\vec{x}'') = \rho(\vec{x}')\text{)}$$

$$\Rightarrow \boxed{q^{new} = q^{old}}$$

$$\vec{p}^{\,new} = \int d^3\vec{x}''\, \rho''(\vec{x}'')\, \vec{x}'' = \int d^3\vec{x}'\, \rho(\vec{x}')\, (\vec{x}' - \vec{R})$$

$$= \int d^3\vec{x}'\, \rho(\vec{x}')\, \vec{x}' - \vec{R}\int \rho(\vec{x}')\, d^3\vec{x}'$$

$$\boxed{\vec{p}^{\,new} = \vec{p}^{\,old} - \vec{R}q}$$

$$Q_{ij}^{old} = \int d^3\vec{x}'\, \rho(\vec{x}')\, (3x_i' x_j' - \delta_{ij} r'^2)$$

$$Q_{ij}^{new} = \int d^3\vec{x}'\, \rho(\vec{x}')\, \left(3(x_i' - R_i)(x_j' - R_j) - \delta_{ij}(x_l' - R_l)^2\right)$$

$$= \int d^3\vec{x}'\, \rho(\vec{x}') \left[3x_i' x_j' - \delta_{ij} r^2 - 3R_i x_j' - 3x_i' R_j + 3R_i R_j + 2\delta_{ij}\vec{R}\cdot\vec{x}' - \delta_{ij}R^2\right]$$

$$= \int d^3\vec{x}'\, \rho(\vec{x}')\, (3x_i' x_j' - \delta_{ij} r^2) - 3R_i \int d^3\vec{x}\; x_j'\, \rho(\vec{x}')$$
$$- 3\left(\int x_i'\, \rho(\vec{x}')\, d^3\vec{x}'\right) R_j + \left[3R_i R_j - \delta_{ij} R^2\right] \int \rho(\vec{x}')\, d^3\vec{x}'$$
$$+ 2\delta_{ij}\vec{R}\cdot\int \vec{x}'\rho(\vec{x}')\, d^3\vec{x}'$$

$$\Rightarrow \boxed{Q_{ij}^{new} = Q_{ij}^{old} - 3R_i p_j^{old} - 3p_i^{old} R_j + (3R_i R_j - \delta_{ij}R^2)\, q^{old} + 2\delta_{ij}\vec{R}\cdot\vec{p}^{\,old}}$$

Note $Q_{ij}^{New} = Q_{ji}^{New}$, $Q_{ii}^{New} = 0$

(over)

(c) If $q \neq 0$

We can chose $\boxed{\vec{R} = \frac{\vec{p}^{old}}{q}}$

$\Rightarrow \vec{p}^{new} = \vec{0}$

If $q \neq 0$, $\vec{p} \neq 0$ or at least $\vec{p} \neq 0$, can we chose $\vec{R}$ so $Q_{ij} = 0$?

Since we are only allowed to make displacements of the origin by vector $\vec{R}$ and not rotations of the coordinate system Q_{ij} will in general have 5 independent components.

However, the equations which determines the choice of $\vec{R}$ that makes $Q_{ij} = 0$ are

$$-Q_{ij}^{old} = -3R_i p_j^{old} - 3p_i R_j^{old} + (3R_i R_j - \delta_{ij} R^2) q^{old} + 2\delta_{ij} \vec{R} \cdot \vec{p}^{old}$$

$\therefore$ These are only three equations. since there are only 3 components R_i

$$\boxed{\text{Thus since } Q_{ij} \text{ has 5 indep. components, we } \underline{\text{cannot}} \text{ in general chose } \vec{R} \text{ to make } Q_{ij}^{new} = 0}$$

Problem 3 (Jackson 4.6)

Quadrapole moment $Q = \frac{1}{e} Q_{33}$, in a cylindrically symmetric electric field with $\left.\frac{\partial E_z}{\partial z}\right|_0$ along z axis

(a) Since the field is cylindrically symmetric we can rotate our coordinate axes to lie along the eigenvectors ~~vector~~ of the Q_{ij} tensor

$$\Rightarrow Q_{ij} = \begin{bmatrix} -\frac{1}{2}Q_{33} & 0 & 0 \\ 0 & -\frac{1}{2}Q_{33} & 0 \\ 0 & 0 & Q_{33} \end{bmatrix} = eQ \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The energy of quadrapole interaction is given in 4.24 as

$$W = -\frac{1}{6} Q_{ij} \left.\frac{\partial E_j}{\partial x_i}\right|_0 = -\frac{eQ}{6}\left[-\frac{1}{2}\left.\frac{\partial E_x}{\partial x}\right|_0 + \frac{1}{2}\left.\frac{\partial E_y}{\partial y}\right|_0 + \left.\frac{\partial E_z}{\partial z}\right|_0\right)$$

For a cylindrically symmetric field $\frac{\partial E_x}{\partial x} = \frac{\partial E_y}{\partial y}$

And since $\vec{\nabla}\cdot\vec{E} = 0$

$$\frac{\partial E_z}{\partial z} = -\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y}\right)$$

$$\Rightarrow \boxed{W = \frac{eQ}{4}\left.\frac{\partial E_z}{\partial z}\right|_0}$$

(b) For $Q = 2\times10^{-24}\,\text{cm}^2$, $\frac{W}{h} = 10\ \text{MHz} \Rightarrow W = 6.63\times10^{-20}\ \text{erg}$
$= 6.63\times10^{-27}\ \text{J}$

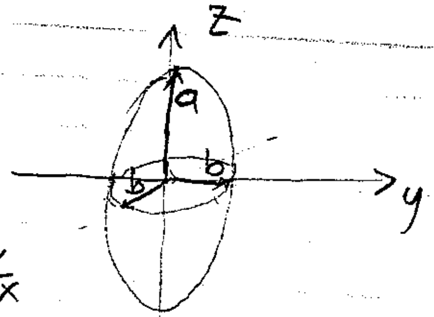
$$\Rightarrow Q = \frac{2\times10^{-24}\,\text{cm}^2}{(0.529\times10^{-8}\,\text{cm})^2}\, a_0^2 = 7.15\times10^{-8}\, a_0^2$$

$$\Rightarrow W = 4.14\times10^{-8}\ \text{eV} = 1.52\times10^{-9}\left(\frac{e^2}{a_0}\right)$$

$\left(\text{Using } \frac{e^2}{2a_0} = \text{Rydberg} = 13.6\ \text{eV}\right)$

$$\therefore 1.52\times10^{-9}\left(\frac{e^2}{a_0}\right) = -\frac{e}{4}\left(7.15\times10^{-8}\, a_0^2\right)\left.\frac{\partial E_z}{\partial z}\right|_0$$

$$\Rightarrow \boxed{\left.\frac{\partial E_z}{\partial z}\right|_0 = -.085\left(\frac{e}{a_0^3}\right)}$$



Charge density : Uniform charge Z_i distributed over the spheroid

$$\frac{x^2}{b^2} + \frac{y^2}{b^2} + \frac{z^2}{a^2} = 1$$

$$\Rightarrow \rho(\vec{x}) = \frac{3Ze}{4\pi a b^2}$$

$$Q = \frac{Q_{zz}}{e} = \frac{1}{e}\int_{\text{spheroid}} (2z^2 - x^2 - y^2)\, \rho(\vec{x})\, d^3x$$

Changing coordinates: $x' = \frac{x}{b}$ $\quad y' = \frac{y}{b}$ $\quad z' = \frac{z}{a}$ $\Rightarrow x'^2 + y'^2 + z'^2 = 1$ ← unit sphere

$$\Rightarrow d^3x = \underbrace{\frac{\partial(x,y,z)}{\partial(x',y',z')}}_{\leftarrow \text{ Jacobian}} d^3x' = ab^2 d^3x'$$

$$\therefore Q = \frac{3Z}{4\pi}\int_{\substack{\text{unit}\\\text{sphere}}} \left(2a^2z'^2 - b^2(x'^2+y'^2)\right) d^3x' = \frac{3Z}{4\pi}\int (2a^2\cos^2\theta' - b^2\sin^2\theta')\, r'^2 dV$$

$$= \frac{3Z}{4\pi}\int_0^{2\pi} d\varphi' \int_0^1 r'^4 dr' \int_{-1}^{1}\left[(2a^2+b^2)\cos^2\theta' - b^2\right] d(\cos\theta')$$

$$= \frac{3Z}{4\pi}(2\pi)\left(\frac{1}{5}\right)\left(\frac{2}{3}(2a^2+b^2) - 2b^2\right)$$

$$\boxed{Q = \frac{2}{5}Z(a^2-b^2)}$$

Example Eu^{153}, $Z = 63$, $Q = 2.5\times10^{-24}\ \text{cm}^2$, $R = \frac{a+b}{2} = 7\times10^{-13}\ \text{cm}$

$$\Rightarrow Q = \frac{2}{5}Z(a+b)(a-b) = \frac{4}{5}ZR(a-b)$$

$$\Rightarrow \boxed{\frac{(a-b)}{R} = \frac{5Q}{4ZR^2} = 0.10}$$

Problem 4

Multipole Moments of an azimuthally symmetric charge distribution

Aside

If ρ is symmetric about the z-axis then:

- $p_x = p_y = 0$ (average x-position = average y-position = 0)
- $Q_{xy} = Q_{xz} = Q_{yz} = 0$ (principle axes x-y-z)
- $Q_{xx} = Q_{yy}$ (by symmetry) $= -\frac{1}{2} Q_{zz}$ since $\mathrm{Tr}(Q_{ij}) = 0$

$\Rightarrow$ Up to quadrapole term:

$$\Phi(\vec{x}) = \frac{q_{net}}{r} + \frac{z\, p_z}{r^3} + \frac{1}{2} Q_{zz}\left(\frac{-x^2-y^2}{2} + z^2\right)\frac{1}{r^5}$$

$$= \frac{q_{net}}{r} + p_z \frac{\cos\theta}{r^2} + \frac{1}{4} Q_{zz} \frac{3z^2 - r^2}{r^5}$$

$$= \frac{q_{net}}{r} + p_z \frac{\cos\theta}{r^2} + \frac{1}{4} Q_{zz} \frac{(3\cos^2\theta - 1)}{r^3}$$

$$= \frac{q_{net}}{r} P_0(\cos\theta) + \frac{p_z}{r^2} P_1(\cos\theta) + \frac{Q_{zz}}{2r^3} P_2(\cos\theta)$$

where $P_0(\mu) = 1$, $P_1(\mu) = \mu$, $P_2(\mu) = \frac{3\mu^2 - 1}{2}$ are the Legendre Polynomials

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This also follows from the spherical multipoles

$$q_{\ell,m} \equiv \int d^3x\, \rho(\vec{x})\, r^\ell\, Y^*_{\ell,m}(\theta,\varphi)$$: If ρ independent of ϕ then $q_{\ell,m} = 0$ if $m \neq 0$

$$\Rightarrow \Phi(\vec{x}) = \sum_\ell \frac{4\pi}{2\ell+1}\, q_{\ell,0}\, \frac{Y_{\ell,0}(\theta)}{r^{\ell+1}}$$

$$Y_{\ell,0}(\theta) = \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\theta)$$

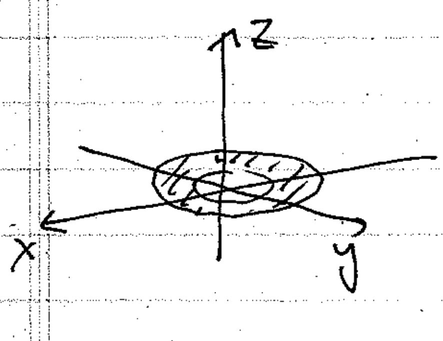
$$\Rightarrow \Phi(\vec{x}) = \sum_\ell Q^{(\ell)}\, \frac{P_\ell(\cos\theta)}{r^{\ell+1}}$$

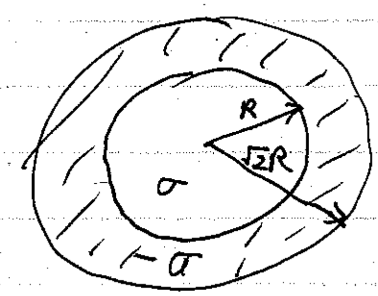
where $Q^{(\ell)} = \sqrt{\frac{4\pi}{2\ell+1}} \int d^3x\, \rho(\vec{x})\, r^\ell\, Y^*_{\ell,0}(\theta,\varphi) = \int d^3x\, \rho(\vec{x})\, r^\ell P_\ell($

It is easy to show that

$$Q^{(\ell)} = \frac{1}{\ell!} Q^{(\ell)}_{\underbrace{33\cdots3}_{\ell\,\text{times}}}$$ (the "z-component" of the ℓ^{th} cartesian tensor)

OK, with that background consider the distribution





$$\boxed{q_{net} = (\pi R^2)\sigma + \left[(\pi 2R^2) - \pi R^2\right](-\sigma) = 0}$$

Problem 4 continued

- $p_x = p_y = 0$ by symmetry.

and $p_z = 0$ since all charge is in x-y plane

- We need $Q_{zz} = \int d^3x\, \rho(\vec{x})\,(3z^2 - r^2) = \int da\, \sigma(r)\,(-r^2)$

all charge at $z=0$

$da = 2\pi r\,dr$ (differential rings)

$$\Rightarrow Q_{zz} = \sigma \int_0^R (-r^2)\, 2\pi r\,dr - \sigma \int_R^{\sqrt{2}R} (-r^2)\, 2\pi r\,dr$$

$$= 2\pi\sigma \left(-\int_0^R r^3\,dr + \int_R^{\sqrt{2}R} r^3\,dr \right)$$

$$= 2\pi\sigma \left(-\frac{r^4}{4}\Big|_0^R + \frac{r^4}{4}\Big|_R^{\sqrt{2}R} \right) = 2\pi\sigma \left(-\frac{R^4}{4} + \frac{4R^4}{4} - \frac{R^4}{4} \right)$$

$$\Rightarrow \boxed{Q_{zz} = \pi\sigma R^4}$$ (units: Charge · Length^2)

$$\Rightarrow Q_{ij} \doteq \pi\sigma R^4 \begin{bmatrix} -1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \boxed{\Phi(r,\theta) = Q_{zz}\frac{P_2(\cos\theta)}{2r^3} = \frac{\pi\sigma R^4}{4}\,\frac{(3\cos^2\theta - 1)}{r^3}}$$

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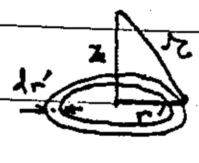
(b) By direct integration, the potential along the z-axis is found by adding up the contribution of rings of charge of radius r' and thickess dr'



$$d\Phi = \frac{\sigma(r')\,da'}{\mathscr{r}}, \quad da' = 2\pi r' dr'$$

$$= \frac{2\pi\sigma(r')dr'}{\sqrt{r'^2+z^2}}$$

$$\Rightarrow \Phi(z) = \int d\Phi = 2\pi\sigma \int \frac{dr'\,\sigma(r')}{\sqrt{r'^2+z^2}}$$

$$= 2\pi\sigma\left[\int_0^R \frac{dr'}{\sqrt{r'^2+z^2}} - \int_R^{\sqrt{2}R} \frac{dr'}{\sqrt{r'^2+z^2}}\right]$$

$$= 2\pi\sigma\left[\sqrt{r'^2+z^2}\Big|_0^R - \sqrt{r'^2+z^2}\Big|_R^{\sqrt{2}R}\right]$$

$$\Rightarrow \Phi(z) = 2\pi\sigma\left(\sqrt{z^2+R^2} - z - \sqrt{z^2+2R^2} + \sqrt{z^2+R^2}\right)$$

$$\Rightarrow \boxed{\Phi(z) = 2\pi\sigma\left(2\sqrt{z^2+R^2} - \sqrt{z^2+2R^2} - z\right)}$$

(c) Since ρ is azimulthally symmetric, we know V is independent of ϕ, and therefore outside the charge distribution

$$\Phi(r,\theta) = \sum_{\ell}' \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

for $r > \sqrt{2}R$ we have the b.c. $V \to 0$ and $r \to \infty$

$$\Rightarrow \quad A_\ell = 0 \quad \forall \ell$$

$$\Rightarrow \quad \Phi(r,\theta) = \sum_{\ell}' \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

We also have the b.c. $V(r,\theta=0)$ (along z-axis)

$$\Phi(r,\theta=0) = 2\pi\sigma\left(2\sqrt{r^2+R^2} - \sqrt{r^2+2R^2} - r\right)$$

To find the expansion coefficients, B_ℓ, expand the above expression in powers of $\frac{R}{r} \ll 1$

$$\Phi(r,\theta=0) = 2\pi\sigma r\left(2\left(1+\frac{R^2}{r^2}\right)^{1/2} - \left(1+\frac{2R^2}{r^2}\right)^{1/2} - 1\right)$$

$$\approx 2\pi\sigma r \left\{ 2\left(1+\frac{R^2}{2r^2} - \frac{1}{8}\left(\frac{R^2}{r^2}\right)^2\right) - \left(1+\frac{1}{2}\frac{2R^2}{r^2} - \frac{1}{8}\left(\frac{2R^2}{r^2}\right)^2\right) - 1 \right\}$$

here I used $(1+\delta)^n \approx 1 + n\delta + \frac{n(n-1)}{2}\delta^2$ for $\delta < 1$

$$\therefore \Phi(r, \theta = 0) \approx 2\pi\sigma r \left(2 + \frac{R^2}{r^2} - \frac{1}{4}\frac{R^4}{r^4} - 1 - \frac{R^2}{r^2} + \frac{1}{2}\frac{R^4}{r^4} - 1 \right)$$

$$= 2\pi\sigma r \left(\frac{R^4}{4r^4}\right) = \frac{\pi\sigma R^4}{2} \frac{1}{r^3}$$

The general expansion is

$$\Phi(r,\theta) = \sum_\ell B_\ell \frac{1}{r^{\ell+1}} P_\ell(\cos\theta)$$

$$\Rightarrow \Phi(r, \theta = 0) = \sum_\ell B_\ell \frac{1}{r^{\ell+1}} P_\ell(0) = \sum_\ell B_\ell \frac{1}{r^{\ell+1}}$$

$$= \frac{B_0}{r} + \frac{B_1}{r^2} + \frac{B_2}{r^3} + \cdots$$

$$\Rightarrow B_0 = B_1 = 0 \qquad B_2 = \frac{\pi\sigma R^4}{2}$$

$\therefore$ Up to order $1/r^3$

$$\boxed{\Phi(r,\theta) = \frac{\pi\sigma R^4}{2} \frac{1}{r^3} P_2(\cos\theta)}$$

(as in part (a))