Physics 511

Problem Set #1: DUE Wed. Jan. 25, 2006
Read: Jackson: Chap. 1, Appendix on units; **Low:** 1.1, Appendix on tensors;
Supplementary notes on tensors.

<u>Problem 1.</u> Vector identities

Use tensor notation and the Einstein summation convention to prove the following:

(a) $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$
(b) $\nabla\times(\psi\mathbf{A}) = \psi(\nabla\times\mathbf{A}) - (\mathbf{A}\times\nabla)\psi$
(c) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

For part (c), show this the “old fashion way”, by explicitly expressing both sides of the equation in Cartesian coordinates (i. e., take the curl, gradient, and divergences)

<u>Problem 2.</u> Irreducible tensors

Generally, a tensor in Cartesian coordinates of rank K can be decomposed into a sum of so called *irreducible* tensors which transform in a simple way under rotations. As we will see, the components of ther irreducible tensor are closely realted to the components of the spherical harmonics.

For example, a rank 2 Cartesian tensor U_{ij} can be decomposed as

$$U_{ij} = U_{ij}^{(0)} + U_{ij}^{(1)} + U_{ij}^{(2)}$$

where $U_{ij}^{(0)} = \frac{\mathrm{Tr}(U)}{3}\delta_{ij}$, is the “scalar” part with 1 component.
$U_{ij}^{(1)} = \frac{1}{2}\left(U_{ij} - U_{ji}\right)$, is the “vector” part with 3 independent components.
$U_{ij}^{(2)} = \frac{1}{2}\left(U_{ij} + U_{ji}\right) - \frac{\mathrm{Tr}(U)}{3}\delta_{ij}$, is the “tensor part” with 5 independent components

In general, the lth irreducible tensor will have $2l+1$ independent components.
(continued next page)

(a) For a tensor formed from the outer product of two vectors, $U_{ij} \equiv A_i B_j$, show that

$$U_{ij}^{(0)} = \frac{\mathbf{A}\cdot\mathbf{B}}{3}\delta_{ij} \qquad U_{ij}^{(1)} = \frac{1}{2}\varepsilon_{ijk}(\mathbf{A}\times\mathbf{B})_k \qquad U_{ij}^{(2)} = \frac{A_i B_j + A_j B_i}{2} - \frac{\mathbf{A}\cdot\mathbf{B}}{3}\delta_{ij}.$$

(b) Show that the "contraction" of two Caretsian tensors decompose as

$$U_{ij}W_{ij} = Tr\left(U\,W^T\right) = U_{ij}^{(0)}W_{ij}^{(0)} + U_{ij}^{(1)}W_{ij}^{(1)} + U_{ij}^{(2)}W_{ij}^{(2)}$$

This is a kind of projection between "orthogonal spaces".

(c) In general, for $l \geq 2$, the irreducible tensors are **symmetric** with respect to exchange of any two indices, and **zero when "traced"** (or "contracted") over any two indices. For example,

$U_{i_1 i_2 \ldots i_l}^{(l)} = U_{i_2 i_1 \ldots i_l}^{(l)}$ (Exchanging the first two indices), $U_{i_1 i_1 \ldots i_l}^{(l)} = 0$ (set $i_1 = i_2$ and sum).
Show that the $l = 3$ part of the of the following outer product satisfies these properties,

$$\left[x_i x_j x_k\right]^{(3)} \equiv x_i x_j x_k - \frac{1}{5}\left(x_i\delta_{jk} + x_k\delta_{ij} + x_j\delta_{ki}\right)r^2,$$

(*Extra credit*: Explicitly give the $2l + 1 = 7$ *independent* components of this tensor)

<u>Problem 3.</u> Helmholtz's theorem.
Any vector field can be decomposed into $\mathbf{V}(\mathbf{x}) = \mathbf{V}_T(\mathbf{x}) + \mathbf{V}_L(\mathbf{x})$, where:
$\mathbf{V}_T(\mathbf{x})$ is the "transverse" or "solenoidal" component, satisfying $\nabla\cdot\mathbf{V}_T(\mathbf{x}) = 0$, and
$\mathbf{V}_L(\mathbf{x})$ is the "longitudinal" or "irrotational" component, satisfying $\nabla\times\mathbf{V}_L(\mathbf{x}) = 0$.
Vector calculus then implies $\mathbf{V}_T(\mathbf{x}) = \nabla\times\mathbf{A}(\mathbf{x})$, $\mathbf{V}_L(\mathbf{x}) = -\nabla\phi(\mathbf{x})$, where **A** and ϕ are the "vector potential" and "scalar potential". According to Helmholtz's theorem,

$$\phi(\mathbf{x}) = \frac{1}{4\pi}\int\frac{\nabla'\cdot\mathbf{V}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}', \qquad \mathbf{A}(\mathbf{x}) = \frac{1}{4\pi}\int\frac{\nabla'\times\mathbf{V}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}'$$

where ∇' means that the derivatives are with respect to the "primed variables"

Prove it!

Hint: Consider $\nabla\times\left(\nabla\times\int\frac{\mathbf{V}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}'\right)$

A corollary to this theorem is that the divergence and curl completely define the vector field.