Physics 511

Problem Set #6: Solutions

Problem 1: Spherical waves: (Wave fronts on Spheres)

Wave equation for a scalar field $\psi(\vec{r},t)$

$$\nabla^2 \psi - \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} = 0$$

(a) Monochromatic field whose amplitude depends only on the radial distance $r = |\vec{r}|$

$$\psi(\vec{r},t) = \tilde{\psi}(r)\, e^{-i\omega t}$$

$$\frac{\partial^2 \psi}{\partial t^2} = -\omega^2 \tilde{\psi}(r)\, e^{-i\omega t}$$

$$\nabla^2 \psi = e^{-i\omega t}\left(\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 \tilde{\psi}(r)) + \underbrace{\cancel{\partial_\theta^2(\cdots)}}_{0} + \underbrace{\cancel{\partial_\phi^2(\cdots)}}_{0}\right)$$

No θ or ϕ dependance

$$\Rightarrow \frac{e^{-i\omega t}}{r^2}\frac{\partial^2}{\partial r^2}\left(r^2 \tilde{\psi}(r)\right) + \frac{\omega^2}{v^2}\tilde{\psi}(r)\, e^{-i\omega t} = 0$$

$$\rightarrow \boxed{\frac{1}{r^2}\frac{\partial^2}{\partial r^2}\left(r^2 \tilde{\psi}(r)\right) + k^2 \tilde{\psi}(r) = 0 \quad \text{where } k = \frac{\omega}{v}}$$

Let $\tilde{\psi}(r) = \frac{u(r)}{r}$ (u is known as the "reduced radial" wave function)

$$\frac{\partial \tilde{\psi}}{\partial r} = \frac{u'}{r} - \frac{u}{r^2} \quad \text{where} \quad u' = \frac{du}{dr}$$

$$\Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \tilde{\psi}}{\partial r}\right) = \frac{1}{r^2}\frac{d}{dr}\left[r u' - u\right]$$

$$= \frac{1}{r^2}\left[r^2 u'' + u' - u'\right] = \frac{u''}{r} = \frac{1}{r}\frac{d^2 u}{dr^2}$$

$$= -k^2 \tilde{\psi} = -k^2 \frac{u(r)}{r}$$

$$\Rightarrow \boxed{\frac{d^2 u}{dr^2} + k^2 u = 0}$$

This is nothing other than the simple harmonic oscillator equation.

General solution $\quad u(r) = u_0 \, e^{\pm ikr} e^{i\phi}$

u_0 and ϕ arbitrary

$$\Rightarrow \psi(r,t) = \mathrm{Re}\left(\frac{u_0}{r} e^{\pm i(kr \pm \omega t + \phi)}\right)$$

$$\Rightarrow \boxed{\psi = u_0 \frac{\cos(kr \pm \omega t + \phi)}{r}}$$

These are spherical waves (ψ constant on a sphere)

$\psi_1 \Rightarrow$ outward propagation, $\psi_2 \Rightarrow$ inward propaga

(b) Vector spherical wave:

Ansatz: $\vec{E} = E_0 \dfrac{\cos(kr-\omega t)}{kr}\,\hat{\phi} = E_\phi(r)\,\hat{\phi}$

Transverse? $\vec{\nabla}\cdot\vec{E} = \dfrac{1}{r\sin\theta}\dfrac{\partial E_\phi}{\partial\phi} = 0$ ✓

$\Rightarrow$ $\vec{E}$ must satisfy the wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = 0$$

$$\Rightarrow \quad \nabla^2\vec{E} = -\frac{\omega^2}{c^2}\vec{E} \quad ?$$

$$\nabla^2\vec{E} = \hat{x}(\nabla^2 E_x) + \hat{y}(\nabla^2 E_y)$$ (since $\hat{x}$, $\hat{y}$ independent of position

$$\hat{\phi} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

$$E_x = \cos\phi\, E_\phi(r) \qquad E_y = \sin\phi\, E_\phi(r)$$

$$\Rightarrow \nabla^2 E_x = \frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r E_x(r,\phi)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 E_x(r,\phi)}{\partial\phi^2}$$

$$= -k^2 E_x(r,\phi) - \frac{1}{r^2\sin^2\theta}E_x(r,\phi)$$

$$\Rightarrow \nabla^2\vec{E} = \left(-k^2 - \frac{1}{r^2\sin^2\theta}\right)\vec{E} \neq -\frac{\omega^2}{c^2}\vec{E}$$

$\Rightarrow$ $\vec{E}$ is not a solution to Maxwell's Eqns in free space

(c) Given $\vec{E} = E_0\left(\frac{\sin\theta}{kr}\right)\left(\cos(kr-\omega t) - \frac{1}{kr}\sin(kr-\omega t)\right)\hat{\phi}$

$$= \mathrm{Re}\left(\vec{E}(r,\theta)\, e^{-i\omega t}\right)$$

where $\vec{E}(r,\theta) = E_0 \sin\theta\left(\frac{1}{kr} + \frac{i}{(kr)^2}\right) e^{ikr}\hat{\phi}$ is the complex amplitude

Given the harmonic time dependence $e^{-i\omega t}$, Maxwell's Equations for the complex amplitudes are (in vacuum

$$\vec{\nabla}\cdot\vec{E} = 0, \quad \vec{\nabla}\cdot\vec{B} = 0, \quad \vec{\nabla}\times\vec{E} = ik\vec{B}, \quad \vec{\nabla}\times\vec{B} = -ik\vec{E}$$

where $k = \frac{\omega}{c}$

We can find the magnetic field associated with this wave through Faraday's Law

$$\vec{B} = \frac{-i}{k}\vec{\nabla}\times\vec{E} = \frac{-i}{k}\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, E_\phi)\hat{\theta} - \frac{1}{r}\frac{\partial}{\partial r}(rE_\phi)\hat{\theta}\right.$$

Aside: $\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, E_\phi) = \frac{\partial E_\phi}{\partial\theta} + \frac{\cos\theta}{\sin\theta}E_\phi = 2E_0\cos\theta\left(\frac{1}{kr} + \frac{i}{(kr)^2}\right)$

$$\frac{\partial}{\partial r}(rE_\phi) = \frac{\partial}{\partial r}\left\{\sin\theta\left(\frac{1}{k} + \frac{i}{k^2 r}\right)e^{ikr}\right\} = \left[i - \frac{1}{kr} - \frac{i}{k^2r^2}\right]\sin\theta\, e$$

$$\Rightarrow \vec{B} = \left\{2\cos\theta\left(\frac{1}{(kr)^2} - \frac{i}{kr}\right)\hat{r} + \sin\theta\left(-1 - \frac{i}{kr} + \frac{1}{(kr)^2}\right)\hat{\theta}\right\} \frac{E_0 e^{ik}}{kr}$$

(Next Page)

Thus, the real magnetic field is

$$\boxed{\vec{B} = (2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})\left(\frac{E_0\cos(kr-\omega t)}{(kr)^3} + \frac{E_0\sin(kr-\omega t)}{(kr)^2}\right) - E_0\sin\theta\,\frac{\cos(kr-\omega t)}{kr}\,\hat{\theta}}$$

What about the other Maxwell eqns!

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi} = 0 \quad (\text{no } \phi \text{ dependence})$$

$$\vec{\nabla}\cdot\vec{B} = -\frac{i}{k}\vec{\nabla}\cdot(\vec{\nabla}\times\vec{E}) = 0 \qquad [\vec{\nabla}\cdot(\vec{\nabla}\times\,) = 0]$$

$$\vec{\nabla}\times\vec{B} = \frac{1}{r}\left[\frac{\partial}{\partial r}(r B_\theta) - \frac{\partial B_r}{\partial\theta}\right]\hat{\phi}$$

Aside:

$$\frac{\partial}{\partial r}(rB_\theta) = B_\theta + r\frac{\partial B_\theta}{\partial r} =$$

$$= E_0\sin\theta\left[\left(-\frac{1}{kr} - \frac{i}{(kr)^2} + \frac{1}{(kr)^3}\right)e^{ikr} + \left(-i + \frac{1}{kr} + \frac{i}{(kr)^2}\right)e \right.$$

$$\left. + \left(\frac{1}{kr} + \frac{2i}{k^2r^2} - \frac{3}{(kr)^3}\right)e^{ikr}\right]$$

$$\Rightarrow \frac{\partial}{\partial r}(rB_\theta) = \left(-i + \frac{1}{kr} + \frac{2i}{(kr)^2} - \frac{2}{(kr)^3}\right)E_0\sin\theta\, e^{ikr}$$

$$\frac{\partial B_r}{\partial\theta} = \left(\frac{-2}{(kr)^3} + \frac{2i}{(kr)^2}\right)E_0\sin\theta\, e^{ikr}$$

$$\therefore \vec{\nabla}\times\vec{B} = E_0\sin\theta\left(-\frac{i}{r} + \frac{1}{kr^2}\right)e^{ikr}\phi = -ik\vec{E}$$

(d) In the lim $kr \ll 1$, $\frac{1}{r^3}$ dominates

$$\vec{B} \to \left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right) \frac{E_0 \cos(\omega t)}{(kr)^3}$$

We recognize this as a dipole field, for a dipole along the z-axis

$$3(\hat{z}\cdot\hat{r})\hat{r} - \hat{z} = 3\cos\theta\,\hat{r} - \left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right)$$
$$= 2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}$$

$$\Rightarrow \boxed{\vec{B} \to \left(3(\hat{z}\cdot\hat{r})\hat{r} - \hat{z}\right)\frac{E_0 \cos\omega t}{(kr)^3} = \frac{3(\vec{m}(t)\cdot\hat{r})\hat{r} - \vec{m}(t)}{r^3}}$$

where $\vec{m}(t) = \frac{E_0}{k^3}\cos\omega t\,\hat{z}$ is the effective magnetic dipole

(e) Time averaged Poynting vector $\langle\vec{S}\rangle = \frac{c}{8\pi}\mathrm{Re}(\vec{E}^*\times\vec{B})$

And $\vec{E}^*\times\vec{B} = E_\phi^*\,\hat{\phi}\times(B_r\hat{r} + B_\theta\hat{\theta}) = E_\phi^* B_r\,\hat{\theta} - E_\phi^* B_\theta\,\hat{r}$

$$E_\phi^* B_r = 2E_0^2\frac{\sin\theta\cos\theta}{kr}\left(\frac{1}{kr} - \frac{i}{(kr)^2}\right)\left(\frac{1}{(kr)^2} - \frac{i}{kr}\right)$$

$$= -2E_0^2\frac{\sin\theta\cos\theta}{(kr)^3}\, i\left(1 + \frac{1}{(kr)^2}\right)$$ Pure imaginary!

$$E_\phi^* B_\theta = \frac{E_0^2\sin^2\theta}{kr}\left(\frac{1}{kr} - \frac{i}{(kr)^2}\right)\left(-1 - \frac{i}{kr} + \frac{1}{(kr)^2}\right)$$

$$= \frac{E_0^2\sin^2\theta}{kr}\left(-\frac{1}{kr} - \frac{2i}{(kr)^2} - \frac{i}{(kr)^4}\right)$$ Next Page

Thus:

$$\langle \vec{S} \rangle = \frac{c}{8\pi} \mathrm{Re}\left(\vec{E}^* \times \vec{B}\right) = -\frac{c}{8\pi} \mathrm{Re}\left(E_\phi^* B_\theta\right) \hat{r}$$

$$\Rightarrow \boxed{\langle \vec{S} \rangle = \frac{c E_0^2 \sin^2\theta}{8\pi (kr)^2} \hat{r}}$$

The Poyting vector points in the $\hat{r}$-direction and falls off a $\frac{1}{r^2}$. This is expected for a spherical wave

(f) Energy flux

$$\frac{dW}{dt} = \oint \vec{S} \cdot d\vec{a} = \oint S_r r^2 d\Omega$$

where $d\Omega = \sin\theta \, d\theta \, d\phi = -d(\cos\theta)\, d\phi$ is the element of solid angle

Since the field is azimuthally symmetric $d\Omega = -2\pi\, d(\cos\theta)$

$$\Rightarrow \frac{dW}{dt} = \int_{-1}^{1} S_r \, 2\pi r^2 d(\cos\theta) = \frac{c}{4} \frac{E_0^2}{k^2} \int_{-1}^{1} (1 - \cos^2\theta)\, d(\cos\theta)$$

$$= \frac{c}{4} \frac{E_0^2}{k^2} \left(\cos\theta - \frac{\cos^3\theta}{3}\right)\Bigg|_{-1}^{1}$$

$$\Rightarrow \boxed{\frac{dW}{dt} = \frac{c E_0^2}{3k^3}}$$

As excepted the energy flux is independent of the radius of the sphere

(2) Standing E&M wave

$\vec{k}_1 = k\hat{z}$ $\qquad\qquad$ $\vec{k}_2 = -k\hat{z}$

$\vec{E}_1 = \hat{x}\, E_0 \cos(kz - \omega t)$ $\qquad\qquad$ $\vec{E}_2 = \hat{x}\, E_0 \cos(kz + \omega t)$

(a) $\vec{E}_3 = \vec{E}_1 + \vec{E}_2 = \hat{x}\, E_0 \left(\cos(kz - \omega t) + \cos(kz + \omega t)\right)$

$$= \hat{x}\, E_0 \left(\cos(kz)\cos(\omega t) + \sin(kz)\sin(\omega t) + \cos(kz)\cos(\omega t) - \sin(kz)\sin(\omega t)\right)$$

$$\Rightarrow \boxed{\vec{E}_3 = \hat{x}\ 2E_0 \cos(kz)\cos(\omega t)}$$

Alternative solution using complex representation

$\vec{E}_3 = Re\left(\tilde{\vec{E}}_3\, e^{-i\omega t}\right)$, $\tilde{\vec{E}}_3 = \tilde{\vec{E}}_1 + \tilde{\vec{E}}_2$

$\vec{E}_1 = Re\left(\tilde{\vec{E}}_1 e^{-i\omega t}\right) = \hat{x}\, E_0 \cos(kz - \omega t) \Rightarrow \tilde{\vec{E}}_1 = \hat{x}\, E_0\, e^{ikz}$

$\vec{E}_2 = Re\left(\tilde{\vec{E}}_2\, e^{-i\omega t}\right) = \hat{x}\, E_0 \cos(-kz - \omega t) \Rightarrow \tilde{\vec{E}}_2 = \hat{x}\, E_0\, e^{-ikz}$

$\Rightarrow \tilde{\vec{E}}_3 = \hat{x}\, E_0 \left(e^{ikz} + e^{-ikz}\right) = \hat{x}\ 2E_0 \sin kz$

$\Rightarrow \vec{E}_3 = Re\left(\tilde{\vec{E}}_3\, e^{-i\omega t}\right) = \hat{x}\, 2E_0 \sin kz\ Re\left(e^{-i\omega t}\right) =$

$= \hat{x}\ 2E_0 \sin kz \cos \omega t$ ✓

(b) Since the <u>total field</u> is not a travelling wave, we cannot use $\vec{B} = \frac{\vec{k}}{\omega} \times \vec{E}$

However $\vec{B}_3$ is monochromatic

$$\Rightarrow \vec{B}_3 = Re\left(\tilde{\vec{B}}_3\, e^{-i\omega t}\right)$$

Plug this into $\vec{\nabla} \times \vec{E} = \frac{-1}{c}\frac{\partial \vec{B}}{\partial t}$

$$\Rightarrow \frac{i\omega}{c} \underset{\sim}{\vec{B}}_3(z) = \vec{\nabla} \times \underset{\sim}{\vec{E}}_3(z) = \hat{y} \frac{\partial}{\partial z}(2E_0 \cos kz)$$

$$= 2kE_0 \sin kz \; \hat{y}$$

$$\therefore \underset{\sim}{\vec{B}}_3(z) = +2i\frac{kc}{\omega} E_0 \sin kz \; \hat{y} = +2i \; E_0 \sin kz \; \hat{y}$$

$$\therefore \vec{B}_3(z,t) = Re\left(\underset{\sim}{\vec{B}}_3(z) e^{-i\omega t}\right) = 2E_0 \sin kz \; Re\left(i e^{-i\omega t}\right) \hat{y}$$

$$\Rightarrow \vec{B}_3(z,t) = 2E_0 \sin kz \sin \omega t \; \hat{y}$$

Alternative solution

Find $\vec{B}_1$ and $\vec{B}_2$ (the mag-field of the two plane waves)

$\vec{B}_1 = \hat{y} \; E_0 \cos(kz - \omega t)$ $\qquad$ $\vec{B}_2 = -\hat{y} \; E_0 \cos(kz + \omega t)$

since $\hat{k}_1 = \hat{z}$ $\qquad$ since $\hat{k}_2 = -\hat{z}$

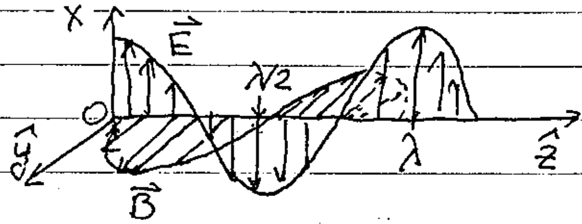
$$\vec{B}_3(z,t) = \vec{B}_1 + \vec{B}_2 = \hat{y} \; E_0 \; Re\left(e^{i(kz-\omega t)} + e^{-i(kz+\omega t)}\right)$$

$$= \hat{y} \; E_0 \; Re\left[\left(e^{ikz} - e^{-ikz}\right) e^{-i\omega t}\right] = \hat{y} \; E_0 \; Re\left[2i \sin kz \; e^{-i\omega t}\right]$$

$$= \hat{y} \; E_0 \; 2 \sin kz \sin \omega t$$

Thus, for a standing wave $\vec{E}$ and $\vec{B}$ are 90° out of phase, both in time and space

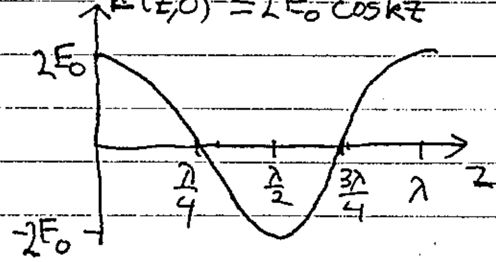
Problem 3 continued



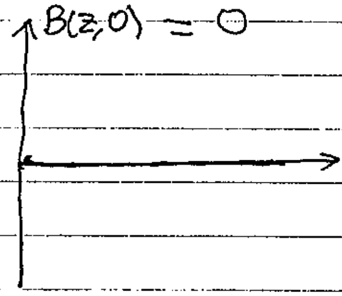
$\vec{E}$ and $\vec{B}$ at $t = T/8$

$t=0$ and $t=T$

$E(z,0) = 2E_0 \cos kz$

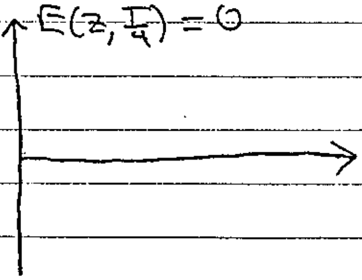


$B(z,0) = 0$

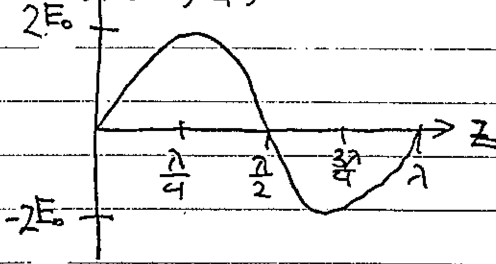


$t = \frac{T}{4}$

$E(z, \frac{T}{4}) = 0$

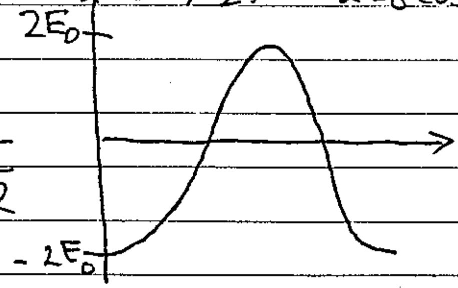


$B(z, \frac{T}{4}) = 2 \sin kz \ E_0$

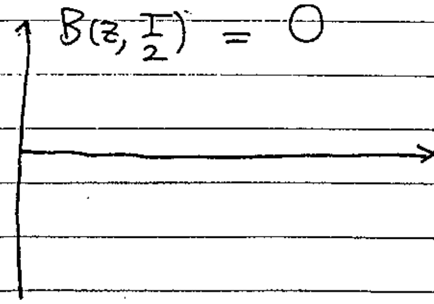


$t = \frac{T}{2}$

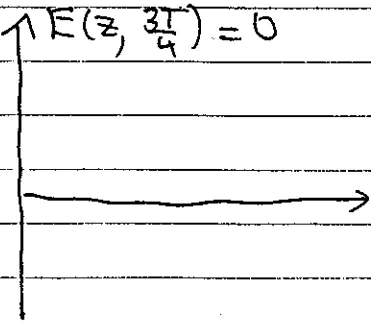
$E(z, \frac{T}{2}) = -2E_0 \cos kz$



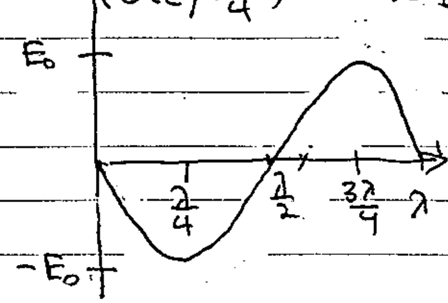
$B(z, \frac{T}{2}) = 0$



$E(z, \frac{3T}{4}) = 0$



$B(z, \frac{3T}{4}) = -2E_0 \sin kz$



Anti-nodes of E at $\frac{m\lambda}{2}$

nodes of E at $\frac{\lambda}{2}(m+\frac{1}{2})$

$m = 0, \pm 1, \ldots$

Anti-nodes of B at $(m+\frac{1}{2})$

nodes of B at $m\frac{\lambda}{2}$

(b) Time averaged electric and magnetic field energy densities

Using real fields:

$$\langle u_E(\vec{x})\rangle = \left\langle \frac{\vec{E}^2}{8\pi}\right\rangle = \frac{1}{8\pi}\, 4\vec{E}_0^{\,2}\cos^2 kz\, \overbrace{\langle \cos^2\omega t\rangle}^{1/2} = \frac{E_0^2}{4\pi}\cos^2 kz$$

$$\langle u_B(\vec{x})\rangle = \left\langle \frac{\vec{B}^2}{8\pi}\right\rangle = \frac{1}{8\pi}\, 4\vec{B}_0^{\,2}\sin^2 kz\, \langle \sin^2\omega t\rangle = \frac{E_0^2}{4\pi}\sin^2 kz$$

Using complex amplitudes $\tilde{\vec{E}} = (2E_0\cos kz)\hat{x}$, $\tilde{\vec{B}} = (i2E_0\sin kz$

$$\langle u_E(\vec{x})\rangle = \frac{1}{16\pi}\,\tilde{\vec{E}}^*\cdot\tilde{\vec{E}} = \frac{1}{4\pi}E_0^2\cos^2 kz$$

$$\langle u_B(\vec{x})\rangle = \frac{1}{16\pi}\,\tilde{\vec{B}}^*\cdot\tilde{\vec{B}} = \frac{1}{4\pi}B_0^2\sin^2 kz$$

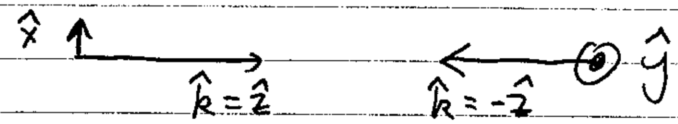
Intensity $\langle\vec{S}\rangle = \frac{c}{4\pi}\langle\vec{E}\times\vec{B}\rangle = \frac{c}{4\pi}4E_0^2\sin kz\cos kz\,\overbrace{\langle\cos\omega t\sin\omega t}^{0}$

(Complex amplitude $\langle\vec{S}\rangle = \mathrm{Re}\left(\frac{c}{8\pi}\tilde{\vec{E}}\times\tilde{\vec{B}}^*\right) = \frac{c}{4\pi}4E_0^2\,\mathrm{Re}(i\sin kz\cos k$

$$\Rightarrow \boxed{\langle\vec{S}\rangle = 0}$$

$\Longrightarrow$ In a standing wave there is no flux of energy as in a traveling wave. Instead, the energy is oscillating back and forth between electric and magnetic fields as in as LC oscillator.

(c) The "lin⊥lin" field



$\vec{E}_1 = \hat{x}\, E_0 \cos(kz - \omega t)$

$\vec{E}_2 = \hat{y}\, E_0 \sin(kz - \omega t)$

Cross polarized counter-propagating plane waves

Complex amplitude representation

- $\vec{E}_1(z,t) = Re\left(\underset{\sim}{\vec{E}}_1(z)\, e^{-i\omega t}\right) = Re\left(\hat{x}\, E_0\, e^{ikz}\, e^{-i\omega t}\right)$

$$\Rightarrow \quad \underset{\sim}{\vec{E}}_1(z) = \hat{x}\, E_0\, e^{ikz}$$

- $\vec{E}_2(z,t) = Re\left(\underset{\sim}{\vec{E}}_2(z)\, e^{-i\omega t}\right) = Re\left(\hat{z}\, i E_0\, e^{-ikz}\, e^{-i\omega t}\right)$

$$\Rightarrow \quad \underset{\sim}{\vec{E}}_2(z) = i\, \hat{z}\, E_0\, e^{-ikz}$$

$\Rightarrow$ Total field $\vec{E}_3(z,t) = Re\left(\underset{\sim}{\vec{E}}_3(z)\, e^{-i\omega t}\right)$

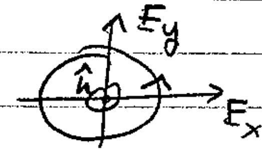
$$\Rightarrow \quad \underset{\sim}{\vec{E}}_3 = \left(e^{ikz}\,\hat{x} + i\, e^{-ikz}\,\hat{y}\right) E_0$$

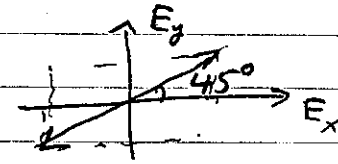
$$= \left(\frac{\hat{x} + i\, e^{-2ikz}\,\hat{y}}{\sqrt{2}}\right) \sqrt{2}\, E_0\, e^{ikz}$$

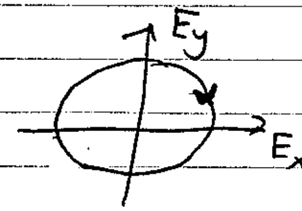
$$\Rightarrow \quad \boxed{\begin{aligned} \underset{\sim}{\vec{E}}_3 &= \vec{\epsilon}(z)\, \sqrt{2}\, E_0\, e^{ik_0 z} \\ \vec{\epsilon}(z) &= \frac{\hat{x} + i\, e^{-2ikz}}{\sqrt{2}}\,\hat{y} \end{aligned}}$$

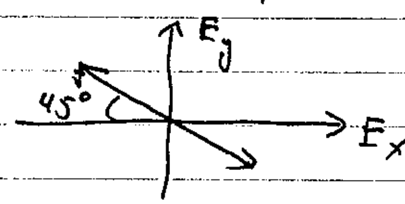
I have written this in such a way that we see the normalized position dependent complex polarization vector $\vec{\epsilon}(z)$

As a function of position

- $\vec{E}(0) = \frac{\hat{x}+i\hat{y}}{\sqrt{2}} = \hat{e}_+$: Positive helicity 

- $\vec{E}\left(\frac{\lambda}{8}\right) = \frac{\hat{x}+i\,e^{-i\pi/2}\,\hat{y}}{\sqrt{2}} = \frac{\hat{x}+\hat{y}}{\sqrt{2}}$: Linear 

- $\vec{E}\left(\frac{\lambda}{4}\right) = \frac{\hat{x}+i\,e^{-i\pi}\,\hat{y}}{\sqrt{2}} = \frac{\hat{x}-i\hat{y}}{\sqrt{2}}$: Negative Helicity 

- $\vec{E}\left(\frac{3\lambda}{8}\right) = \frac{\hat{x}+i\,e^{-i\frac{3\pi}{2}}\,\hat{y}}{\sqrt{2}} = \frac{\hat{x}-\hat{y}}{\sqrt{2}}$: Linear 

- $\vec{E}\left(\frac{\lambda}{2}\right) = \frac{\hat{x}+i\,e^{-i2\pi}\,\hat{y}}{\sqrt{2}} = \frac{\hat{x}+i\hat{y}}{2}$. Positive Helicity

⋮

etc.

(d) Let us reexpress $\hat{x}$ and $\hat{y}$ in terms of $\hat{e}_\pm = \frac{\hat{x}\pm i\hat{y}}{\sqrt{2}}$

$$\Rightarrow \hat{x} = \frac{\hat{e}_+ + \hat{e}_-}{\sqrt{2}}, \quad \hat{y} = \frac{\hat{e}_+ - \hat{e}_-}{i\sqrt{2}}$$

$$\Rightarrow \underset{\sim}{E}_3 = \left(e^{ikz}\left(\frac{\hat{e}_+ + \hat{e}_-}{\sqrt{2}}\right) + i e^{-ikz}\left(\frac{\hat{e}_+ - \hat{e}_-}{i\sqrt{2}}\right)\right) E_0$$

$$= \left[\hat{e}_+\left(\frac{e^{ikz}+e^{-ikz}}{\sqrt{2}}\right) + \hat{e}_-\left(\frac{e^{ikz}-e^{-ikz}}{\sqrt{2}}\right)\right] E_0$$

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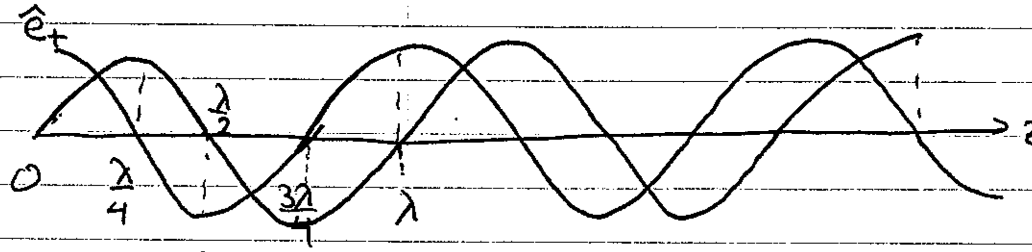
Thus $\underset{\sim}{\vec{E}}_3 = \sqrt{2}\,E_0\left(\hat{e}_+ \cos kz + i\,\hat{e}_- \operatorname{sen} kz\right)$

$\vec{E}_3(z,t) = Re\left(\underset{\sim}{\vec{E}}_3\, e^{-i\omega t}\right) =$

$$\boxed{\vec{E}_3(z,t) = \hat{e}_+ \sqrt{2}E_0 \cos kz \cos\omega t + \hat{e}_- \sqrt{2}E_0 \sin kz \sin\omega t}$$

Superposition of standing waves of $\hat{e}_\pm$



Node of one standing wave corresponds to anti-nodes of other
(places where $\hat{e}_+$ wave $=$ $\hat{e}_-$ wave $\Rightarrow$ linear polarization)

(c) Intensity as a function of position

$I = \langle \vec{S} \rangle \quad , \quad \vec{S} = \frac{c}{4\pi}(\vec{E}\times\vec{B})$

$\underset{\sim}{\vec{B}}_3 = \frac{c\vec{k}_1}{\omega}\times\underset{\sim}{\vec{E}}_1 + \frac{c\vec{k}_2}{\omega}\times\underset{\sim}{\vec{E}}_2 = \hat{z}\times\left(\underset{\sim}{\vec{E}}_1 - \underset{\sim}{\vec{E}}_2\right)$

$\Rightarrow \quad \underset{\sim}{\vec{B}}_3(z) = \hat{z}\times\left(e^{ikz}\hat{x} - i e^{-ikz}\hat{y}\right)E_0 = \left(e^{ikz}\hat{y} + i e^{-ikz}\hat{x}\right)E_0$

$\Rightarrow I = \frac{c}{8\pi} Re\left(\underset{\sim}{\vec{E}}_3 \times \underset{\sim}{\vec{B}}_3^{*}\right) = \frac{cE_0^2}{8\pi} Re\left[\left(\hat{x} + i e^{-2ikz}\hat{y}\right)\times\left(\hat{y} - i e^{2ikz}\hat{x}\right)\right]$

$= \frac{cE_0^2}{8\pi} Re\left[\hat{z} - \hat{z}\right] = \boxed{0}$ As expected

Note: Since $\vec{E}_1 \perp \vec{E}_2 \qquad \vec{S} = \vec{S}_1 + \vec{S}_2 = 0$: Orthogonal waves do not interfere

Problem 3: Paraxial Wave Equation:

Wave in vacuum: $\vec{E}$-field satisfies wave eqn $\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec{E} = 0$

(a) Monochromatic paraxial Ansatz:

$\vec{E}(\vec{x},t) = \hat{e}\ \mathcal{E}(\vec{x})\, e^{ik_0 z - \omega_0 t}$, $\mathcal{E}(\vec{x})$: Slowly varying envelope

Plug into W.E., with $\nabla^2 = \nabla_\perp^2 + \frac{\partial^2}{\partial z^2}$, $\frac{\partial^2}{\partial t^2} \Rightarrow -\omega_0^2$

$$\Rightarrow (\nabla_\perp^2 \mathcal{E})\, e^{ik_0 z} + \frac{\partial^2}{\partial z^2}(\mathcal{E} e^{ik_0 z}) + \frac{\omega_0^2}{c^2}\mathcal{E}(\vec{x})\, e^{ik_0 z} = 0$$

Aside:
$$\left\{\begin{aligned} \frac{\partial^2}{\partial z^2}(\mathcal{E}(\vec{x}) e^{ik_0 z}) &= \frac{\partial}{\partial z}\left[ik_0\, \mathcal{E}(\vec{x}) e^{ik_0 z} + \frac{\partial \mathcal{E}}{\partial z} e^{ik_0 z}\right] \\ &= \left(2ik_0 \frac{\partial \mathcal{E}}{\partial z} + \frac{\partial^2 \mathcal{E}}{\partial z^2} - k_0^2 \mathcal{E}\right) e^{ik_0 z} \end{aligned}\right.$$

$$\Rightarrow \nabla_\perp^2 \mathcal{E} + 2ik_0 \frac{\partial \mathcal{E}}{\partial z} + \frac{\partial^2 \mathcal{E}}{\partial z^2} = 0 \qquad \left(\text{using } k_0 = \frac{\omega_0}{c}\right)$$

So far exact

Now make the S.V.E.A. $\left|\frac{\partial^2 \mathcal{E}}{\partial z^2}\right| \ll k_0 \left|\frac{\partial \mathcal{E}}{\partial z}\right|$
$\ll |\nabla_\perp^2 \mathcal{E}|$

(Envelope varies much more slowly in z-direction than x and variations along z ≪ variations along x-y)

$$\Rightarrow \boxed{i\frac{\partial \mathcal{E}}{\partial z} = -\frac{1}{2k_0}\nabla_\perp^2 \mathcal{E}}$$ the paraxial wave equation

(b) Fourier transform of the complex field

$$\tilde{E}(\vec{k},\omega) = \int d^3x\, dt\, \underbrace{\mathcal{E}(\vec{x})\, e^{i(k_0 z - \omega_0 t)}}_{\tilde{E}(\vec{x},t)}\, e^{-i(\vec{k}\cdot\vec{x} - c|\vec{k}|t)}$$

$$= \underbrace{\int dt\, e^{-i(\omega_0 - c|\vec{k}|)t}}_{=2\pi\delta(\omega_0 - c|\vec{k}|)} \underbrace{\int \mathcal{E}(\vec{x})\, e^{-i(\vec{k} - k_0\vec{e}_z)\cdot\vec{x}}\, d^3\vec{x}}_{\tilde{\mathcal{E}}(\vec{k} - k_0\vec{e}_z)}$$

$$\therefore \boxed{\tilde{E}(\vec{k},\omega) = \tilde{\mathcal{E}}(\vec{k} - k_0\vec{e}_z)\, 2\pi\, \delta(\omega_0 - c|\vec{k}|)}$$

Because the field is monochromatic, all the wave vectors have the same magnitude $|\vec{k}| = \frac{\omega_0}{c}$. However, because of the envelope, there is a spread in the direction of the wave vectors whose distribution is given by $\tilde{\mathcal{E}}(\vec{k} - k_0\vec{e}_z) = \tilde{\mathcal{E}}(\vec{q})$ where $\vec{q} = \vec{k} - \vec{k}_0$

In the SVEA: $\left|\frac{\partial^2 \mathcal{E}}{\partial z^2}\right| << k_0 \left|\frac{\partial \mathcal{E}}{\partial z}\right| \qquad \left|\frac{\partial^2 \mathcal{E}}{\partial z^2}\right| << \left|\nabla_\perp^2 \mathcal{E}\right|$

In the Fourier domain this translates into

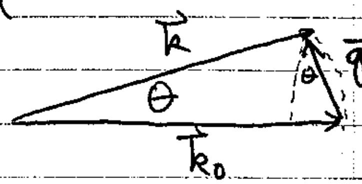
$$|q_z^2\, \tilde{\mathcal{E}}| << |k_0 q_z \tilde{\mathcal{E}}| \quad \text{and} \quad |q_z^2\, \tilde{\mathcal{E}}| << |\vec{q}_\perp|^2\, \tilde{\mathcal{E}}$$

$$\text{Or } |q_z| << k_0 \quad \text{and} \quad |q_z| << |\vec{q}_\perp|$$

(Next page)

This can be seen geometrically as a "paraxial approximation", i.e. all the wave vectors make a small angle θ with respect to the main propagation direction, $\vec{k}_0$

Exagerated



For small θ

- $q_z = k_0 - k\cos\theta \approx k_0(1-\cos\theta)$
  $\approx \frac{1}{2} k_0 \theta^2$
- $|\vec{q}_\perp| = k\sin\theta \approx k_0\theta$

Thus the paraxial approximation amounts to keeping terms to order θ^2 in the small angle

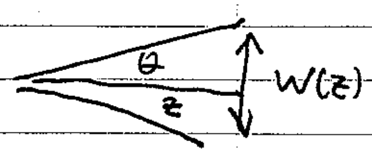
(c) If the beam has a characteristic width W_0, then by the uncertainty principle the spread in transverse wave vectors

$$|\Delta \vec{k}_T| = |\Delta \vec{q}_T| \sim \frac{1}{W_0}$$

But according to the geometrical construction above, the characteristic $|\vec{q}_\perp|$ is of order $k_0\theta = \theta/\lambda\!\!\!^-$

$$\Rightarrow \boxed{\theta \sim \frac{1}{k_0 W_0} = \frac{\lambda\!\!\!^-}{W_0}}$$ The "diffraction angle"

Because of the finite transverse components the beam will spread, or diffract



The width at z is approximately
$W(z) \simeq z\sin\theta \approx \theta z$

The characteristic spreading distance is when $W(z) = W_0$

$$\Rightarrow \theta z_c \cong W_0 \Rightarrow \boxed{z_c \sim \frac{W_0}{\theta} = k W_0^2}$$

(d) Given the envelope function at $z=0$, we can propagate forward to $z>0$ the same as we propagate the wave function forward in time

Note the analogy:

$$i\frac{\partial}{\partial t}\psi = \frac{-\hbar}{2m}\nabla_\perp^2\psi \ : \quad \text{Schrödinger equation for free particle in 2D}$$

$$i\frac{\partial}{\partial z}\mathcal{E} = -\frac{1}{2k_0}\nabla_\perp^2\mathcal{E} \ : \quad \text{Paraxial wave equation}$$

$$t \Leftrightarrow z \qquad \left(\text{If we let } z=ct \text{ then the "effective mass" is } m_{eff} = \frac{\hbar\omega}{c^2}\right)$$

Thus $\mathcal{E}(\vec{x}_\perp, z) = U(z)\,\mathcal{E}(\vec{x}_\perp, 0) = e^{+\frac{i}{2k_0}\nabla_\perp^2 z}\,\mathcal{E}(\vec{x}_\perp, 0)$

$$= \int\frac{d^2q_\perp}{(2\pi)^2}\, e^{-i\frac{|\vec{q}_\perp|^2}{2k_0}z}\,\tilde{\mathcal{E}}(\vec{q}_\perp)\, e^{i\vec{q}_T\cdot\vec{x}_T}$$

Where $\tilde{\mathcal{E}}(\vec{q}_\perp) = \int d^2x_\perp\, \tilde{\mathcal{E}}(\vec{x}_\perp, 0)\, e^{-i\vec{q}_\perp\cdot\vec{x}_\perp}$ (F.T. of incident envelope)

Given initial profile: $\mathcal{E}(\vec{x}_T, 0) = E_0\, e^{-\frac{x^2+y^2}{w_0^2}}$ Gaussian beam

$$\tilde{\mathcal{E}}(\vec{q}_\perp) = E_0 \underbrace{\int_{-\infty}^{\infty}dx\, e^{-\frac{x^2}{w_0^2}}\, e^{-iq_x x}}_{\sqrt{\pi w_0^2}\; e^{-\left(q_x^2\frac{w_0^2}{4}\right)}} \underbrace{\int_{-\infty}^{\infty}dy\, e^{-\frac{y^2}{w_0^2}}\, e^{-iq_y y}}_{\sqrt{\pi w_0^2}\; e^{-\left(q_y^2 w_0^2/4\right)}}$$

$$= E_0\pi w_0^2 \exp\left\{-|\vec{q}_T|^2\frac{w_0^2}{4}\right\}$$

(Gaussian in Fourier domain)

Now we add the propagation phase for each $\vec{q}_T$ and integrate:

$$\mathcal{E}(\vec{x}_T, z) = \int_{-\infty}^{\infty} \frac{d^2 q_T}{(2\pi)^2}\, \tilde{\mathcal{E}}(\vec{q}_T)\, \exp\left\{ i\vec{q}_T\cdot\vec{x}_T - i\frac{|\vec{q}_T|^2}{2k_0} z \right\}$$

$$= E_0\, \pi W_0^2 \left[\int_{-\infty}^{\infty} \frac{d^2 q_T}{(2\pi)^2} \exp\left\{ i\vec{q}_T\cdot\vec{x}_T - \frac{|\vec{q}_T|^2}{4} W_0^2 - i\frac{|\vec{q}_T|^2}{2k_0} z \right\} \right]$$

$$= E_0\, \pi W_0^2\, \underbrace{\left[\text{Integral over } q_x\right]}_{\mathcal{I}_x} \times \underbrace{\left[\text{Integral over } q_y\right]}_{\mathcal{I}_y}$$

<u>Aside</u>

$$\mathcal{I}_x = \int_{-\infty}^{\infty} \frac{dq_x}{2\pi} \exp\left\{ iq_x x - q_x^2\left(\frac{W_0^2}{4} + i\frac{z}{2k_0}\right) \right\}$$

Aside again: Complete the square

$$-q_x^2\left(\frac{W_0^2}{4} + i\frac{z}{2k_0}\right) + iq_x X = -\frac{a^2}{4}\left(q_x - 2i\frac{X}{a}\right) - \frac{X^2}{a^2}$$

where $a^2 = W_0^2 + i\frac{2z}{k_0}$ (complex)

$$\Rightarrow \mathcal{I}_x = \frac{e^{-\frac{X^2}{a^2}}}{2\pi} \underbrace{\int_{-\infty}^{\infty} dq_x \exp\left\{ -\frac{a^2}{4}\left(q_x - \frac{2iX}{a}\right) \right\}}$$

Magic of Gaussians: everything as if real $= \sqrt{\frac{4\pi}{a^2}}$

$$\Rightarrow \mathcal{I}_x = \frac{\sqrt{\pi}}{a} e^{-\frac{X^2}{a^2}}$$

Similarly $\mathcal{I}_y = \frac{\sqrt{\pi}}{a} e^{-\frac{y^2}{a^2}}$ (Next Page)

Thus, $$\boxed{\mathcal{E}(\vec{x}_T, z) \simeq E_0 \frac{W_0^2}{a^2} \exp\left\{-\frac{x^2-y^2}{a^2}\right\}}$$

The rest is getting it in the given form

- **First** $$a^2 = W_0^2\left(1 + i\frac{2z}{kW_0^2}\right) = W_0^2\left(1 + i\frac{z}{z_0}\right)$$

where $z_0 = \frac{1}{2}kW_0^2$ (Rayleigh range)

And $$1 + i\frac{z}{z_0} = \sqrt{1 + \frac{z^2}{z_0^2}}\, e^{i\phi(z)}, \quad \phi(z) = \tan^{-1}\left(\frac{z}{z_0}\right)$$

(Polar decomposition)

$$\Rightarrow \frac{W_0^2}{a^2} = \frac{W_0}{W(z)} e^{-i\phi(z)} \text{ where } W(z) = W_0\sqrt{1 + \frac{z^2}{z_0^2}}$$

- **Second** $$\frac{1}{a^2} = \frac{1}{W_0^2}\frac{1}{1 + i\frac{z}{z_0}} = \frac{1 - i z/z_0}{W_0^2\left(1 + \frac{z^2}{z_0^2}\right)}$$

$$= \frac{1}{W^2(z)} - \frac{i}{\frac{W_0^2}{z_0}\left(z + \frac{z_0^2}{z}\right)}$$

$$= \frac{1}{W^2(z)} - i\frac{k}{2R(z)}, \quad R(z) = z + \frac{z_0^2}{z}$$

Putting it all together:

$$\boxed{\mathcal{E}(\vec{x}_T, z) = E_0 e^{-i\phi(z)} \frac{W_0}{W(z)} e^{\frac{-x^2-y^2}{W^2(z)}} e^{ik_0\frac{(x^2+y^2)}{2R(z)}}}$$

$$z_0 = \frac{kW_0^2}{2}, \quad W(z) = W_0\left(1 + \frac{z^2}{z_0^2}\right)^{1/2}, \quad \phi(z) = \tan^{-1}\left(\frac{z}{z_0}\right),$$

$$R(z) = z + \frac{z_0^2}{z}$$

Whew!

Interpretation:

The solution we found is well known in laser physics. It's the Gaussian beam solution. Because of the finite width w_0 at $z=0$, there must be a finite spread of transverse wave vectors. The resulting beam spreads as it propagates with a width

$$W(z) = W_0\sqrt{1+\frac{z^2}{z_0^2}}$$

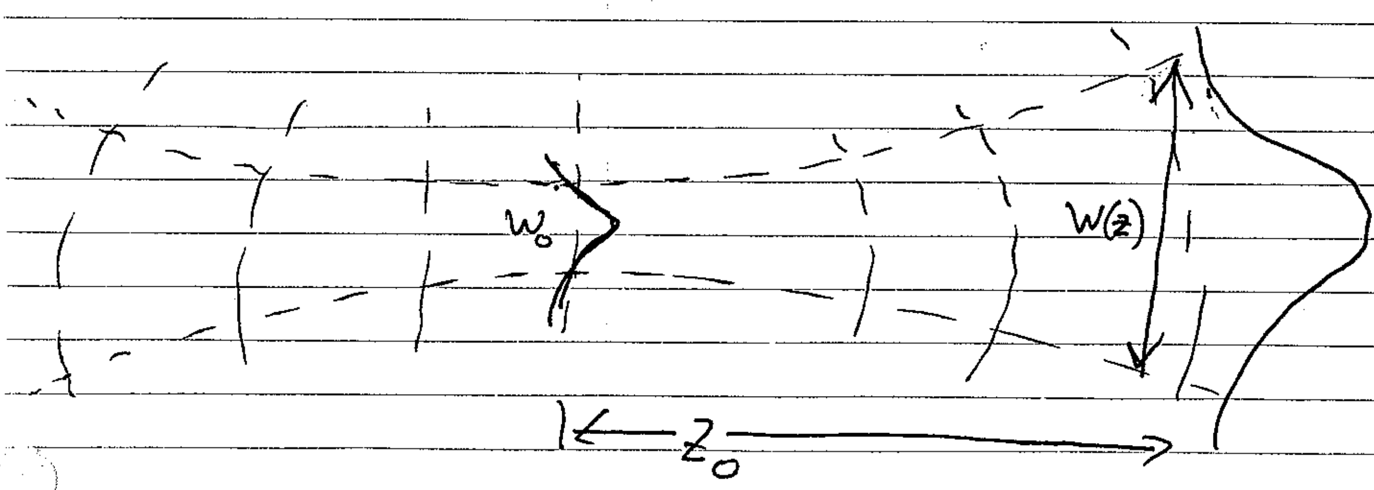
Note when $z = z_0 = \frac{k w_0^2}{2}$ $\quad W_0 \Rightarrow \sqrt{2}\,W_0$

i.e. noticeable spread, as we estimated in part (c)

In addition to the spreading of the intensity the initially uniform phase front $\perp$ to z becomes curved surfaces

$$\text{phase}(x,y,z) = k_0 z + \phi(z) + \frac{k_0 (x^2+y^2)}{2R(z)}$$

We can sketch this beam as



Note: for an optical field with $\lambda \sim 0.5\,\mu m$

and $w_0 = 1$ mm, $\quad z_0 = \left(\frac{1}{2}\right)\frac{2\pi}{\lambda} w_0^2 = \frac{\pi w_0^2}{\lambda} \cong 6$ meters

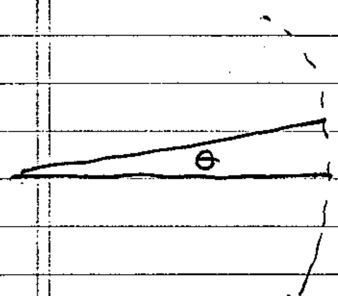
(e) For $z \gg z_0$

$R(z) \approx z \quad , \quad W(z) \approx W_0 \frac{z}{z_0} = \frac{2}{k_0 W_0}\, z = \frac{2}{k W_0} R(z)$

$$\Rightarrow \mathcal{E}(\vec{x}_T, z) \approx \mathcal{E}_0 \, e^{-\frac{x^2+y^2}{w^2(z)}} \, \frac{e^{i k_0 (x^2+y^2)/2z}}{kz}$$

Recall the spherical wave: $E = E_0 \frac{e^{ik_0 r}}{k_0 r} = \frac{e^{ik_0\sqrt{x^2+y^2+z^2}}}{k_0\sqrt{x^2+y^2+z^2}}$

Because the propagation ~~is~~ paraxial we need only consider the wave front for small angles, so $x^2+y^2 \ll z^2$



$$k_0 r = k_0\sqrt{x^2+y^2+z^2} = k_0 z\left(1 + \frac{x^2+y^2}{z^2}\right)^{1/2}$$

$$\approx k_0 z + \frac{k_0(x^2+y^2)}{2z} \qquad \left(z \gg \sqrt{x^2+y^2}\right)$$

$$\Rightarrow E \approx \underbrace{E_0 \frac{e^{ik_0\frac{(x^2+y^2)}{2z}}}{z}}_{\text{Envelope function}} e^{ik_0 z}$$

$\Rightarrow$ For $z \gg z_0$, the wave fronts are approximately spherical

(f) The fudge: $\vec{\nabla}\cdot\vec{E} = 0$? No since we assumed uniform polarization, yet $E(\vec{x}_T, z)$ dependence in z-direction

Resolution: The neglected part of $\vec{\mathcal{E}}$ which would make $\vec{\nabla}\cdot\vec{E} = 0$ in higher order in the paraxial approximation

See M. Lax et al., Phys Rev A 11, 1365 (1975).

Problem 4: Angular Momentum in Electromagnetic ~~Waves~~ Waves

$$\vec{L} = \frac{1}{4\pi c}\int d^3x \; \vec{x}\times(\vec{E}\times\vec{B}) \quad \text{(ang. mom. in field)}$$

(a) $$L_i = \epsilon_{ijk}\,\epsilon_{klm}\,\frac{1}{4\pi c}\int d^3x \; x_j E_l B_m$$

Use $\vec{B} = \vec{\nabla}\times\vec{A} \Rightarrow B_m = \epsilon_{mnp}\,\partial_n A_p$

$$L_i = \epsilon_{ijk}\,\underbrace{\epsilon_{klm}\,\epsilon_{mnp}}_{\delta_{kn}\delta_{lp}-\delta_{kp}\delta_{ln}}\,\frac{1}{4\pi c}\int d^3x \; x_j E_l \partial_n A_p$$

$$\Rightarrow L_i = \epsilon_{ijk}\,\frac{1}{4\pi c}\int d^3x\left[E_l(x_j\partial_k)A_l - x_j E_l \partial_l A_k\right]$$

Aside: $\epsilon_{ijk}\, x_j \partial_k = (\vec{x}\times\vec{\nabla})_i$

Aside: $$\int x_j E_l \partial_l A_k \underset{\text{by parts}}{=} -\int\left[\partial_l(x_j E_l)\right]A_k$$

$$= -\int \delta_{lj} E_l A_k - \int x_j(\partial_l E_l)\,A_k$$

$$= -\int E_j A_k - \int x_j \underbrace{(\vec{\nabla}\cdot\vec{E})}_{0 \text{ in free space}} A_k$$

$$\Rightarrow L_i = \frac{1}{4\pi c}\int d^3x\left[E_l\,(\vec{x}\times\vec{\nabla})_i\, A_l + (\vec{E}\times\vec{A})_i\right]$$

(Next Page)

Thus: $\vec{L} = \vec{L}_{orbital} + \vec{L}_{spin}$

$$\boxed{\begin{aligned} \vec{L}_{orbital} &\equiv \frac{1}{4\pi c}\int d^3x \left(E_\ell \, (\vec{x}\times\vec{\nabla})\, A_\ell \right) \quad \text{(sum over } \ell\text{)} \\ \vec{L}_{spin} &\equiv \frac{1}{4\pi c}\int d^3x \left(\vec{E}(\vec{x}) \times \vec{A}(\vec{x}) \right) \end{aligned}}$$

- The "orbital" angular momentum depends on the spatial variation of the field via $\vec{x}\times\vec{\nabla}$
- The "spin" angular momentum depends on the vector nature of the field ("intrinsic" angular momentum)

(b) Given plane wave decomposition

$$\vec{A}(\vec{x},t) = \sum_{\mu}\int \frac{d^3k}{(2\pi)^3} \left[\tilde{A}_\mu(\vec{k})\, \vec{e}_\mu(\hat{k})\, e^{i(\vec{k}\cdot\vec{x}-\omega_k t)} + c.c. \right]$$

$$\text{where } \vec{e}_\pm(\hat{k}) = \frac{\vec{e}_1 \pm i\,\vec{e}_2}{\sqrt{2}} \qquad \vec{e}_1, \vec{e}_2 \perp \text{ to } \hat{k}$$

$$\vec{E}(\vec{x},t) = \frac{-1}{c}\frac{\partial \vec{A}}{\partial t}$$ (In radiation gauge. The Longitudinal part of $\vec{E}$ does not contribute to $\vec{L}$)

$$\Rightarrow \vec{E}(\vec{x},t) = i\sum_{\mu}\int \frac{d^3k}{(2\pi)^3}\, k \left[\tilde{A}_\mu(\vec{k})\, \vec{e}_\mu(\hat{k})\, e^{i(\vec{k}\cdot\vec{x}-\omega_k t)} \right] + c.c.$$

$$\vec{A}(\vec{x},t) = \vec{A}^{(+)}(\vec{x},t) + \vec{A}^{(-)}(\vec{x},t)$$

$$\vec{E}(\vec{x},t) = \vec{E}^{(+)}(\vec{x},t) + \vec{E}^{(-)}(\vec{x},t)$$

$$\text{where}\quad \vec{A}^{(+)} = \sum_{\mu}\int\frac{d^3k}{(2\pi)^3}\,\tilde{A}_{\mu}(\vec{k})\,\vec{e}_{\mu}(\hat{k})\,e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

$$= \vec{A}^{(-)\,*}$$

$$\vec{E}^{(+)} = i\sum_{\mu}\int\frac{d^3k}{(2\pi)}\,k\,\tilde{A}_{\mu}(\vec{k})\,\vec{e}_{\mu}(\hat{k})\,e^{i(\vec{k}\cdot\vec{x}-\omega t)} = \vec{E}^{(-)\,*}$$

$$\Rightarrow \vec{L}_{spin} = \int d^3x\left(\frac{\vec{E}^{(-)}\times\vec{A}^{(+)}}{4\pi c} + \frac{\vec{E}^{(+)}\times\vec{A}^{(+)}}{4\pi c}\right) + c.c.$$

Aside:

$$\int d^3x\,\frac{\vec{E}^{(-)}\times\vec{A}^{(+)}}{4\pi c} = \frac{-i}{4\pi c}\int\frac{d^3k\,d^3k'}{(2\pi)^6}\sum_{\mu,\mu'}k\,\tilde{A}_{\mu}(\vec{k})\,\tilde{A}^{*}_{\mu'}(\vec{k}')\,\vec{e}_{\mu}(\hat{k})\times\vec{e}^{\,*}_{\mu'}(\hat{k}')$$

$$\underbrace{\int d^3x\;e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}}}_{(2\pi)^3\,\delta(\vec{k}-\vec{k}')}\quad e^{i(\omega_k-\omega_{k'})t}$$

$$= \frac{-i}{4\pi c}\int\frac{d^3k}{(2\pi)^3}\,k\sum_{\mu}\tilde{A}_{\mu}(\vec{k})\,\tilde{A}^{*}_{\mu'}(\vec{k})\quad\vec{e}_{\mu}(\hat{k})\times\vec{e}^{\,*}_{\mu'}(\hat{k})$$

Double Aside:

$$\vec{e}_{\mu}(\hat{k})\times\vec{e}_{\mu'}(\hat{k}) = \pm i\,\hat{k}\;\delta_{\mu\mu'}$$

$$\vec{e}_{\mu=\pm 1}(\hat{k})\times\vec{e}_{\mu'=\pm}(\hat{k}) = \pm i\,\hat{k}$$

$$\Rightarrow \int d^3x\,\frac{\vec{E}^{(-)}\times\vec{A}^{(+)}}{4\pi c} = \frac{1}{4\pi c}\int\frac{d^3k}{(2\pi)^3}\,\vec{k}\left[|\tilde{A}_{+}(\vec{k})|^2 - |\tilde{A}_{-}(\vec{k})|^2\right]$$

Note: $\vec{e}_\mu(\hat{k}) \times \vec{e}_\mu(-\hat{k}) = 0$

$$\Rightarrow \int \vec{E}^{(+)} \times \vec{A}^{(+)} \, d^3x = 0$$

$$\therefore \quad \vec{L}_{spin} = \int d^3x \, \frac{\vec{E}^{(-)} \times \vec{A}^{(+)}}{4\pi c} + c.c$$

$$\Rightarrow \boxed{\vec{L}_{spin} = \frac{1}{2\pi c} \int \frac{d^3k}{(2\pi)^3} \, \hat{k} \left[|\tilde{A}_+(\vec{k})|^2 - |\tilde{A}_-(\vec{k})|^2 \right]}$$

Thus, a circularly polarized field with $\vec{e}_\pm = \frac{\vec{e}_1 \pm i\vec{e}_2}{\sqrt{2}}$ carries angular momentum in the $\pm\hat{k}$ direction (positive/negative helicity)