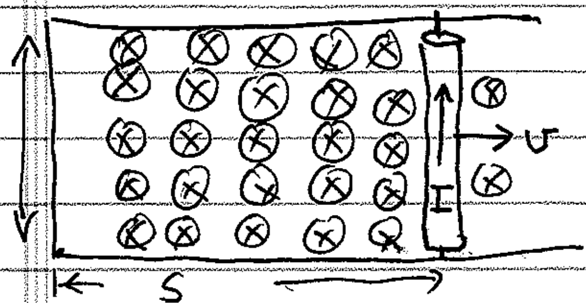
Physics 511

P.S. #5 Solutions



(a) As the rod moves the Lorentz Force pushes current I. This "motional EMF" can be calculated from Faraday's Law

$$\mathcal{E} = -\frac{1}{c}\frac{d}{dt}\Phi_B \qquad \Phi_B = \int \vec{B}\cdot\hat{n}\,da = B\ell\, s(t)$$

(magnetic flux through circuit)

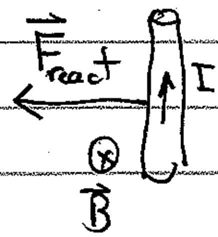
$\Rightarrow$ Voltage drop across rod: $\mathcal{E} = +\frac{B\ell}{c}\frac{ds}{dt} = +\frac{Bv\ell}{c}$

Current $I = \frac{\mathcal{E}}{R}$, $R = \frac{\ell}{a\sigma}$

$$\Rightarrow \boxed{I = +\frac{v}{c}a\sigma B}$$

(flowing counter clockwise) (Lenz's Law)

(b) Once current flows, there is a back reaction force



$$\boxed{|\vec{F}_{react}| = \frac{I\ell}{c}B}$$

$$\boxed{\vec{F}_{react} = -\frac{\vec{v}}{c^2}a\ell\sigma B^2}$$

Note, this is a damping force, proportional, but opposed to $\vec{v}$

(c) Newtons Law:

$$m\frac{d\vec{v}}{dt} = \vec{F}_{react} = -al\frac{\sigma}{c^2}B^2\vec{v}$$

$$\Rightarrow \frac{d\vec{v}}{dt} = -\underbrace{\left(\frac{B^2\sigma}{\rho_M c^2}\right)}_{\equiv \Gamma}\vec{v} \qquad \text{where } \rho_M = \frac{m}{al} \left(\begin{matrix}\text{mass}\\ \text{density}\end{matrix}\right)$$

We can integrate this easily with initial condition $\vec{v}(0) = \vec{v}_0$

$$\Rightarrow \boxed{\vec{v}(t) = \vec{v}_0\, e^{-\Gamma t} \qquad \Gamma = \frac{B^2\sigma}{\rho_m c^2}}$$

(d) Where does the energy go?

<u>Ans</u>: It is dissipated by the resistor into heat

Dissipation rate $I^2R = \left(\frac{v^2}{c^2}a^2\sigma^2B^2\right)\left(\frac{l}{a\sigma}\right)$

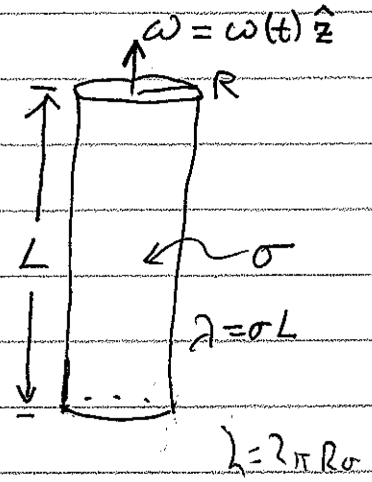
$$\frac{dU_{diss}}{dt} = \frac{v_0^2}{c^2}al\sigma B^2\, e^{-2\Gamma t}$$

Integrate $0 \to \infty$ $\quad U_{\substack{total\\ dissipated}} = \int_0^\infty \frac{dU}{dt}dt = \frac{v_0^2}{c^2}al\sigma B^2\int_0^\infty e^{-2\Gamma t}dt$

$$= \frac{v_0^2}{c^2}\frac{al\sigma B^2}{2\Gamma} = \frac{1}{2}\rho_m al\, v_0^2 = \frac{1}{2}mv_0^2$$

Problem 2

$L \gg R$



$$\vec{E}_0 = \begin{cases} 0 & r_c < R \\ \frac{2\lambda}{r_c}\hat{r}_c & r_c > R \end{cases}$$

$$\vec{B}_0 = \begin{cases} \frac{4\pi\sigma\omega_0 R}{c}\hat{z} & r_c < R \\ 0 & r_c > R \end{cases}$$

$$\vec{J} = \sigma\omega R\,\delta(r_c - R)\,\hat{\phi}$$

We spin up the cylinder from rest to ω_0 over a time $T \ll \frac{R}{c} \Rightarrow \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \simeq 0$

$$\Rightarrow \vec{B}(\vec{x},t) \approx \begin{cases} \frac{4\pi}{c}\sigma\omega(t) R\hat{z} & r_c < R \\ 0 & r_c > R \end{cases}$$

(a) The time dependent part of $\vec{E}$ arises from Faraday's Law $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$

$$\Rightarrow \oint \vec{E}\cdot d\vec{l} = -\frac{1}{c}\int\left(\frac{\partial \vec{B}}{\partial t}\right)\cdot d\vec{a}$$

As in Ampère's law we take a loop over which $\vec{E}\cdot d\vec{l}$ is constant. Taking $\vec{E}$ in $\hat{\phi}$ direction and loop at radius r_c, normal in $\hat{z}$

$$\oint \vec{E}\cdot d\vec{l} = E(r_c,t)\,2\pi r_c$$

$$-\frac{1}{c}\int\left(\frac{\partial \vec{B}}{\partial t}\right)\cdot d\vec{a} = \begin{cases} r_c > R & -\frac{\pi R^2}{c}\frac{\partial B_z}{\partial t} = -\frac{4\pi^2 R^2\sigma}{c}\frac{d\omega}{dt} \\ r_c < 0 & 0 \end{cases}$$

Thus the total $\vec{E}$ field (including static part)

$$\boxed{\vec{E} = \begin{cases} -\dfrac{2\pi\sigma}{c^2}\dfrac{d\omega}{dt} R\, r_c\, \hat{\phi} & r_c < R \\[2ex] \dfrac{2\lambda}{r_c}\hat{r}_c - \dfrac{2\pi\sigma}{c^2}\dfrac{d\omega}{dt}\dfrac{R^3}{r_c}\hat{\phi} & r_c > R \end{cases}}$$

(b) Energy supplied to spin up cylinder = Work done against the back EMF generated

$$-\int_{\text{all space}} \vec{J}\cdot\vec{E}\,\frac{d^3x}{dz} = -\int \sigma\omega R\,\delta(r_c - R)\,E(r_c)\,2\pi r_c\,dr_c$$

per unit length $\Rightarrow dz$

$$= -\sigma\omega R\ 2\pi R\ E_\phi(R)$$

$$= -(\sigma\omega R)(2\pi R)\left(-\frac{2\pi\sigma}{c^2}\frac{d\omega}{dt}R^2\right)$$

$$= \boxed{\frac{4\pi^2\sigma^2 R^4}{c^2}\omega\frac{d\omega}{dt}}$$

$$\frac{\partial}{\partial t}\frac{U_B}{dz} = \frac{\partial}{\partial t}\int_{\text{all space}}\frac{B^2}{8\pi}\,dx\,dy = \left(\frac{B}{4\pi}\frac{\partial B}{\partial t}\right)(\pi R^2)$$

$$= \frac{R^2}{4}B\frac{\partial B}{\partial t} = \boxed{\frac{4\pi^2\sigma^2 R^4}{c^2}\omega\frac{d\omega}{dt}}$$

Thus for a bounding surface at "∞"

$$\frac{\partial}{\partial t} U_B = -\int (\vec{J} \cdot \vec{E})\, d^3x$$

Note $\oint \vec{S} \cdot \hat{n}\, da = 0$ at this surface

$\boxed{\text{\underline{Question}: What happened to } \frac{\partial}{\partial t} U_E \text{ ?}}$

(c) Flow of energy/length <u>into</u> cylinder: just inside $r<R$

$$\oint \vec{S} \cdot (-\hat{r}_c) \frac{da}{dz} \qquad da = 2\pi R\, dz$$

$$= -\hat{r}_c \cdot \vec{S}(r_c = R)\, 2\pi R$$

$$-\hat{r}_c \cdot \vec{S} = \frac{c}{4\pi}(\vec{E} \times \vec{B}) = \frac{-c}{4\pi}(E_\phi \hat{\phi} \times B \hat{z}) \cdot \hat{r}_c \quad \left(\begin{array}{c}\text{Note } \hat{r}_c \\ \text{component} \\ \text{only}\end{array}\right)$$

$$= -\frac{c}{4\pi} E_\phi(R) B = -\frac{c}{4\pi}\left(-\frac{2\pi\sigma}{c^2} R^2 \frac{d\omega}{dt}\right)\left(\frac{4\pi}{c}\sigma\omega R\right)$$

$$= \frac{2\pi\sigma^2 R^3}{c^2} \omega \frac{d\omega}{dt}$$

$$\therefore \oint \vec{S} \cdot (-\hat{r}_c) \frac{da}{dz} = \boxed{\frac{4\pi^2\sigma^2 R^4}{c^2} \omega \frac{d\omega}{dt}}$$

$$= \boxed{\frac{\partial U_B}{\partial t\, dz}}$$

Energy flows from edge of cylinder into interior

(d) Additional torque to spin up cylinder / unit volume

$$\frac{\vec{\tau}}{Vol} = -\vec{x} \times \left(\rho \vec{E} + \frac{1}{c}\vec{J}\times\vec{B}\right)$$

$\vec{x} \Downarrow r_c \hat{r}_c$ (lever arm along cylindrical radial direction)

$$\vec{E} = E_{r_c}\hat{r}_c + E_\phi \hat{\phi} \qquad \vec{J}\times\vec{B} = J_\phi B_z \underbrace{\hat{\phi}\times\hat{z}}_{\hat{r}_c}$$

$$\Rightarrow \frac{\vec{\tau}}{Vol} = -r_c \rho E_\phi \hat{z} \qquad \rho = \sigma\,\delta(r_c - R)$$

To find extra torque/length, integrate in x-y plane

$$\vec{\tau}/_{length} = \int \frac{\vec{\tau}}{Vol}\, d^2x = -\hat{z}\int \rho r_c E_\phi \,(2\pi r_c\, dr_c)$$

$$= -2\pi\sigma\hat{z} \int \delta(r_c - R)\, E_\phi\, r_c^2\, dr_c$$

$$= -2\pi R^2 \sigma\, E_\phi(R)\, \hat{z} = -2\pi R^2 \sigma \left(-\frac{2\pi\sigma}{c^2} R^2 \frac{d\omega}{dt}\hat{z}\right)$$

$$\Rightarrow \boxed{\vec{\tau}/_{length} = \frac{4\pi^2\sigma^2 R^4}{c^2}\frac{d\omega}{dt}\hat{z}}$$

$$\Rightarrow \boxed{\left|\vec{\tau}/_{length}\right| = \frac{1}{\omega}\frac{\partial}{\partial t} U_{B/length}}$$

Rate of creation of E/M angular momentum

= Rate of creation of energy / ω

(e) Where does E/M angular momentum go?

Well if doesn't seem to be anywhere when we're finished spinning. At that time $\frac{d\omega}{dt} = 0$, $\omega = \omega_0$

and $\vec{E} = 0$ inside, $\vec{B} = 0$ outside $\Rightarrow \vec{E} \times \vec{B} = 0$!

$$\Rightarrow \vec{\mathcal{L}}_{em} = 0 \ !$$

Let's consider the flux density of angular momentum, $-\vec{r}_c \times (\overleftrightarrow{T} \cdot \hat{n})$, with $\hat{n} = \hat{r}_c$, then as we are building up $\vec{B}$ we have:

$$-\vec{r}_c \times (\overleftrightarrow{T} \cdot \hat{r}_c) = -r_c \, \hat{r}_c \times \left(T_{r_c r_c} \hat{r}_c + T_{\phi r_c} \hat{\phi} + T_{z r_c} \hat{z} \right)$$

$$= -r_c T_{\phi r_c} \hat{z} + r_c T_{z r_c} \hat{\phi}$$

Since $\overleftrightarrow{T} = \frac{1}{4\pi}\left(\vec{E}\vec{E} + \vec{B}\vec{B} + \frac{1}{2}\mathbb{1}(E^2 + B^2) \right)$

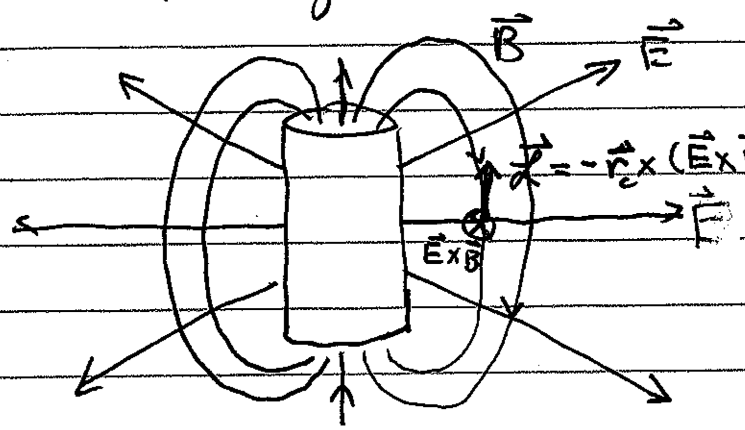
$$T_{\phi r_c} = \frac{1}{4\pi} E_\phi E_{r_c} \qquad T_{z r_c} = 0$$

$$\Rightarrow \boxed{-\vec{r}_c \times (\overleftrightarrow{T} \cdot \hat{r}_c) = -\frac{r_c}{4\pi} E_\phi E_{r_c} \hat{z}}$$

$\Rightarrow$ During the spinning from result, E/M $\measuredangle$-mom ~~[illegible]~~ in the $\hat{z}$ direction is flowing radially away from cylinder.

So, where did it go!?

This is tricky. Consider a finite cylinder. After spinning we have fields outside



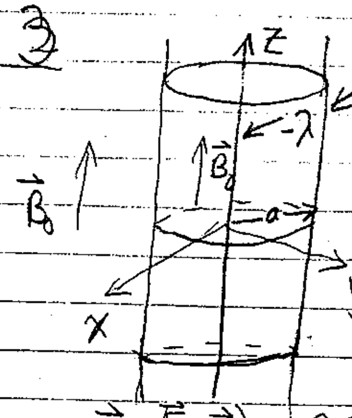
We see that there IS angular momentum, but it is stored in the fringing fields

For the infinite cylinder we have artificially moved these fringing fields to ∞!

But for any finite distribution the energy and angular momentum are stored locally and can be retrieved by despinning the cylinder.

Moral of the story: In the quasi-static approx. we can be how energy, angular momentum etc flow from the sources to local fields. These quantities can be ~~retrieved~~ recovered by sources. This is in constrast to radiation which flows to ∞ even for ~~finite sources~~ and cannot be recovered

Problem 3



$\sigma = \frac{\lambda}{2\pi a}$ at $t=0$

$\vec{\mathcal{L}} = \vec{r}_c \times \vec{g}$

$\vec{g} = \frac{1}{4\pi c}\vec{E}\times\vec{B}$

$t=0 \quad \vec{E} = \begin{cases} -\frac{2\lambda}{r_c}\vec{e}_\rho & r_c < a \\ 0 & r_c > a \end{cases}$

$\vec{B} = B_0 \vec{e}_z$

$\vec{r}_c \times (\vec{E}\times\vec{B}) = -2\lambda B_0 \vec{e}_z$

(a) The angular momentum stored / unit length at $t=0$ is

$$\vec{L}_{em} = \frac{1}{4\pi c}\int_0^{r_c} +2\lambda B_0 \vec{e}_z \, 2\pi r_c \, dr_c = \boxed{-\frac{\lambda B_0 a^2}{2c}\vec{e}_z} \; \vec{L}_{em}$$

(b) The external field is reduced over a time $T \gg \frac{a}{c}$ (No radiation), so by angular momentum conservation, the cylinder will begin to rotate will and angular velocity $\omega(t)$

$$\Rightarrow \vec{B} = \vec{B}_{ext} + \frac{4\pi}{c}\sigma a \omega(t) = \boxed{\left(B_{ext}(t) + \frac{2\lambda}{c}\omega(t)\right)\vec{e}_z = \vec{B}_{total}}$$

(c) $$\frac{d}{dt}\vec{L}_{mech} = \vec{r}_c \times \left(\lambda \vec{E} + \frac{1}{c}\vec{K}\times\vec{B}\right) = \vec{r}_c \times \lambda \vec{E}$$

since $\vec{K}\times\vec{B}$ in $-\vec{e}_\rho$ direct.

$$\Rightarrow \frac{d}{dt}(I\omega) = (\vec{r}_c \times \lambda\vec{E})_z$$

moment of inertial / unit length

From Faraday's Law, we know that the changing flux in the cylinder will produce an electric field

$$E_\varphi = -\frac{r_c}{2c}\frac{dB_z}{dt} \Rightarrow E_\varphi = -\frac{r_c}{2c}\frac{dB_{ext}}{dt} - \frac{r_c \lambda}{c^2}\frac{d\omega}{dt}$$

$$\Rightarrow I\frac{d\omega}{dt} = -\frac{r_c^2\lambda}{2c}\frac{dB_{ext}}{dt} - \frac{r_c^2\lambda^2}{c^2}\frac{d\omega}{dt}$$

$$\frac{d\omega}{dt} = \frac{-r_c^2\lambda}{2c\left(I + \frac{r_c^2\lambda^2}{c^2}\right)}\frac{dB_{ext}}{dt} = \frac{-r_c^2\lambda c}{2c^2 I + 2r_c^2\lambda^2}\frac{dB_{ext}}{dt}$$

(next page)

∴ Conservation of angular momentum at the surface of the cylinder $r_c = a$

$$\boxed{\frac{d\omega}{dt} = \frac{-a^2\lambda c}{2c^2 I + 2a^2\lambda^2}\frac{dB_{ext}}{dt}}$$

(d) $$\omega = \frac{-a^2\lambda c}{2c^2 I + 2a^2\lambda^2}\int_0^T \frac{dB_{ext}}{dt}$$

$$\omega = \frac{a^2\lambda c}{2c^2 I + 2a^2\lambda^2}\left[B_0 - \cancelto{0}{B(T)}\right]$$

$$\Rightarrow \boxed{\vec{\omega}_{final} = \frac{a^2\lambda c}{2c^2 I + 2a^2\lambda^2} B_0 \vec{e}_z}$$

The final angular momentum in the system/unit length is

$$\vec{L}_{total} = \vec{L}_{em} + \vec{L}_{mech}$$

$$\vec{L}_{mech} = I\vec{\omega}_{final} = \frac{I a^2 \lambda c B_0}{2c^2 I + 2a^2\lambda^2}\vec{e}_z$$

$$\vec{L}_{em} = \frac{1}{4\pi c}\int_0^a \vec{r}_c \times (\vec{E}\times\vec{B})_{final} 2\pi r_c dr_c = \frac{1}{2c}\int_0^a \vec{r}_c \times\left[\frac{2\lambda}{r_c}\vec{e}_{r_c} \times \frac{2\lambda}{c}\omega_{final}\vec{e}_z\right] r_c dr_c$$

$$= \frac{2}{c}\int_0^a \frac{\lambda^2}{c}\vec{e}_z\left[\frac{a^2\lambda c B_0}{2c^2 I + 2a^2\lambda^2}\right] r_c dr_c$$

$$\vec{L}_{em} = \frac{a^4\lambda^3 B_0 \vec{e}_z}{c(2c^2 I + 2a^2\lambda^2)}$$

$$\Rightarrow \vec{L}_{\substack{total \\ final}} = \frac{I a^2 B_0 \lambda c^2 + a^4\lambda^3 B_0}{c(2c^2 I + 2a^2\lambda^2)}\vec{e}_z = \frac{\lambda B_0 a^2 (\cancel{c^2 I + \lambda^2 a^2})}{2c(\cancel{c^2 I + \lambda^2 a^2})}\vec{e}_z$$

$$\boxed{\vec{L}_{\substack{total \\ final}} = \frac{\lambda B_0 a^2}{2c} = \vec{L}_{\substack{total \\ initial}} = \vec{L}_{\substack{em \\ initial}}}$$